The subject of this talk is:

Trying to think about quantum field theory geometrically ...

though it is not obvious, or generally accepted, that a geometric viewpoint is appropriate.

Classical physics usually studies systems with
a finite number of degrees of freedom.

The states form a finite dimensional symplectic manifold Y.

I shall return to this

Usually $Y = T(X_t)$

X_t = configurations at time t

i.e. a state is determined by the instantaneous configuration and the speed with which it is changing

In QFT the state-space $Y = Y_M$ is ∞-dimensional, constructed functorially from space-time $M = \bigcup_t M_t$ (M_t = space at time t, assumed compact in this talk).

In QFT space-time is given and is passive: the particles and fields move in a fixed gravitational background. To describe the real world one must incorporate the dynamics of M itself.

$$Y_M = T(X_{M_t})$$

QFT is the quantum version of the classical picture in which the world consists of point particles interacting via fields defined in the space between the particles.

$$(\text{fields on } (M_t \setminus \sigma)) \longrightarrow X_{M_t} \longrightarrow C(M_t)$$

σ

fields

particles

Space of finite subsets of M_t

The state-spaces Y of QFT are "TAMED" by the energy function $H : Y \to \mathbb{R}_+$ ("Hamiltonian")

$$Y = \bigcup_{E \geq 0} Y_E \, , \qquad Y_E = \{ y \in Y : H(y) \leq E \}$$

Morally, each Y_E is compact and finite dimensional.

(assuming M_t is compact]

This makes QFT accessible to the viewpoint of algebraic topology.

CONTRAST : fluid flow / Navier-Stokes equation

But, literally, Y_E is neither compact nor finite dimensional.

I shall say a little about each property.

"Compactness" of Y_E :

Palais - Smale condition — every downwards gradient trajectory of E goes to a critical point.

Y_E is compact in a coarser topology on Y

Usually this condition is nearly but not precisely satisfied in QFT.

Example 1 $Y = \mathcal{L} M$ = smooth loops in a Riemannian manifold M

$$E(\gamma) = \frac{1}{2} \int_0^1 \| \dot{\gamma}(t) \|^2 dt$$

The critical points of E are the closed geodesics.

In this case the Palais-Smale condition holds, and the topology of $\mathcal{L}M$ is related to the closed geodesics by Morse theory.

(Non-) example

Perelman's Ricci flow on the space Y of compact Riemannian 3-manifolds

This is a gravitational rather than a QFT example

The critical points are the manifolds of constant curvature

Starting from an arbitrary manifold
the flow does not usually lead to a manifold of constant curvature
rather
to a singular manifold consisting of constant curvature manifolds
stuck together according to Thurston's geometrization picture.

<u>Example 2</u> $\quad Y = \mathrm{Map}(S^2; S^2)$ with usual energy.

critical points are harmonic maps

either holomorphic or anti-holomorphic (rational functions)

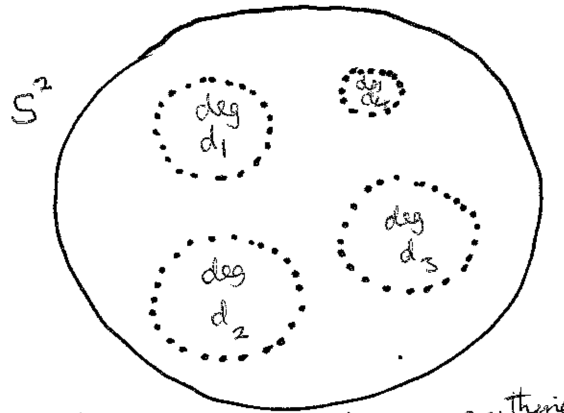
positive degree . negative degree

— the absolute minima in their respective components

BUT <u>not</u> compact

To have a Morse-theoretical description of Y

we must adjoin higher "virtual" critical levels



$$\text{index} = |d_1| + |d_2| + |d_3| + \cdots$$

$$\text{degree} = d_1 + d_2 + d_3 + \cdots$$

A virtual critical map takes everything outside the dotted discs to a single point, and maps each disc to the whole of S^2 by a holomorphic or antiholomorphic map.

There are virtual descending flow-lines along which a disc of degree $+d$ annihilates a disc of degree $-d$.

... of holomorphic

Y_E is "finite dimensional": The classical state-space Y is replaced by a Hilbert space $\mathfrak{H}$.

Here QUANTUM THEORY enters for the first time.

$$Y \xrightarrow{H} \mathbb{R}_+ \quad \longleftrightarrow \quad \mathfrak{H} \xrightarrow{H} \mathfrak{H}$$

The energy function H becomes a positive self-adjoint operator in $\mathfrak{H}$.

$$Y_E \subset Y \quad \longleftrightarrow \quad \mathfrak{H}_E \subset \mathfrak{H}$$

spanned by eigenvectors of H with eigenvalue $\leq E$

$$\mathrm{vol}(Y_E) \sim \dim(\mathfrak{H}_E)$$

Liouville measure

as $E \to \infty$

How can we make sense of this when $\dim(Y) = \infty$?

<u>Example</u> Waves on a compact manifold M_0

$$X = C^\infty(M_0 ; \mathbb{R}) \quad \text{— real vector space}$$

$$Y = TX = X \oplus X$$

$$H(\varphi, \dot\varphi) = \frac{1}{2} \int_{M_0} \left(\|d\varphi\|^2 + \dot\varphi^2 + m^2 \varphi^2 \right) dx = \sum E_k y_k^2$$

$$(E_k \uparrow \infty)$$

$$Y_E = \text{ellipsoid with axes } \frac{1}{E_k^{1/2}}$$

Planck (1900): measure $\mathrm{vol}(Y_E)$ by counting lattice points

"High energy degrees of freedom are not activated"

$$(y_k = 0 \quad \text{if} \quad E_k > E)$$

Y_E behaves like a finite-dimensional ellipsoid whose dimension increases with E

A "degree of freedom" is a symplectic projection to $\mathbb{R}^2$.

"Gromov squeezing"

$$\pi : Y \longrightarrow \mathbb{R}^2$$

$$\pi(Y_E) \subset \mathbb{R}^2$$

It is not activated if $\mathrm{vol}(\pi(Y_E)) < 1$.

QUANTUM THEORY :

(Y, ω) is "really" an approximation to a non commutative algebra $\mathcal{A} = \mathcal{A}_h$ (topological $*$-algebra)

symplectic form

ω describes the deformation of $\mathcal{A}_0 = C^\infty(Y)$ to $\mathcal{A}_h$

— in classical physics the symplectic structure arises ~~from~~ because the trajectories obey a principle of LEAST ACTION.

Example of waves :

$$Y = C^\infty(M_0) \oplus C^\infty(M_0) \qquad (\varphi, \dot\varphi) \in Y$$

What do we mean by $\mathcal{A}_0 = C^\infty(Y)$?

$\varphi \longmapsto \varphi(x)$ is NOT in $\mathcal{A}_0$

$\varphi \longmapsto \varphi_f = \int_{M_0} f(x)\,\varphi(x)\,dx$ IS in $\mathcal{A}_0$.

Thus $S(Y) \hookrightarrow \mathcal{A}_0$ (map of algebras extending the linear map $f \mapsto \varphi_f$)

with dense image

BUT the non-commutativity completely changes the geometric picture.

$$
\begin{tikzcd}
 & C^\infty(\mathbb{R}) \arrow[dr] & \\
\mathbb{C}[a] \arrow[ur] \arrow[rr, hook] & & C^\infty(\mathbb{N})
\end{tikzcd}
\quad \left(= \underset{e_0}{\mathbb{C}} \oplus \underset{e_1}{\mathbb{C}} \oplus \underset{e_2}{\mathbb{C}} \oplus \cdots \quad e_k^2 = e_k \right)
$$

<u>Theorem</u> If $a = bb^*$ with $b^*b - bb^* = 1$

then a generates $C^\infty(\mathbb{N})$

$\mathbb{C}[a] \subset \mathbb{C}\langle b, b^* \rangle$

If a $*$-action of $\mathbb{C}[a]$ on $\mathcal{H}$ extends to $\mathbb{C}\langle b, b^*\rangle$

then it factorizes through $\mathbb{C}[a] \to C^\infty(\mathbb{N})$.

<u>Generalization</u> When $Y = C^\infty(M_0) \oplus C^\infty(M_D)$

and $S(Y)$ is deformed

we have commuting elements $a_f \in \mathcal{A}_h$ for $f \in C^\infty(M_0)$

which generate $C^\infty\left(\coprod_{n \geq 0} \left(M_0^n / \mathrm{Symm}_n \right) \right)$.

$$a_f \longleftrightarrow \left((x_1, \ldots, x_n) \longmapsto \sum f(x_i) \right)$$

Thus we see particles not <u>waves</u>.

The algebra $\mathcal{A}_M$ of QFT has a subtle structure induced by its relation to the space-time M.

QFT is about the functor $M \mapsto \mathcal{A}_M$.

It describes the way an algebra can be "organized" by a space.

— perhaps it is only an approximation to say that $\mathcal{A}_M$ is an algebra

String theory perhaps suggests that $\mathcal{A}_M$ should not be precisely associative.

Noncommutative geometry

A commutative algebra $\mathcal{A}$ determines a space,
and hence a homotopy type.

What if $\mathcal{A}$ is noncommutative?

Category $\mathcal{C}_{\mathcal{A}}$ of finite-dimensional $\mathcal{A}$-modules

— consider $|\mathcal{C}_{\mathcal{A}}^{\mathrm{virt}}|$.

$\mathcal{C}_{\mathcal{A}}^{\mathrm{virt}}$ = category of virtual objects of $\mathcal{C}_{\mathcal{A}}$.

$\pi_*(|\mathcal{C}_{\mathcal{A}}^{\mathrm{virt}}|)$ = algebraic K-theory of $\mathcal{A}$

$\mathcal{A} \mapsto \mathcal{C}_{\mathcal{A}}$ is contravariant

If $\mathcal{A} = C^\infty(M)$ then $|\mathcal{C}_{\mathcal{A}}^{\mathrm{virt}}| = k_M = (M \wedge bU)_0$

i.e. $\pi_i(k_M) = k_i(M)$.

(In fact $k_M \simeq$ space of finite sets of (labelled) points in M.)

QUESTION:

Is $|\mathcal{C}_{\mathcal{A}}^{\mathrm{virt}}|$ always $(X \wedge bU)_0$ for some space X?

NO!

The functor $\mathcal{A} \mapsto |\mathcal{C}_{\mathcal{A}}^{\mathrm{virt}}|$ is not stable under deformation.

$\mathcal{A}_0 = C_0^\infty(\mathbb{R}^2)$ deforms to $\mathcal{A}_h$, and if $h \neq 0$ then

$$\mathcal{A}_h \cong (\text{smoothing operators on } L^2(\mathbb{R}))$$

i.e. $$(f \circ g)(x,y) = \int_{\mathbb{R}} f(x,t)\, g(t,y)\, dt.$$

So $(\mathcal{A}_h\text{- modules}) \simeq (\text{vector spaces})$

↑ STONE – von NEUMANN theorem

i.e. $\mathcal{A}_h$ describes a <u>point</u> if $h \neq 0$.

To have deformation-invariance we must identify $(\mathbb{R}^2)^+$ with a point,

and, more generally, must stabilize $M \wedge bU$ to

$$M \wedge BU = \varinjlim \{ M \wedge bU \xrightarrow[\text{Bott}]{} S^{-2}(M \wedge bU) \xrightarrow[\text{Bott}]{} S^{-4}(M \wedge bU) \xrightarrow[\text{Bott}]{} \cdots \}$$

Cf. KASPAROV defined a BU-module-spectrum of morphisms between two C^*-algebras

KITCHLOO " " " " " " " " two symplectic manifolds

Bott stabilization of the homotopy type of a noncommutative algebra
has an algebraic analogue :

to make cyclic homology deformation invariant
we must invert the S operator of Connes,
forming PERIODIC cyclic homology.

Proved by Goodwillie, Getzler, ...

A noncommutave ring thus has a stable homotopy type, but not a homotopy type.

One place in ordinary geometry where this phenomenon arises
is FLOER THEORY.

FLOER HOMOTOPY TYPES

$Y \xrightarrow{H} \mathbb{R}_+$ ∞-dimensional manifold tamed by H

Floer theory arises when

$H(y) = \frac{1}{2} \| dh(y) \|^2$ for some $h : Y \to \mathbb{R}/\mathbb{Z}$

Then $\mathrm{crit}(h) \subset \mathrm{crit}(H)$,

but, more important,

$\mathrm{Hessian}(H) = \mathrm{Hessian}(h)^2$

and $\mathrm{Hessian}(h)$ defines a polarization of Y,

i.e. a decomposition $T_y Y = T_y^+ Y \oplus T_y^- Y$

up to finite-dimensional adjustment.

— The polarization determines a class of subspaces of $T_y Y$ which are "close to" $T_y^- Y$

— the "restricted Grassmannian" $Gr_{res}(T_y Y)$.

Any two points of $Gr_{res}(T_y Y)$ have a finite relative dimension

The space of downwards directions at a critical point of h belongs to $Gr_{res}(T_y Y)$.

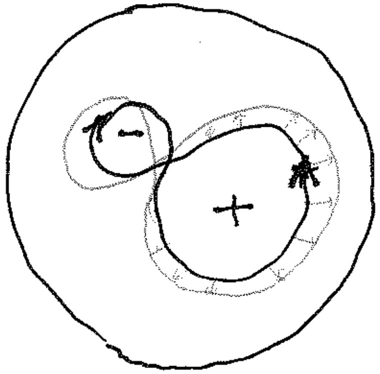
Hence two critical points have a finite relative index.

Example 1

$$Y = \mathcal{L} S^2$$

"Signed" area is defined modulo area$(S^2) = 4\pi$.

$$h(\gamma) = \text{area}(\gamma) \in \mathbb{R}/4\pi\mathbb{Z}$$



$$dh(\gamma; \delta\gamma) = \int_{S^1} \langle \dot{\gamma}, \delta\gamma \rangle \, dt$$

$$\text{So} \quad \tfrac{1}{2}\|dh\|^2 = \tfrac{1}{2}\int_{S^1} \|\dot{\gamma}\|^2 \, dt = H(\gamma)$$

The only critical points of h are the point-loops.
A gradient line from one point-loop to another is a holomorphic map $S^2 \to S^2$.

More generally,

$$Y = \mathcal{L} M$$

with M integral symplectic

$T_\gamma Y = \Gamma(S^1; \gamma^* TM)$ is polarized by the positive/negative eigenspaces of

$$J\frac{D}{Dt} : T_\gamma Y \to T_\gamma Y$$

J = almost-complex structure of M

Example 2

Compact Lie group G with $\langle\,,\rangle$ on $\mathfrak{g} = \mathrm{Lie}(G)$

$$Y = \{ G\text{- bundles } E \text{ with connection } A \text{ on } M^3 \} / \text{isomorphisms}$$

$$h = \text{Chern-Simons} : Y \longrightarrow \mathbb{R}/\mathbb{Z}$$

such that $dh(A) = F_A =$ curvature of A

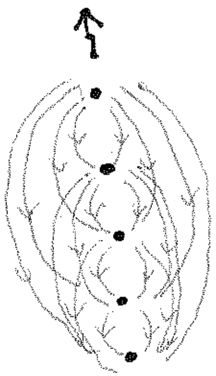
$$\tfrac{1}{2} \| dh(A) \|^2 = \tfrac{1}{2} \int_M \| F_A \|^2$$

$$= \text{Yang-Mills}(A)$$

$$T_{(E,A)} Y = \Omega^1(M; \mathrm{End}(E)) / (\mathrm{im}\ D_A : \Omega^0 \to \Omega^1)$$

$$F_A \in T^*_{(E,A)} Y = \cancel{\ker} D_A : \Omega^2(M; \mathrm{End}(E)) \longrightarrow \Omega^3(M; \mathrm{End}(E))$$

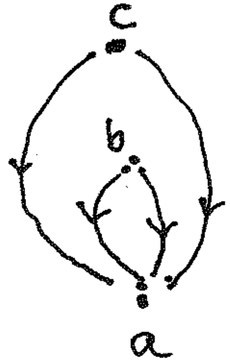
$T_{(E,A)} Y$ is polarized by the operator $*D_A + D_A *$.



A flow-line of h between two critical points has a finite index-difference.

Let Y^e = points on flow-lines with index-difference $\leq e$.

The $\{Y^e\}$ play the same role as the $\{Y_E\}$.



$$Y_{ab} \subset Y_{ac}$$

points on flow line from b to a

In good cases, can assemble the Y_{ab} by these maps to form a pro-spectrum

$$Y_{ac} \longrightarrow Y_{bc}^{E} \quad \left(= \text{Thom space of } E\right)$$

E = normal bundle of Y_{bc} in Y_{ac}

Recent work of M. Lipyanskiy constructs "semi-infinite" bordism classes of Y.

Semi-infinite de Rham complex ???

A polarized vector space has well-defined semi-infinite exterior forms.

Can one define a semi-infinite ~~exter~~ de Rham complex which gives the Floer cohomology?

Final remarks

(noncommutative rings) $\longrightarrow$ (BU-module-spectra)

$\updownarrow$ essentially the same

(1-dimensional QFTs)

(2-dimensional QFTs) $\xrightarrow{??}$ (module spectra over elliptic cohomology spectrum)

$\uparrow$

suggested by string theory