Highlights in Algebraic Topology

Ulrike Tillmann, Oxford

2012 Stanford Symposium in honour of Gunnar Carlsson, Ralph Cohen and Ib Madsen

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- anything published more than 30 years ago

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- Kervaire conjecture
- Floer homotopy type

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What to choose? So much to pick from!

Sins of omission:

- anything published more than 30 years ago
- Rat_k and Atiyah-Jones conjecture
- Kervaire conjecture
- Floer homotopy type
- controlled topology
- anything I am not *aware of*

Annals of Mathematics, 120 (1984), 189–224

Equivariant stable homotopy and Segal's Burnside ring conjecture

By GUNNAR CARLSSON*

Introduction

In 1960, M. F. Atiyah proved the following:

THEOREM [7]. *Let BG denote the classifying space of a finite group G, and let KU^* denote representable complex periodic K theory (so $KU^0(X) = [X, BU \times \mathbf{Z}]$). Then we have*

$$KU^0(BG) \cong \hat{R}[G]$$

and

$$KU^1(BG) = 0$$

where $\hat{R}[G]$ denotes the completion of the complex representation ring at its augmentation ideal.

Analogous results were proved later for $K0$, in the generality of compact Lie groups, by Atiyah and Segal [8], and for $K\mathbf{F}_q$, the algebraic K-theory spectrum associated to the finite field $\mathbf{F}_q$, by Rector [29] using Quillen's [26] computation of $\pi_*(K\mathbf{F}_q)$.

In each case, the answer involves an appropriate completed representation ring of G, and the cohomology theory in question is constructed from the permutative category of finite dimensional vector spaces over a field (see [31]). If one considers cohomology theories constructed from other permutative categories, one expects to find analogous computations in terms of a "completed representation ring" of G in the given category, appropriately defined. In particular, stable cohomotopy, π_S^*, is constructed from the category of finite sets [31], and for this category the analogue of the representation ring is a well-known object, the *Burnside ring* $A(G)$ [13]. $A(G)$ is a commutative ring with augmentation, so one may speak of $\hat{A}(G)$, the completed Burnside ring. Moreover, there is

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Annals of Mathematics 1984

Segal Conjecture

G a finite group.

Atiyah: $R[G] \to K^0(BG) = [BG, \mathbb{Z} \times BU]$ taking $\rho : G \to U_n$ to $B\rho : BG \to \{n\} \times BU$ induces an isomorphism after completion of the complex representation ring $R[G]$ with respect to its augmentation ideal, the kernel of $R[G] \to \mathbb{Z}$.

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Segal conjecture:
$A(G) \to \pi_s^0(BG) = [BG, Q(S^0)] = [BG, \mathbb{Z} \times B\Sigma_\infty^+]$
taking $\rho : G \to \Sigma_n$ to $B\rho : BG \to \{n\} \times B\Sigma_\infty$ induces an isomorphism after completion of the Burnside ring $A(G)$ with respect to its augmentation ideal.

Lin: $G = \mathbb{Z}_2$ (Adams SS arguments)
Gunawardea: $G = \mathbb{Z}_p$, p odd (following Lin)

Adams, Gunawardea, Miller: $G = (\mathbb{Z}_p)^k$
(much more complicated Adams SS arguments)

May, McClure: if the conjecture holds for all p-groups then it holds for any group

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May, McClure: if the conjecture holds for all p-groups then it holds for any group

Carlsson: if the conjecture holds for $(\mathbb{Z}_p)^k$ for all k then it holds for any p group

New input: reformulation in terms of equivariant stable homotopy theory

Homotopy limit problem:

when is the map below a homotopy equivalence?

$$X^G = \mathrm{maps}(p, X)^G \to \mathrm{maps}(EG, X)^G$$

When it is, we get a spectral sequence

$$E_2^{*,*} = H^*(G, \pi_{-*}(X)) \implies \pi_*(\mathrm{maps}(EG, X)^G)$$

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Segal conjecture: $X = Q^G(S^0)$ and $(Q^G(S^0))^G = A(G)$

Sullivan conjecture: X a finite complex with trivial G-action, and $\text{maps}(EG, X)^G = \text{maps}(BG, X)$

Conjectural **descent spectral sequence** for algebraic K-theory: For Galois extensions L of L_0,

$$E_2^{*,*} = H^*(G, K_{-*}(L)) \Longrightarrow K_*(L_0)$$

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Carlsson proved:

• generalized Sullivan conjecture for finite p-groups and X a finite G-space; need p-completion.
(*Invent. 1991*)

• descent for certain finite Galois groups.
(*Amer. J. Math. 1991*)

• Segal conjecture for tori with **Douglas** and **Dundas**.
(*Adv. Math. 2011*)

Annals of Mathematics, 122 (1985), 237–328

The immersion conjecture for differentiable manifolds

By Ralph L. Cohen*

To Fran

TABLE OF CONTENTS

Introduction

An old problem in differential topology is to find the smallest integer k_n with the property that every compact, C^∞ n-dimensional manifold M^n immerses

*The research in this paper was partially supported by N.S.F. grant MCS 79-06085-A01 and by a fellowship from the A. P. Sloan Foundation.

Annals of Mathematics 1985

Immersion Conjecture

Whitney: $M^n \hookrightarrow \mathbb{R}^{2n}$ and $M^n \looparrowright \mathbb{R}^{2n-1}$

Massey: $w_i(\nu M) = 0$ for $i > n - \alpha(n)$
and $w_i(\nu \mathbb{R}P^{i+1}) \neq 0$ for $i = 2^j - 1$.

Immersion Conjecture: $M^n \looparrowright \mathbb{R}^{2n-\alpha(n)}$

Here $\alpha(n) =$ length of 2-adic expansion of n.

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Immersion Conjecture: $M^n \looparrowright \mathbb{R}^{2n-\alpha(n)}$

Here $\alpha(n)$ = length of 2-adic expansion of n.
Put $I_n := \bigcap_{M^n} ker(\nu^*)$.

$$\begin{array}{ccc} H^*BO(n-\alpha(n)) & \longrightarrow & (H^*BO)/I_n \\ \uparrow & & \downarrow \\ H^*BO & \xrightarrow{\nu^*} & H^*M \end{array}$$

Hirsch-Smale: conjecture is equivalent to existence of lifting

$$\begin{array}{ccc} . & & BO(n-\alpha(n)) \\ & & \downarrow \\ M & \xrightarrow{\nu} & BO \end{array}$$

Brown-Petersen construct spaces BO/I_n with cohomology $(H^*BO)/I_n$ and maps at the level of Thom spaces

$$\begin{array}{ccc} MO/I_n & \longrightarrow & MO(n-\alpha(n)) \\ \uparrow & & \downarrow \\ Th(\nu(M)) & \xrightarrow{\nu} & MO \end{array}$$

Ralph Cohen proved the conjecture by "de-Thomification": uses Snaith splitting theory, Brown-Petersen obstruction, ... some 90 pages.

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Questions:

- Is there a more geometric proof of the conjecture?
- Can the theorem be improved for special classes of manifolds?

Novikov conjecture

M a $4n$ dimensional, smooth, closed, oriented manifold;
$L(M) =$ polynomial in $p_i(TM)$;
$sig(M) =$ signature of the intersection form

$$H^{2i}M \otimes H^{2i}M \to H^{4i}M = \mathbb{Q}$$

Hirzebruch signature formula:

$$\langle L(M), [M] \rangle = sig(M)$$

Novikov conjecture:
Let Γ discrete group, $\alpha : M \longrightarrow B\Gamma$, and $x \in H^{4k}B\Gamma$.
Then $\langle L(M) \cup \alpha^*(x), [M] \rangle$ is a homotopy invariant.

This is equivalent to the assembly map for surgery theory being a $\mathbb{Q}$-injection:

$$B\Gamma \wedge \mathbb{L}(\mathbb{Z}) \longrightarrow \mathbb{L}(\mathbb{Z}[\Gamma])$$

Analogues in K-theory: of C^*-algebras, rings, spaces.

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Analogues in K-theory: of C^*-algebras, rings, spaces.

Connes cyclic homology:

$$HC_*A = H_*(\ldots \xrightarrow{b} A^{\otimes * +1}/(1-t) \xrightarrow{b} \ldots)$$

Long exact sequence:

$$\ldots HC_{*+1}A \xrightarrow{B} HH_*A \xrightarrow{I} HC_*A \xrightarrow{S} HC_{*+2}A \ldots$$

For group rings there is a topological interpretation of these algebraic invariants:

$HH_*\,\mathbb{Z}[\Gamma] = H_*\,LB\Gamma$ and $HC_*\,\mathbb{Z}[\Gamma] = H_*^{S^1}LB\Gamma$

Injectivity of the assembly for HC_* corresponds to the S^1-equivariant split inclusion $B\Gamma \subset LB\Gamma$ as constant loops.

Cohen, Jones (*Topology 1990*)

Idea: translate K-theory assembly map to cyclic homology assembly map via a "lift" of the Dennis trace map $D : K_*A \to HH_*A$ through $HC_{*+1}A$.

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Bökstedt, Hsiang, Madsen define the

$$\text{cyclotomic trace} \qquad \text{Trc} : A(X) \longrightarrow TC(X;p).$$

and prove:
$B\Gamma_+ \wedge A(*) \longrightarrow A(B\Gamma)$ and $B\Gamma_+ \wedge K(\mathbb{Z}) \longrightarrow K(\mathbb{Z}[\Gamma])$ are rational split injections for a large classes of Γ.

Invent. math. 111, 465–540 (1993)

Inventiones mathematicae

The cyclotomic trace and algebraic K-theory of spaces

M. Bökstedt[1,*], W.C. Hsiang[2], and I. Madsen[1]

[1] Department of Mathematics, Aarhus University, DK-8000 Aarhus, Denmark
[2] Department of Mathematics, Princeton University, Princeton, NJ 08544, USA

Oblatum VI-1990 & 3-VII-1992

Introduction

The cyclotomic trace is a map from algebraic K-theory of a group ring to a certain topological refinement of cyclic homology. The target is naturally mapped to topological Hochschild homology, and the cyclotomic trace lifts the topological Dennis trace. Our cyclic homology can be defined also for "group rings up to homotopy", and in this setting the cyclotomic trace produces invariants of Waldhausen's A-theory.

Our main applications go in two directions. We show on the one hand that the K-theory assembly map is rationally injective for a large class of discrete groups, including groups which have finitely generated Eilenberg-MacLane homology in each degree. This is the analogue in algebraic K-theory of Novikov's conjecture about homotopy invariance of higher signatures. It implies for Quillen's K-groups the inclusion

$$H_i(\Gamma;\mathbb{Q}) \oplus \sum_{k\geq 1}^{\oplus} H_{i-4k-1}(\Gamma;\mathbb{Q}) \subset K_i(\mathbb{Z}\Gamma)\otimes\mathbb{Q}\,. \tag{0.1}$$

On the other hand, the cyclotomic trace gives information about $A(*)$. We show that its p-adic completion contains $\Omega^\infty S^\infty(\Sigma BO(2)) \times \Omega^\infty S^\infty$ as a direct factor, at least if p is a regular prime (in terms of number theory). This in turn gives

$$\operatorname{holim}_{\longrightarrow} BC^{\mathrm{Diff}}(D^n)^\wedge_p \simeq \Omega^\infty S^\infty(BO(2))^\wedge_p \times T_p \tag{0.2}$$

(after p-adic completion, p regular) where $C^{\mathrm{Diff}}(D^n)$ denotes the space of differentiable pseudo-isotopies of the n dimensional disc, and T_p is a torsion space (possibly contractible), cf. [W4].

The topological cyclic homology space $TC(F, p)$ can be defined for any "functor with smash product" in the sense of [B] and for any prime p. Such functors include group rings, $R\Gamma$ and homotopy group rings, $\Omega^\infty S^\infty(\Gamma_+)$. At the time of writing only limited information is available about $TC(R\Gamma, p)$ in the group ring case, and anyhow this is not the subject of the present paper; here we give, for any group-like topological momoid Γ, an explicit calculation of $TC(\Omega^\infty S^\infty(\Gamma_+), p)$ in terms of more familiar objects in homotopy theory.

* Partially supported by an NSF-Grant

Inventiones mathematicae 1993

Bökstedt-Carlsson-Cohen-Goodwillie-Hsiang-Madsen:
Trc is a homotopy equivalence on the reduced theories after p-completion.
(*Duke 1996*)

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Madsen and Hesselholt

- extend Trc to rings
- compute: $\tilde{K}_*(A[x]/(x^d))$ for A a smooth algebra over a perfect field k a perfect field of characteristic p.
- prove the Lichtenbaum-Quillen conjecture for certain complete discrete valuation fields of characteristic zero.

(*Annals 2003*)

Carlsson-Pedersen prove the splitting of assembly map of spectra

$$B\Gamma \wedge \mathbb{L}(\mathbb{Z}) \to L(\mathbb{Z}[\Gamma]) \text{ and } B\Gamma \wedge K(\mathbb{Z}) \to K(\mathbb{Z}[\Gamma]).$$

using controlled topology for groups Γ with $B\Gamma$ finite plus some condition on $E\Gamma$. Essentially the idea is to use an "infinite transfer construction" for the splitting. (*Topology 1996*)

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Carlsson-Goldfarb prove split injectivity in algebraic K-theory for geometrically finite groups with finite asymptotic dimension (this includes all Gromov hyperbolic groups) using coarse topology - extending rational results by G. Yu.
(*Invent. Math. 2004*)

Persistent homology

Persistent homology

Let $\mathcal{D} \subset \mathbb{R}^N$ be a point cloud data set.
At scale ϵ consider

$$X(\epsilon) := \bigcup_{x \in \mathcal{D}} B_\epsilon(x)$$

Varing scales leads to filtered associated chain complexes:

$$\ldots \xrightarrow{t} C(\epsilon_i) \xrightarrow{t} C(\epsilon_{i+1}) \xrightarrow{t} \ldots$$

Want to track when homology classes are born and when they die.

Carlsson - Zomorodian:
Persistent homology $PH_n := \bigoplus_i H_n(C(i))$ with coefficients in a field $\mathbb{F}$ can be interpreted as homology over the ring $\mathbb{F}[t]$.

Key: $\mathbb{F}[t]$ is a PID
$\Longrightarrow$ finitely generated, graded modules are sums of modules of the form $\Sigma^i \mathbb{F}[t]/(t^{j-i})$
$\Longrightarrow$ easy visualisation by barcodes of length $j - i$ $\Longrightarrow$ algorithmic computation of persistent homology.

Nicolau-Levine-Carlsson: “Topological based data analysis identifies a subgroup of breast cancers with a unique mutational profile and excellent survival”
(*Proceedings of the National Academy of Sciences 2011*)

Discrete Comput Geom 33:249–274 (2005)
DOI: 10.1007/s00454-004-1146-y

Discrete & Computational
Geometry

Computing Persistent Homology*

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Abstract. We show that the persistent homology of a filtered d-dimensional simplicial complex is simply the standard homology of a particular graded module over a polynomial ring. Our analysis establishes the existence of a simple description of persistent homology groups over arbitrary fields. It also enables us to derive a natural algorithm for computing persistent homology of spaces in arbitrary dimension over any field. This result generalizes and extends the previously known algorithm that was restricted to subcomplexes of $\mathbb{S}^3$ and $\mathbb{Z}_2$ coefficients. Finally, our study implies the lack of a simple classification over non-fields. Instead, we give an algorithm for computing individual persistent homology groups over an arbitrary principal ideal domain in any dimension.

1. Introduction

In this paper we study the homology of a filtered d-dimensional simplicial complex K, allowing an arbitrary principal ideal domain D as the ground ring of coefficients. A filtered complex is an increasing sequence of simplicial complexes, as shown in Fig. 1. It determines an *inductive system* of homology groups, i.e., a family of Abelian groups $\{G_i\}_{i \geq 0}$ together with homomorphisms $G_i \to G_{i+1}$. If the homology is computed with field coefficients, we obtain an inductive system of vector spaces over the field. Each vector space is determined up to isomorphism by its dimension. In this paper we obtain a simple classification of an inductive system of vector spaces. Our classification is in

* Research by the first author was partially supported by NSF under Grants CCR-00-86013 and ITR-0086013. Research by the second author was partially supported by NSF under Grant DMS-0101364. Research by both authors was partially supported by NSF under Grant DMS-0138456.

Discrete and Computational Geometry 2005

String topology

For a smooth, closed, oriented, n-dimensional manifold M, Chas-Sullivan constructed at the chain level map giving a product of degree $-n$

$$H_* LM \otimes H_* LM \longrightarrow H_* LM$$

which extends to a BV-algebra structure.

String topology

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which extends to a BV-algebra structure.

Cohen-Jones lift the chain-level construction to a product of ring spectrum:

$$LM^{-TM} \wedge LM^{-TM} \longrightarrow LM^{-TM}$$

and thus the Chas-Sullivan structure can be studied via homotopy theory.

Math. Ann. 324, 773–798 (2002) **Mathematische Annalen**
DOI: 10.1007/s00208-002-0362-0

A homotopy theoretic realization of string topology

Ralph L. Cohen · John D. S. Jones

Received: 31 July 2001 / Revised version: 11 September 2001
Published online: 5 September 2002 –

Abstract. Let M be a closed, oriented manifold of dimension d. Let LM be the space of smooth loops in M. In [2] Chas and Sullivan defined a product on the homology $H_*(LM)$ of degree $-d$. They then investigated other structure that this product induces, including a Batalin -Vilkovisky structure, and a Lie algebra structure on the S^1 equivariant homology $H_*^{S^1}(LM)$. These algebraic structures, as well as others, came under the general heading of the "string topology" of M. In this paper we will describe a realization of the Chas–Sullivan loop product in terms of a ring spectrum structure on the Thom spectrum of a certain virtual bundle over the loop space. We also show that an operad action on the homology of the loop space discovered by Voronov has a homotopy theoretic realization on the level of Thom spectra. This is the "cactus operad" defined in [6] which is equivalent to operad of framed disks in $\mathbb{R}^2$. This operad action realizes the Chas - Sullivan BV structure on $H_*(LM)$. We then describe a cosimplicial model of this ring spectrum, and by applying the singular cochain functor to this cosimplicial spectrum we show that this ring structure can be interpreted as the cup product in the Hochschild cohomology, $HH^*(C^*(M); C^*(M))$.

Introduction

Let M^d be a smooth, closed d-dimensional manifold, and let $LM = C^\infty(S^1, M)$ be the space of smooth loops in M. In [2] Chas and Sullivan described an intersection product on the homology $H_*(LM)$ having total degree $-d$,

$$\circ : H_q(LM) \otimes H_r(LM) \to H_{q+r-d}(LM).$$

In this paper we show that this product is realized by a geometric structure, not on the loop space itself, but on the Thom spectrum of a certain bundle over LM. We describe this structure both homotopy theoretically and simplicially, and in so doing describe the relationship of the Chas-Sullivan product to the cup product in Hochschild cohomology. We now make these statements more precise.

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The first author was partially supported by a grant from the NSF.

Mathematische Annalen 2002

More detail:

Let $M \subset \mathbb{R}^{N+n}$ and ν^N be the normal bundle. Set $M^{-TM} := \Sigma^{-(N+n)} Th(\nu^N) = \text{maps}(M_+, S^0)$.

Let $ev : LM \to M$ be defined by $\gamma \mapsto \gamma(1)$. Then

$$LM^{-TM} := \Sigma^{-(N+n)} Th(ev^*(\nu)).$$

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Let $ev : LM \to M$ be defined by $\gamma \mapsto \gamma(1)$. Then

$$LM^{-TM} := \Sigma^{-(N+n)} Th(ev^*(\nu)).$$

Key-construction:

$$\begin{array}{ccc} LM \times_M LM & \longrightarrow & LM \times LM \\ {\scriptstyle ev}\downarrow & & \downarrow{\scriptstyle ev \times ev} \\ M & \xrightarrow{\triangle} & M \times M \end{array}$$

gives

$$\begin{aligned} H_* LM \times LM &\longrightarrow H_* Th((ev)^*(TM)) \\ &\simeq H_{*-n} LM \times_M LM \longrightarrow H_{*-n} LM. \end{aligned}$$

This product extends to an algebra structure of Voronov's cactus operad, and hence there is a BV-algebra structure on the homology.

Cohen-Godin extend the BV-algebra structure to a TQFT (without trace), i.e. identify $H_* LM$ as a finite dimenional graded commutative Frobenius algebra (without counit).

Cohen-Klein-Sullivan show that the product is an oriented homotopy invariant. (Independently also shown by Gruher-Salvatore.)

Question: Are there any operations on $H_* LM$ that are not oriented homotopy invariants?

Mumford Conjecture

$F_{g,1}$ orientable surface of genus g and 1 boundary circle:

$$B\mathrm{Diff}(F_{g,1}) \simeq B\Gamma_{g,1} \simeq \mathcal{M}_{g,1}$$

$\Gamma_{g,1} = \mathrm{Diff}(F_{g,1})$. Put $\Gamma_\infty := \lim_{g\to\infty} \Gamma_{g,1}$.

Early 1980s:
Harer: $H_* B\Gamma_{g,1} = H_* B\Gamma_\infty$ for $g << *$.

E. Miller: $H_* B\Gamma_\infty \otimes \mathbb{Q} \supset \mathbb{Q}[\kappa_i]$, $\deg \kappa_i = 2i$

Mumford conjecture: $H_* B\Gamma_\infty \otimes \mathbb{Q} = \mathbb{Q}[\kappa_i]$.

Miller-Morita-Mumford classes:
For a bundle of smooth, closed surfaces $\pi : E \to B$ with vertical tangent space T^vE,

$$\kappa_i := \pi_*(e(T^vE)^{i+1})) \in H^{2i}B$$

Mid 1990s:
Interpreted at the space level:

$$B_+ \xrightarrow{trf} Q(E_+) \xrightarrow{T^vE} Q(\mathbb{C}P^\infty_+)$$

Generalized Mumford conjecture:

$$\mathbb{Z} \times B\Gamma^+_\infty \simeq \Omega^\infty(\mathbb{C}P^\infty_{-1}) = \Omega^\infty MTSO(2).$$

Proved by **Madsen and Weiss** in 2002. Gives also integral information, in particular lots of torsion classes.

Annals of Mathematics, **165** (2007), 843–941

The stable moduli space of Riemann surfaces: Mumford's conjecture

By Ib Madsen and Michael Weiss*

Abstract

D. Mumford conjectured in [33] that the rational cohomology of the stable moduli space of Riemann surfaces is a polynomial algebra generated by certain classes κ_i of dimension $2i$. For the purpose of calculating rational cohomology, one may replace the stable moduli space of Riemann surfaces by $B\Gamma_\infty$, where Γ_∞ is the group of isotopy classes of automorphisms of a smooth oriented connected surface of "large" genus. Tillmann's theorem [44] that the plus construction makes $B\Gamma_\infty$ into an infinite loop space led to a stable homotopy version of Mumford's conjecture, stronger than the original [24]. We prove the stronger version, relying on Harer's stability theorem [17], Vassiliev's theorem concerning spaces of functions with moderate singularities [46], [45] and methods from homotopy theory.

Contents

*I.M. partially supported by American Institute of Mathematics. M.W. partially supported by the Royal Society and by the Engineering and Physical Sciences Research Council, Grant GR/R17010/01.

Annals of Mathematics 2007

Galatius-Madsen-Tillmann-Weiss:
Let $\mathcal{C}ob_d$ be the topological cobordism category of embedded oriented d-manifolds in $\mathbb{R}^\infty$. Then

$$\Omega B\mathcal{C}ob_d \simeq \Omega^\infty MTSO(d)$$

Galatius: The inclusion $\Sigma_n \to \mathsf{Aut}F_n$ induces a homotopy equivalence

$$\mathbb{Z} \times B\Sigma_\infty^+ \simeq \mathbb{Z} \times B\mathsf{Aut}_\infty^+$$

Hopkins-Lurie: Baez-Dolan cobordism hypothesis.

Also see talks by:
Berglund, Galatius, Randal-Williams, Hatcher.

Hopkins' joke

Hopkins' joke

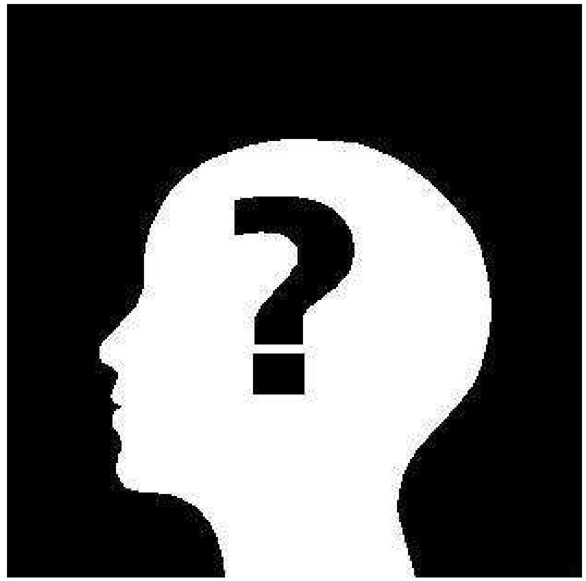
Three wishes:

1. money

2. beautiful wife

3. big giant round orange head

(*Homology, Homtopy Appl. 2008*)



Try again:

Try again:

1. money

Try again:

1. money

2. beautiful ideas

Try again:

1. money

2. beautiful ideas

3. deep understanding!

Try again:

1. money

2. beautiful ideas

3. deep understanding!

Gunnar, Ralph, and Ib

HAPPY BIRTHDAY!