Reducing complexes in Multidimensional Persistence

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joint work with M. Allili, T. Kaczynski and F. Masoni

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Motivation

- The most common algorithm used for computing 1-D persistent homology has complexity $O(n^3)$ [Zomorodian-Carlsson 2005]
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  ◦ use reductions that preserve persistent homology groups
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- This talk: apply this strategy for multiD persistence in any degree
Outline

Introduction
  Lefschetz Complexes
  Combinatorial Morse theory
  Multidimensional Persistent Homology
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Numerical tests


Lefschetz Complexes

1. $S$ a finite set with a gradation $S_q$ such that:
   - $S_q = \emptyset$ for $q < 0$
   - For every $\sigma \in S$ there exists a unique $q$, called dimension, s. t. $\sigma \in S_q$. 

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2. An incidence function $\kappa : S \times S \rightarrow R$, $R$ a PID, such that, if $\kappa(\sigma, \tau) \neq 0$, then $\dim \sigma = \dim \tau + 1$. 
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4. $(S, \kappa)$ is a Lefschetz complex if $(C_* (S), \partial^\kappa_*)$ with $\partial^\kappa_q : C_q(S) \rightarrow C_{q-1}(S)$ defined on generators $\sigma \in S$ by

   $$\partial^\kappa_q(\sigma) := \sum_{\tau \in S} \kappa(\sigma, \tau) \tau$$

   is a free chain complex with base $S$. 
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2. An *incidence function* $\kappa : S \times S \to R$, $R$ a PID, such that, if $\kappa(\sigma, \tau) \neq 0$, then $\dim \sigma = \dim \tau + 1$.

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4. $(S, \kappa)$ is a *Lefschetz complex* if $(C_*(S), \partial_*)$ with $\partial_q : C_q(S) \to C_{q-1}(S)$ defined on generators $\sigma \in S$ by

$$\partial^\kappa(\sigma) := \sum_{\tau \in S} \kappa(\sigma, \tau) \tau$$

is a free chain complex with base $S$.

5. The homology of a Lefschetz complex $(S, \kappa)$ is the homology of $(C_*(S), \partial^\kappa)$. 
Partial matchings

A *partial matching* on \((S, \kappa)\) is a quadruplet \((A, B, C, m)\) where

- \(A, B, C\) is a partition of \(S\),
- \(m : A \rightarrow B\) is a map such that, for each \(\tau \in A\), \(\kappa(m(\tau), \tau)\) is invertible.
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- For all pairs \((m(\tau), \tau)\), draw an arrow from \(\tau\) to \(m(\tau)\)
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\[
\sigma_0 \xrightarrow{m} \tau_0 \xrightarrow{m} \sigma_1 \xrightarrow{m} \tau_1 \rightarrow \ldots \rightarrow \sigma_n \xrightarrow{m} \tau_n \rightarrow \sigma_0
\]

- A partial matching is called *acyclic* if there is no non-trivial sequence of simplices
One-step reductions [KMS 1998]

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Let \((A, B, C, m)\) be a partial matching (not necessarily acyclic) on \((S, \kappa)\). A single pair \((m(\sigma), \sigma)\) can be removed so to obtain again a Lefschetz complex:

- For \(\sigma \in A\), define \((\overline{S}, \overline{\kappa})\) where \(\overline{S} = S \setminus \{m(\sigma), \sigma\}\), and \(\overline{\kappa} : \overline{S} \times \overline{S} \to R\),

\[
\overline{\kappa}(\eta, \xi) = \kappa(\eta, \xi) - \frac{\kappa(\eta, \sigma)\kappa(m(\sigma), \xi)}{\kappa(m(\sigma), \sigma)}.
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- \((\overline{S}, \overline{\kappa})\) is a Lefschetz complex.
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- Define \(\pi : C_*(S) \to C_*(\overline{S})\) and \(\iota : C_*(\overline{S}) \to C_*(S)\) on generators by setting

\[
\pi(\tau) = \begin{cases} 
0 & \text{if } \tau = m(\sigma) \\
- \sum_{\xi \in \overline{S}} \frac{\kappa(m(\sigma), \xi)}{\kappa(m(\sigma), \sigma)} \xi & \text{if } \tau = \sigma \\
\tau & \text{otherwise}
\end{cases}
\]

and

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\iota(\tau) = \tau - \frac{\kappa(\tau, \sigma)}{\kappa(m(\sigma), \sigma)} m(\sigma).
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  \tau & \text{otherwise}
  \end{cases}
  \]

  and

  \[
  \iota(\tau) = \tau - \frac{\kappa(\tau, \sigma)}{\kappa(m(\sigma), \sigma)} m(\sigma).
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- \(\pi\) and \(\iota\) are chain equivalences.
Multi-filtration of a Lefschetz complex

• In $\mathbb{R}^k$ consider the partial order $a = (a_i) \preceq b = (b_i)$ if and only if $a_i \leq b_i$ for all $i = 1, 2, \ldots, k$;
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- A $k$-filtration of $S$ is a family $\mathcal{F} = \{S^a\}_{a \in \mathbb{R}^k}$ of subsets of $S$ with the following properties:
  - $\mathcal{F}$ is nested with respect to inclusions:
    $$S^a \subseteq S^b, \text{ for every } a \preceq b$$
  - $\mathcal{F}$ is non-increasing on faces:
    $$\text{if } \sigma \in S^a \text{ and } \tau \text{ is a face of } \sigma \text{ then } \tau \in S^a$$
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    \]
- Given a function $f : S_0 \to \mathbb{R}^k$, the sublevel set filtration is defined by
  \[ S^a = \{ \sigma = [v_0, v_1, \ldots, v_q] \in S \mid f(v_i) \preceq a, \ i = 0, \ldots, q \}. \]
Multidimensional Persistent Homology

Persistence analyzes the homological changes of the filtration as $a$ varies:
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- for $a \leq b$, consider the homomorphism

$$H_*(j^{(a,b)}) : H_*(S^a) \to H_*(S^b).$$

induced by the inclusion map $j^{(a,b)} : S^a \hookrightarrow S^b$. 

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\[
H_\ast(j^{(a,b)}) : H_\ast(S^a) \rightarrow H_\ast(S^b).
\]

induced by the inclusion map \( j^{(a,b)} : S^a \hookrightarrow S^b \).

- The \( i \)-th persistent homology group of the filtration at \( (a, b) \) is image of the map \( H_i(j^{(a,b)}) \):

\[
H_i^{a,b}(S) := \text{im } H_i(j^{(a,b)})
\]
Filtration-preserving matching

A partial matching \((A, B, C, m)\) on a filtered Lefschetz complex is said to \textit{preserve the filtration} when, for every \(a \in \mathbb{R}^k\),

\[
\text{If } \sigma \in S^a \text{ then } m(\sigma) \in S^a.
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A filtration-preserving partial matching on \(S\) naturally induces a filtration on \(\overline{S}\): for each \(\tau \in \overline{S}\),

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\tau \in \overline{S}^a \iff \tau \in S^a.
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**Proposition**

Let \((\overline{S}, \overline{\kappa})\) be obtained from \((S, \kappa)\) by reduction of the pair \((m(\sigma), \sigma)\). Then, for each \(q \in \mathbb{Z}\),

\[
\pi(C_q(S^a)) \subseteq C_q(S^a), \quad \iota(C_q(S^a)) \subseteq C_q(S^a), \quad D_q(C_q(S^a)) \subseteq C_{q+1}(S^a)
\]
Reductions preserve persistent homology

Theorem

Let \( \sigma \in \mathbb{A} \) and let \((\mathcal{S}, \kappa)\) be obtained from \((\mathcal{S}, \kappa)\) by reduction of the pair \((m(\sigma), \sigma)\). Then the diagram

\[
\begin{array}{ccc}
H_*^a(\mathcal{S}^a) & \xrightarrow{H_*^{j(a,b)}} & H_*^b(\mathcal{S}^b) \\
\downarrow \cong & & \downarrow \cong \\
H_*^a(\overline{\mathcal{S}}^a) & \xrightarrow{H_*^{j(a,b)}} & H_*^b(\overline{\mathcal{S}}^b)
\end{array}
\]

commutes and \( H_*^{a,b}(\mathcal{S}) \) is isomorphic to \( H_*^{a,b}(\overline{\mathcal{S}}) \).
Proposition

If \((A, B, C, m)\) is acyclic then, for any \(\tau \in A \setminus \{\sigma\}\), \(\kappa(m(\tau), \tau)\) is invertible. Furthermore, \(\kappa(m(\tau), \tau) = \kappa'(m(\tau), \tau)\).
Iterated reductions

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If \((A, B, C, m)\) is acyclic then, for any \(\tau \in A \setminus \{\sigma\}\), \(\kappa(m(\tau), \tau)\) is invertible. Furthermore, \(\kappa(m(\tau), \tau) = \kappa(m(\tau), \tau)\).

**Corollary**
Let \((A, B, C, m)\) be an acyclic partial matching on \((S, \kappa)\). Given a fixed \(\sigma \in A\), define \(\overline{A} = A \setminus \{\sigma\}\), \(\overline{B} = B \setminus \{m(\sigma)\}\), \(\overline{m} = m|_{\overline{A}}\), and \(\overline{C} = C\). Then \((\overline{C}, \overline{m} : \overline{A} \rightarrow \overline{B})\) is an acyclic partial matching on \((\overline{S}, \overline{\kappa})\).

**Corollary**
For every \(a \preceq b \in \mathbb{R}^k\), \(H_{*}^{a,b}(C) \cong H_{*}^{a,b}(S)\). Moreover, the diagram

\[
\begin{array}{ccc}
\mathbb{R}^k & \xrightarrow{\mathbb{R}} & \mathbb{R}^k \\
H_{*}(S^a) & \xrightarrow{H_{*}(j^{(a,b)})} & H_{*}(S^b) \\
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H_{*}(C^a) & \xrightarrow{H_{*}(j^{(a,b)})} & H_{*}(C^b)
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commutes.
The matching algorithm: introduction

- We start from the matching algorithm of [Robins-Wood-Sheppard 2010] for 1D persistent homology
  - input: a 3D cubical complex and an $\mathbb{R}$-valued injective function on its vertices
  - output: a filtration preserving acyclic partial matching
  - vertex based
  - lower-star based
The matching algorithm: introduction

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- and extend it for multidimensional persistent homology
  - input: a simplicial complex, and a component-wise injective $\mathbb{R}^k$-valued function on its vertices and an ordering on its vertices
  - output: a multi-filtration-preserving acyclic partial matching
  - simplex based
  - lower-star based
Component-wise injective function on the vertices

\[ f = (f_1, \ldots, f_k) : S_0 \to \mathbb{R}^k \] is componentwise injective if each \( f_i \) is injective.
Component-wise injective function on the vertices

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Given \( \tilde{f} : S_0 \to \mathbb{R}^k \), we obtain a component-wise injective \( f : S_0 \to \mathbb{R}^k \) by perturbing \( f_i \):

- set \( \eta_i = \min \{|\tilde{f}_i(v) - \tilde{f}_i(w)| : v, w \in S_0 \land f_i(v) \neq f_i(w)\} \);
- for each \( i \), order the \( n \) vertices of \( S_0 \) by an integer index \( j \) increasing with \( \tilde{f}_i \);
- let \( f_i(v_j) = \tilde{f}_i(v_j) + j\eta_i/n^s \), with \( s \geq 1 \)
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Extend \( f \) to a function \( f : S \rightarrow \mathbb{R}^k \) as follows.

\[
    f(\sigma) = (f_1(\sigma), \ldots, f_k(\sigma)) \quad \text{with} \quad f_i(\sigma) = \max_{v \in S_0(\sigma)} f_i(v).
\]
Indexing map for vertices

Lemma

There exists an injective function $l : S \to \mathbb{N}$ such that, for each $\sigma, \tau \in S$ with $\sigma \neq \tau$, if $\sigma \subseteq \tau$ or $f(\sigma) \preceq f(\tau)$ then $l(\sigma) < l(\tau)$. 
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Lemma

There exists an injective function \( I : S \to \mathbb{N} \) such that, for each \( \sigma, \tau \in S \) with \( \sigma \neq \tau \), if \( \sigma \subseteq \tau \) or \( f(\sigma) \preceq f(\tau) \) then \( I(\sigma) < I(\tau) \).

Construction of \( I \):
Indexing map for vertices

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Construction of \( I \):

- consider the poset \((S, \sqsubseteq)\) with \( \sigma \sqsubseteq \tau \) if and only if either \( \sigma = \tau \) or \( \sigma \neq \tau \) but in the latter case \( \sigma \) is a face of \( \tau \) or \( f(\sigma) \preceq f(\tau) \)
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Lemma

There exists an injective function \( I : S \to \mathbb{N} \) such that, for each \( \sigma, \tau \in S \) with \( \sigma \neq \tau \), if \( \sigma \subseteq \tau \) or \( f(\sigma) \nless f(\tau) \) then \( I(\sigma) < I(\tau) \).

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- represent it by a Directed Acyclic Graph
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- represent it by a Directed Acyclic Graph
- apply the topological sorting algorithm
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- represent it by a Directed Acyclic Graph
- apply the topological sorting algorithm
- this algorithm has linear complexity
The lower star of a simplex
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Given $f : S \to \mathbb{R}^k$, the lower star of a simplex is the set

$$L(\sigma) = \{ \alpha \in S \mid \sigma \subseteq \alpha \quad \text{and} \quad f(\alpha) \preceq f(\sigma) \},$$

and the reduced lower stars is the set $L_*(\sigma) = L(\sigma) \setminus \{\sigma\}$. 
The lower star of a simplex

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**Lemma**

If $f$ is component-wise injective on the vertices, then the following statements hold:

1. If $\tau \in L(\sigma)$, then $f(\tau) = f(\sigma)$.
2. If $\tau \in L_*(\sigma)$, then $I(\sigma) < I(\tau)$.
3. If $f(\sigma) = f(\tau)$ then there exists $\alpha \subseteq \sigma \cap \tau$ with $f(\alpha) = f(\sigma) = f(\tau)$.
4. Assume that $\sigma_1$ and $\sigma_2$ are two distinct simplices of $\mathcal{K}$ such that $L(\sigma_1) \cap L(\sigma_2) \neq \emptyset$. Then, there exists a simplex $\beta \in \mathcal{K}$ such that $L(\sigma_1) \cup L(\sigma_2) \subseteq L(\beta)$ and $I(\beta) \leq \min\{I(\sigma_1), I(\sigma_2)\}$. 
The matching algorithm for multiD persistence

**Input:** A finite simplicial complex \( S \) with a function \( f : S \to \mathbb{R}^k \) component-wise injective on the vertices, and an indexing \( I : S \to \mathbb{N} \).

**Output:** Three lists \( A, B, C \) of simplices of \( S \), and a function \( m : A \to B \).
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Set $A, B, C = \emptyset$; $\text{classified}(\sigma) = \text{false}$ $\forall \sigma \in S$; $\text{PQzero}, \text{PQone} = \emptyset$
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Set $A, B, C = \emptyset$; $\text{classified}(\sigma) = \text{false} \ \forall \sigma \in S$; $\text{PQzero}, \text{PQone} = \emptyset$ for $i = 1$ to $\#S$
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Set $A, B, C = \emptyset$; $\text{classified}(\sigma) = \text{false}$ $\forall \sigma \in S$; $\text{PQzero}, \text{PQone} = \emptyset$; for $i = 1$ to $\#S$

- set $\sigma := I^{-1}(i)$
The matching algorithm for multiD persistence

**Input:** A finite simplicial complex $S$ with a function $f : S \rightarrow \mathbb{R}^k$ component-wise injective on the vertices, and an indexing $l : S \rightarrow \mathbb{N}$.

**Output:** Three lists $A, B, C$ of simplices of $S$, and a function $m : A \rightarrow B$.

Set $A, B, C = \emptyset$; $\text{classified}(\sigma) = \text{false} \ \forall \sigma \in S$; $\text{PQzero}, \text{PQone} = \emptyset$

for $i = 1$ to $#S$

set $\sigma := l^{-1}(i)$

if $\text{classified}(\sigma) = \text{false}$

if $L_*(\sigma)$ contains no cells

add $\sigma$ to $C$, $\text{classified}(\sigma) = \text{true}$
The matching algorithm for multiD persistence

**Input:** A finite simplicial complex $S$ with a function $f : S \to \mathbb{R}^k$ component-wise injective on the vertices, and an indexing $I : S \to \mathbb{N}$.

**Output:** Three lists $A, B, C$ of simplices of $S$, and a function $m : A \to B$.

Set $A, B, C = \emptyset$; $\text{classified}(\sigma) = \text{false} \ \forall \sigma \in S$; $\text{PQzero}, \text{PQone} = \emptyset$

for $i = 1$ to $\# S$

set $\sigma := I^{-1}(i)$

if $\text{classified}(\sigma) = \text{false}$

if $L_*(\sigma)$ contains no cells

add $\sigma$ to $C$, $\text{classified}(\sigma) = \text{true}$

else

set $\delta :=$ the (primary) coface in $L_*(\sigma)$ of minimal index $I(\delta)$

add $\sigma$ to $A$ and $\delta$ to $B$ and define $m(\sigma) = \delta$, $\text{classified}(\sigma) = \text{true}$, $\text{classified}(\delta) = \text{true}$

add all $\alpha \in L_*(\sigma)$ to $\text{PQzero}$ if $\text{num\_unclass\_faces}_\sigma(\alpha) = 0$

add all $\alpha \in L_*(\sigma)$ to $\text{PQone}$ if $\text{num\_unclass\_faces}_\sigma(\alpha) = 1 \land \alpha > \delta$
\textbf{while } PQone \neq \emptyset \textbf{ or } PQzero \neq \emptyset \\
\textbf{while } PQone \neq \emptyset \\
\text{set } \alpha := PQone.\text{pop\_front}
while \( \text{PQone} \neq \emptyset \) or \( \text{PQzero} \neq \emptyset \)
  while \( \text{PQone} \neq \emptyset \)
    set \( \alpha \) := \text{PQone.pop_front} 
    if \( \text{num_unclass_faces}_\sigma(\alpha) = 0 \)
      add \( \alpha \) to \( \text{PQzero} \)
while $\text{PQone} \neq \emptyset$ or $\text{PQzero} \neq \emptyset$
  while $\text{PQone} \neq \emptyset$
    set $\alpha := \text{PQone}.\text{pop_front}$
    if $\text{num\_unclass\_faces}_\sigma(\alpha) = 0$
      add $\alpha$ to $\text{PQzero}$
    else
      add $\lambda \in \text{unclass\_faces}_\sigma(\alpha)$ to $A$, add $\alpha$ to $B$ and define $m(\lambda) = \alpha$, $\text{classified}(\alpha) = \text{true}$, $\text{classified}(\lambda) = \text{true}$, remove $\lambda$ from $\text{PQzero}$
      add all $\beta \in \mathbb{L}_\sigma^*$ to $\text{PQone}$ if $\text{num\_unclass\_faces}_\sigma(\beta) = 1$ and either $\beta > \alpha$ or $\beta > \lambda$
  endwhile
\[ \text{while } \text{PQone} \neq \emptyset \text{ or } \text{PQzero} \neq \emptyset \] 
\[ \text{while } \text{PQone} \neq \emptyset \] 
\[ \quad \text{set } \alpha := \text{PQone.pop_front} \] 
\[ \quad \text{if } \text{num_unclass_faces}_\sigma(\alpha) = 0 \] 
\[ \quad \quad \text{add } \alpha \text{ to PQzero} \] 
\[ \quad \text{else} \] 
\[ \quad \quad \text{add } \lambda \in \text{unclass_faces}_\sigma(\alpha) \text{ to A, add } \alpha \text{ to B and define} \] 
\[ \quad \quad \quad m(\lambda) = \alpha, \text{classified}(\alpha) = \text{true}, \text{classified}(\lambda) = \text{true}, \text{remove } \lambda \text{ from PQzero} \] 
\[ \quad \quad \text{add all } \beta \in L_*(\sigma) \text{ to PQone if } \text{num_unclass_faces}_\sigma(\beta) = 1 \text{ and} \] 
\[ \quad \quad \quad \text{either } \beta > \alpha \text{ or } \beta > \lambda \] 
\[ \quad \text{ endwhile} \] 
\[ \text{if } \text{PQzero} \neq \emptyset \] 
\[ \quad \gamma := \text{PQzero.pop_front} \] 
\[ \quad \text{add } \gamma \text{ to C, classified}(\gamma) = \text{true} \] 
\[ \quad \text{add all } \tau \in L_*(\sigma) \text{ to PQone if } \text{num_unclass_faces}_\sigma(\tau) = 1 \land \] 
\[ \quad \quad \tau > \gamma \] 
\[ \text{ endif} \] 
\[ \text{ endwhile} \]
Correctness and complexity

**Proposition**

*Each cell is processed exactly once by the algorithm and it is paired with some other cell or classified as critical. Hence the matching algorithm always terminates.*

**Theorem**

*The matching algorithm produces a partial matching \((A, B, C, m)\) that is acyclic and filtration-preserving.*
Correctness and complexity

**Proposition**

*Each cell is processed exactly once by the algorithm and it is paired with some other cell or classified as critical. Hence the matching algorithm always terminates.*

**Theorem**

*The matching algorithm produces a partial matching \((A, B, C, m)\) that is acyclic and filtration-preserving.*

**Complexity**

The matching algorithm produces \((A, B, C, m)\) in \(O(\gamma \cdot \log \gamma \cdot (\#S))\) steps, where \(\gamma\) is the maximum number of cofaces of simplices in \(S\).

The computational complexity of the reductions is \(O((\#S) \cdot \gamma \cdot (\#C)^2)\)  

[Mishaikow-Nanda 2013]
Numerical tests

- Considered 2D simplicial complexes.
- Filtered each complex by the $\mathbb{R}^2$-valued function defined on vertices.
- Present the results in a table where
  - row 1 shows the number of vertices, edges, faces, and the total number of cells of each considered mesh $S$,
  - while row 2 shows the same quantities referred to the cell complex $C$ obtained by using our matching algorithm to reduce $S$.
  - Finally, row 3 shows the ratio between the second and the first rows, expressing them in percentage points. In other words, the lower are those ratios, the higher is the reduction rate.
Numerical tests on synthetic data

**Table:** Reduction performance on five different triangulations of the sphere.

<table>
<thead>
<tr>
<th>$f = (x, y)$</th>
<th>sphere_1</th>
<th>sphere_2</th>
<th>sphere_3</th>
<th>sphere_4</th>
<th>sphere_5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$#S$</td>
<td>38</td>
<td>242</td>
<td>962</td>
<td>1538</td>
<td>2882</td>
</tr>
<tr>
<td>$#C$</td>
<td>4</td>
<td>20</td>
<td>98</td>
<td>178</td>
<td>278</td>
</tr>
<tr>
<td>%</td>
<td>10.5263</td>
<td>8.2645</td>
<td>10.1871</td>
<td>11.5735</td>
<td>9.6461</td>
</tr>
</tbody>
</table>
Table: Reduction performance on different triangulations of the torus.

<table>
<thead>
<tr>
<th></th>
<th>torus_96</th>
<th>torus_4608</th>
<th>torus_7200</th>
</tr>
</thead>
<tbody>
<tr>
<td>#S</td>
<td>96</td>
<td>4608</td>
<td>7200</td>
</tr>
<tr>
<td>#C</td>
<td>8</td>
<td>128</td>
<td>156</td>
</tr>
<tr>
<td>%</td>
<td>8.3333</td>
<td>2.7778</td>
<td>2.1667</td>
</tr>
</tbody>
</table>
**Table:** Reduction performance on different triangulations approximating an immersion of the Klein bottle.

<table>
<thead>
<tr>
<th>$f = (x, y)$</th>
<th>klein_89</th>
<th>klein_187</th>
<th>klein_491</th>
<th>klein_1881</th>
</tr>
</thead>
<tbody>
<tr>
<td>#$\mathcal{K}$</td>
<td>89</td>
<td>187</td>
<td>491</td>
<td>1881</td>
</tr>
<tr>
<td>#C</td>
<td>19</td>
<td>35</td>
<td>59</td>
<td>257</td>
</tr>
<tr>
<td>%</td>
<td>21.3483</td>
<td>18.7166</td>
<td>12.0163</td>
<td>13.6629</td>
</tr>
</tbody>
</table>
Numerical tests on real data

\[ f = (|x|, |z|) \]

<table>
<thead>
<tr>
<th></th>
<th>tie</th>
<th>space_shuttle</th>
<th>x_wing</th>
<th>space_station</th>
</tr>
</thead>
<tbody>
<tr>
<td>#S</td>
<td>11785</td>
<td>12658</td>
<td>18365</td>
<td>31935</td>
</tr>
<tr>
<td>#C</td>
<td>2287</td>
<td>484</td>
<td>2449</td>
<td>5149</td>
</tr>
<tr>
<td>%</td>
<td>19.4060</td>
<td>3.8237</td>
<td>13.3351</td>
<td>16.1234</td>
</tr>
</tbody>
</table>

C

![Images of tie, space_shuttle, x_wing, and space_station]