Topological Field Theories in Homotopy Theory

I

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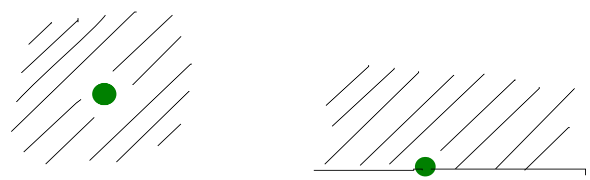
Manifolds

M is a manifold of dimension d if locally it is diffeomorphic to

$$\mathbb{R}^d \quad \text{or} \quad \mathbb{R}^d_{\geq 0}.$$



M is closed if it is compact and has no boundary.

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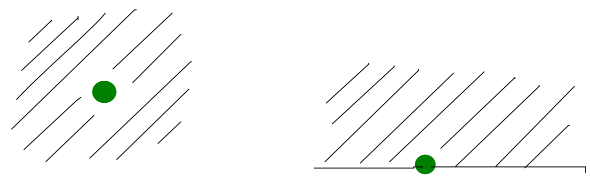


M is closed if it is compact and has no boundary.

Fundamental problem:

– classify compact smooth manifolds M of dim d;

– understand their groups of diffeomorphisms Diff(M).

d any : the empty $\varnothing$ set is a manifold of any dimension

$d = 0$: M is a collection of finitely many points

$d = 1$: M is a collection of circles S^1 and intervals $[0, 1]$

$d = 2$: M is a collection of orientable surfaces $F_{g,n}$ and non-orientable surfaces $N_{g,n}$ of genus g and with n boundary components

$F_{1,0} =$ torus $\qquad N_{1,1} =$ Möbius band

Leitmotif = Understanding Manifolds

1. Classical Cobordism Theory (Thom, ...)

2. Topological Field Theory (Witten, Atiyah, Segal, ...)

3. Cobordism Hypothesis (Baez-Dolan, Lurie, ...)

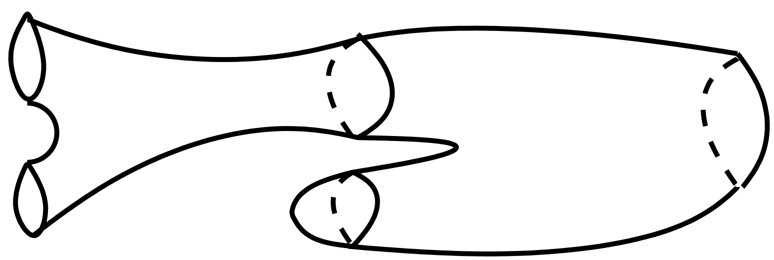
4. Classifying spaces of cobordism categories
—— classifying space of cobordism categories
. (Galatius–Madsen–Tillmann–Weiss)
—— the cobordism hypothesis for invertible TQFTs
—— filtration of the classical theory

5. Extracting information on Diff(M)
—— Mumford conjecture (Madsen–Weiss, ...)
—— higher dimensional analogues
. (Galatius–Randal-Williams, ...)

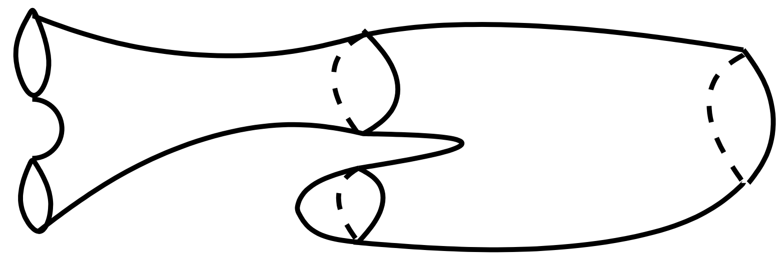
1. Classical Cobordism Theory

Definition: Two closed oriented $(d-1)$-dimensional manifolds M_0 and M_1 are cobordant if there exists a compact oriented d-dimensional manifold W with boundary

$$\partial W = \bar{M}_0 \sqcup M_1$$



$$M_0 \longrightarrow W \longleftarrow M_1 \longrightarrow W' \longleftarrow M_2$$



$$M_0 \longrightarrow W \longleftarrow M_1 \longrightarrow W' \longleftarrow M_2$$

Cobordism is an equivalence relation; denote equivalence classes by

$$\mathfrak{N}^+_{d-1}$$

It is a group with product $\coprod$ and inverse

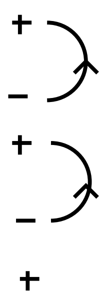
$$M^{-1} = \bar{M}$$

Together they form a graded ring with multiplication $\times$

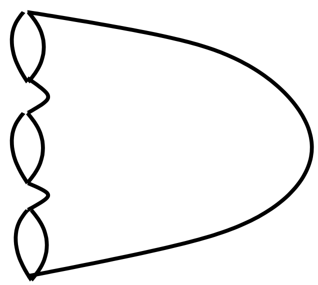
$$\bigoplus_{d>0} \mathfrak{N}^+_{d-1}$$

Examples:

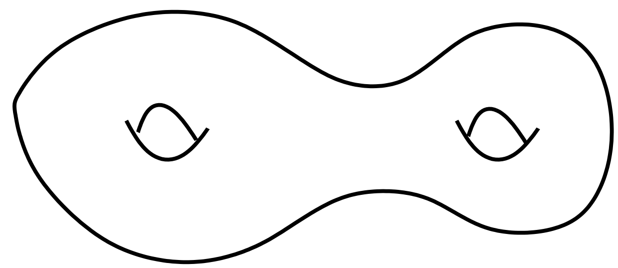
$\mathfrak{N}_0^+ = \mathbb{Z}$



$\mathfrak{N}_1^+ = \{0\}$



$\mathfrak{N}_2^+ = \{0\}$



Theorem **[Thom]** $\mathfrak{N}_d^+ = \pi_d(\Omega^\infty \mathbf{MSO})$

Recall: For any space X,

$$\pi_d(X)$$

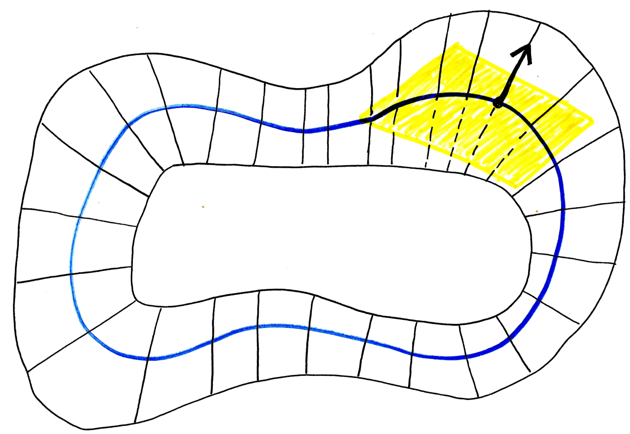
is the group of homotopy classes of based maps from S^d to X

Theorem [Thom] $\mathfrak{N}_d^+ = \pi_d(\Omega^\infty \mathbf{MSO})$

where

$$\Omega^\infty \mathbf{MSO} := \lim_{n\to\infty} \lim_{k\to\infty} \mathsf{maps}_*(S^n, (U_{n,k})^c)$$

and $U_{n,k} \to Gr^+(n,k)$ is the universal n-dimensional bundle over the Grassmannian manifold of oriented n-planes in $\mathbb{R}^{n+k}$.



$M^d \subset$ tubular neighbourhood $N(M) \subset \mathbb{R}^{d+n}$

$\mapsto$

$\mathsf{S}^{d+n} = (R^{d+n})^c \xrightarrow{collapse} (N(M))^c \xrightarrow{\phi_{N(M)}} (U_{n,d})^c$

. $(x, v) \mapsto (N_x M, v).$

Theorem **[Thom]** $\mathfrak{N}_*^+ \otimes \mathbb{Q} \simeq \mathbb{Q}\,[\mathbb{C}P^2, \mathbb{C}P^4, \ldots]$.

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Proof: For fixed $*$ and large n and k,

$$\begin{aligned}
\pi_*(\Omega^\infty \mathbf{MSO}) \otimes \mathbb{Q} &= \pi_*(\lim_{n\to\infty} \lim_{k\to\infty} \text{maps}_*(S^n, (U_{n,k})^c)) \otimes \mathbb{Q} \\
&= \pi_*(\text{maps}_*(S^n, (U_{n,k})^c)) \otimes \mathbb{Q} \\
&= \pi_{*+n}((U_{n,k})^c) \otimes \mathbb{Q} \\
&= H_{*+n}((U_{n,k})^c) \otimes \mathbb{Q} \quad \text{by Serre} \\
&= H_*(Gr^+(n,k)) \otimes \mathbb{Q} \quad \text{by Thom.}
\end{aligned}$$

Réné Thom (1923–2002); Fields Medal 1958

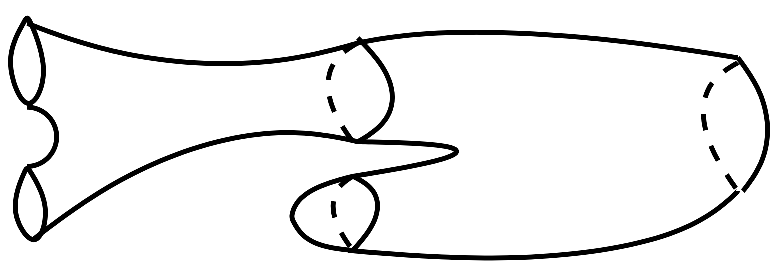
2. Topological Field Theory

$\mathcal{C}ob_d^\delta$ is the discrete cobordism category with

Objects: closed oriented $d-1$ dimensional manifolds M

Morphisms from M_0 to M_1:
d-dimensional cobordism W with $\partial W = \bar{M}_0 \sqcup M_1$
— modulo diffeomorphisms rel. boundary

Composition: gluing of cobordisms.



$$W' \circ W : M_0 \longrightarrow M_1 \longrightarrow M_2$$

Definition: A d-dimensional TFT is a functor

$$\mathcal{F} : \mathcal{C}ob_d^\delta \longrightarrow \mathcal{V}$$

to the category $\mathcal{V}$ of vector spaces that takes disjoint union of manifolds to tensor products of vector spaces.

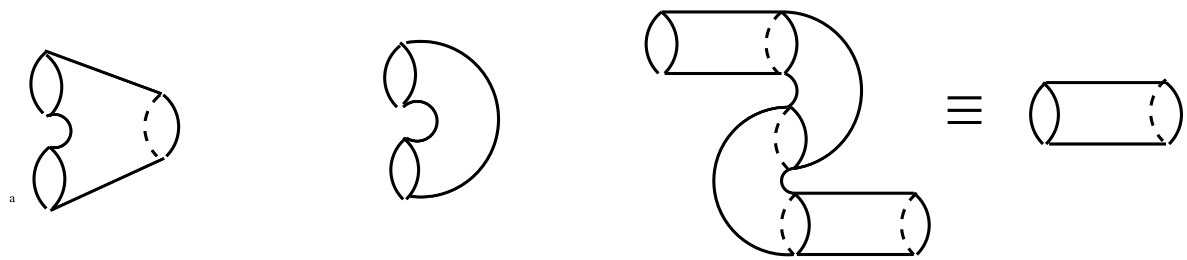
Example: the unit has to be mapped to the unit; hence

$$\mathcal{F}(\varnothing) = \mathbb{C}$$

Folk Theorem: 2-dimensional TFTs are in one-to-one correspondence with finite dimensional, commutative Frobenius algebras:

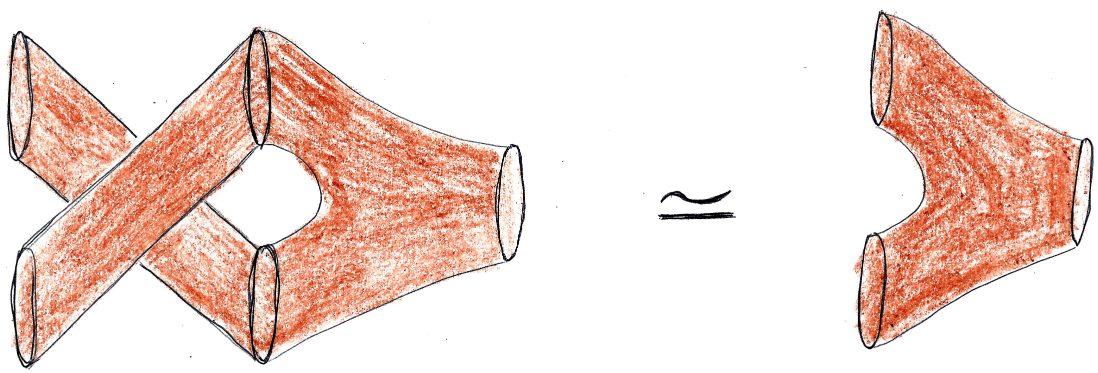
$$\mathcal{F} \quad \longleftrightarrow \quad A := \mathcal{F}(S^1)$$

and hence $\mathcal{F}(\coprod_n S^1) = A^{\otimes n}$



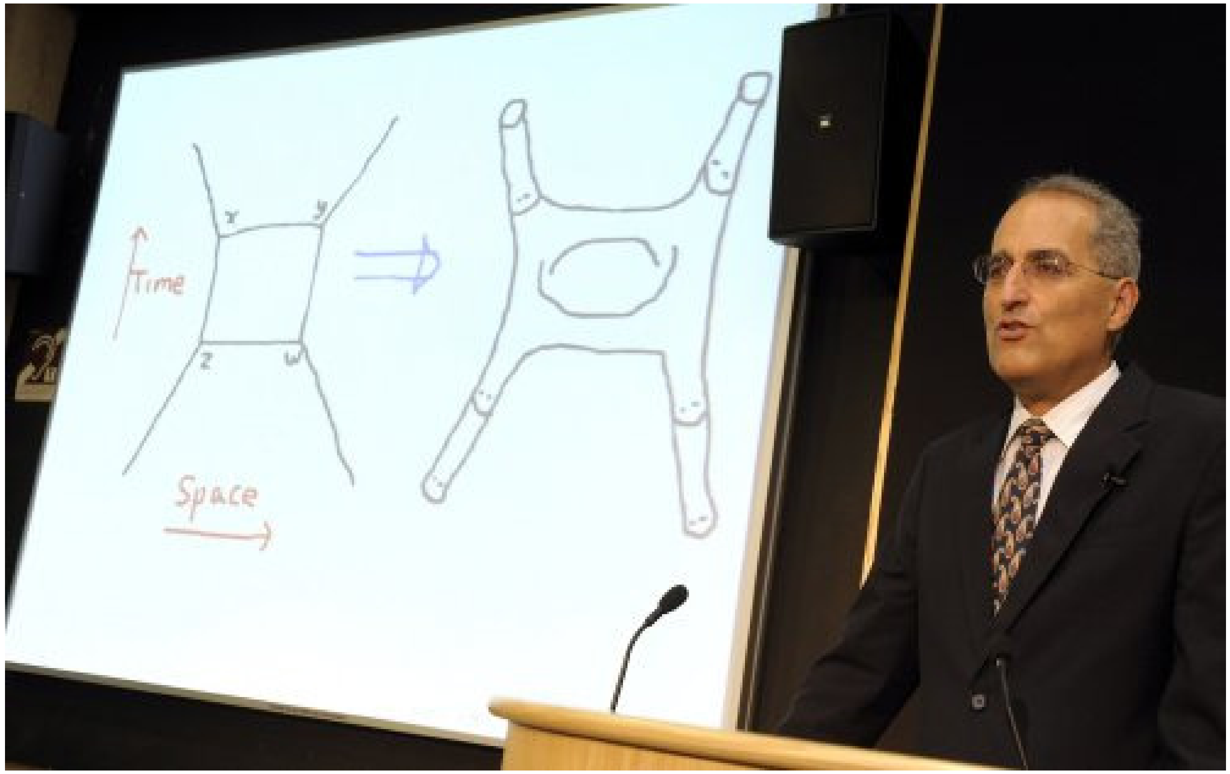
$$\mu : A \otimes A \to A \qquad <,>: A \otimes A \to \mathbb{C} \qquad Id : A \to A$$

Commutativity:



$$\mu \circ \tau = \mu : A \otimes A \longrightarrow A$$

Associativity and unitality are similar.



Edward Witten

Michael Atiyah

Motivation:

d-dimensional TFTs define topological invariants for d-dimensional closed manifolds:

If $\partial W = \varnothing$ then it defines a morphisms $W : \varnothing \to \varnothing$, and $\mathcal{F}$ assigns a number to W depending only on its topology:

$$\mathcal{F}(W) : \mathcal{F}(\varnothing) = \mathbb{C} \longrightarrow \mathcal{F}(\varnothing) = \mathbb{C}$$

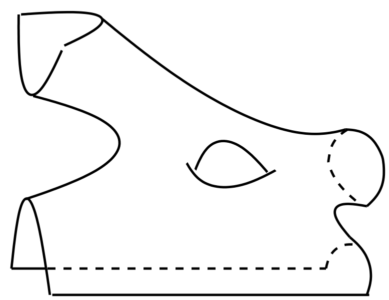
Physical inspiration: locality!

3. Cobordism Hypothesis

Physical inspiration: locality!

Categorification:

points, cobordisms, cobordisms of cobordisms, ...



$\mathcal{C}ob_d^\delta$ is replaced by d-fold category $ex\mathcal{C}ob_d^\delta$
$\mathcal{V}$ replaced by a d-fold symmetric monoidal category $\mathcal{V}_d$

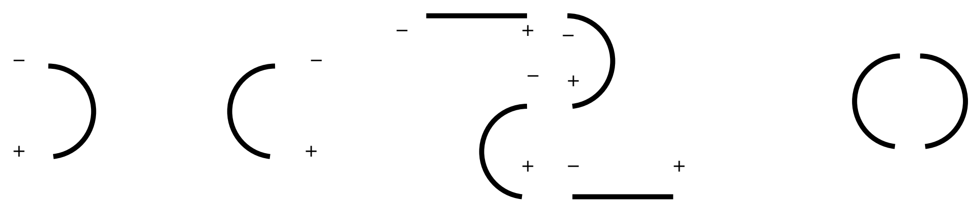
Study extended TFTs

$$\mathcal{F} : ex\mathcal{C}ob_d^\delta \longrightarrow \mathcal{V}_d$$

Cobordism hypothesis (weak) [Baez-Dolan]
Extended TFTs are determined by $\mathcal{F}(*)$.

Example: 1-dimensional theories

Let $\mathcal{F}(*_+) = V$ and $\mathcal{F}(*_-) = V'$.



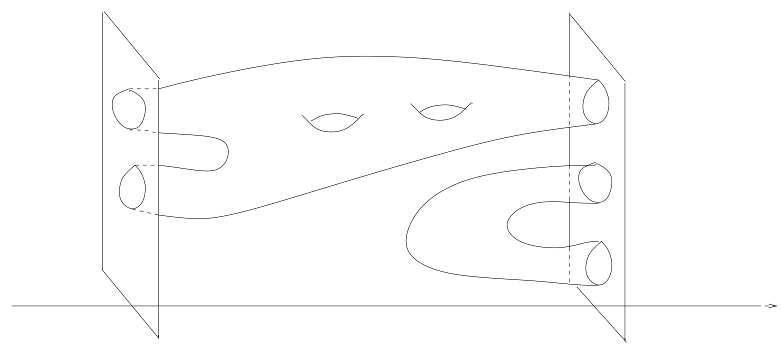
- evaluation $e : V \otimes V' \to \mathbb{C}$
- co-evaluation $e^* : \mathbb{C} \to V' \otimes V$
- V is finite dimensional as

$$id : V \xrightarrow{id \otimes e^*} V \otimes V' \otimes V \xrightarrow{e \otimes id} V$$

- $e \circ e^* = \dim(V) : \mathbb{C} \to \mathbb{C}$

Enriched TFTs

Consider moduli spaces of all compact $(d-1)$- and d-manifolds embedded in $\mathbb{R}^{d+n}, n \to \infty$, to form the topological category $\mathcal{C}ob_d$.



$$M_0 \longrightarrow W \longleftarrow M_1$$

The homotopy type of the space of morphisms:

$$\begin{aligned} \mathit{mor}_{\mathcal{C}ob_d}(M_0, M_1) &\simeq \coprod_W Emb^{\partial}(W, \mathbb{R}^{d+\infty})/\mathrm{Diff}(W;\partial) \\ &\simeq \coprod_W B\mathrm{Diff}(W;\partial) \end{aligned}$$

where the disjoint union is taken over all diffeomorphism classes of cobordisms W.

Note: $Emb^{\partial}(W, \mathbb{R}^{d+\infty})$ is weakly contractible, and $\mathrm{Diff}(W;\partial)$ acts freely

Theorem [Hopkins-Lurie, Lurie]:

$$\mathcal{F} : ex\mathcal{C}ob_d^{fr} \to \mathcal{V}_d$$

is determined by $\mathcal{F}(*)$, the value on a point.
Vice versa, any object in $\mathcal{V}_d$ satisfying certain duality and non-degeneracy properties gives rise to a TFT.

More general: for non-orientable, oriented, ..., $\mathcal{F}$ is still determined by $\mathcal{F}(*)$ but there are group actions that have to be considered.

Excursion: tangential structures

Recall:

$$\text{Vect}_n(W) = [W, BO(n)] = [W, Gr(n, \infty)]$$

$$E \leftrightarrow \phi_E$$

Definition: Let $\theta(n) : \mathcal{X}(n) \to BO(n)$ be a fiber bundle. A $\theta(n)$-structure on W^d is a lift of $\phi_E : W \to BO(n)$ to $\mathcal{X}(n)$ for $E = TW \oplus \mathbb{R}^{n-d}$

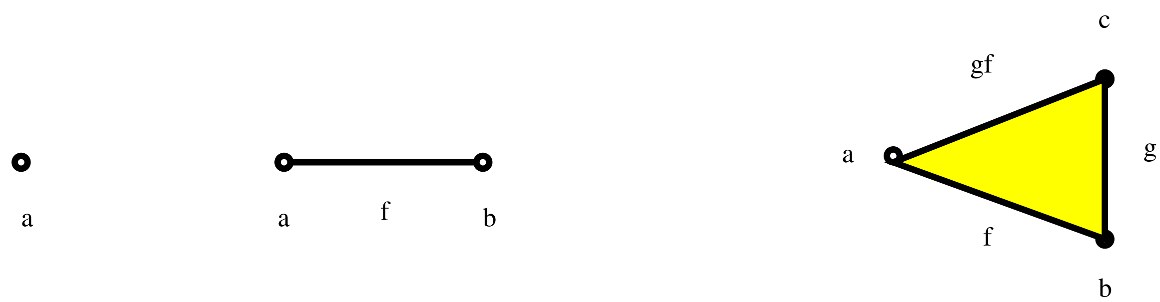
Oriented $+$: $\mathbb{Z}/2\mathbb{Z} \to BSO(n) \to BO(n)$

Framed fr: $O(n) \to EO(n) \to BO(n)$

Example: S^d is $EO(n)$-framed if $n > d$

4. Classifying space of cobordism categories

B : Topological Categories $\longrightarrow$ Spaces, $\mathcal{C} \mapsto B\mathcal{C}$



Example: for a group G get BG

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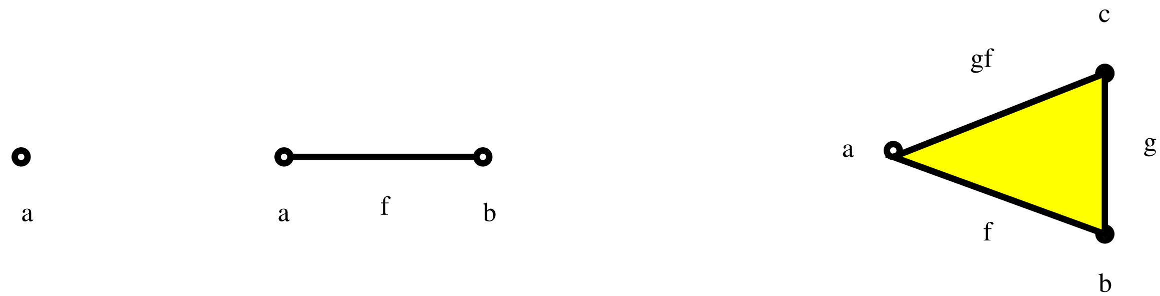


- morphisms $\mapsto$ paths which are homotopy invertible!

. for every $a \in ob_{\mathcal{C}}$, there is a characteristic map

$$\alpha : mor_{\mathcal{C}}(a,a) \longrightarrow \text{maps}([0,1],\partial; B\mathcal{C}, a) = \Omega B\mathcal{C}$$

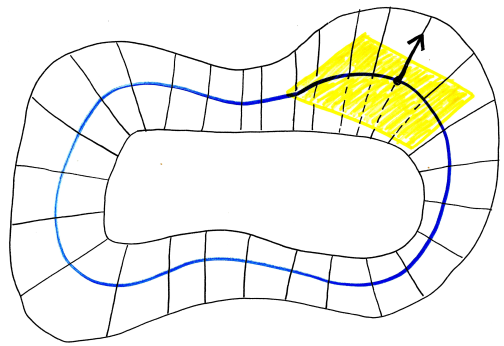
- monoidal cats $\mapsto$ E_1-spaces (Ω-spaces)
- symmetric monoidal cats $\mapsto$ E_∞-spaces (Ω^∞-spaces)

Theorem [Galatius, Madsen, Tillmann, Weiss]

$$\Omega B(\mathcal{C}ob_d) \simeq \Omega^\infty \mathbf{MTSO}(d) = \lim_{n\to\infty} \Omega^{d+n}((U_{d,n}^\perp)^c)$$

where $U_{d,n}^\perp$ is the orthogonal complement of the universal bundle $U_{d,n} \to Gr^+(d,n)$.

Note: the Thom class is in dimension $-d$



The characteristic map:

$mor_{\mathcal{C}ob_d}(\varnothing, \varnothing) \ni W \subset N(W) \subset \mathbb{R}^{d+n}$

$\mapsto$

$\alpha(W) : S^{d+n} = (R^{d+n})^c \xrightarrow{collapse} N(W)^c \xrightarrow{\phi_{T(W)}} (U_{d,n}^{\perp})^c$

. $(x, v) \mapsto (T_x W, v)$

In Thom's theory: $(x, v) \mapsto (N_x W, v) \in (U_{n,d})^c.$

Filtration of classical cobordism theory

The inclusion of multi-categories

$$ex\mathcal{C}ob_1 \subset \cdots \subset ex\mathcal{C}ob_{d-1} \subset ex\mathcal{C}ob_d \subset \dots$$

induces on taking multi-classifying spaces a filtration

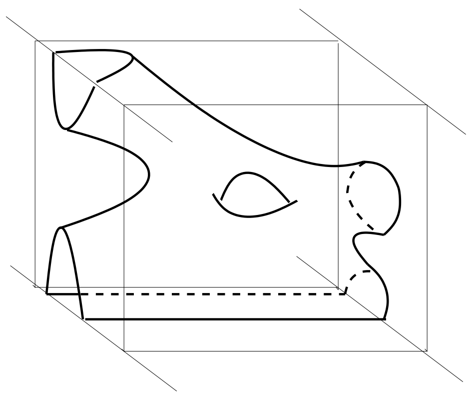
$$\Omega^\infty S^\infty \to \cdots \to \Omega^{\infty-(d-1)}\mathbf{MTSO}(d{-}1) \to \Omega^{\infty-d}\mathbf{MTSO}(d)\dots$$

of Thom's space $\Omega^\infty\mathbf{MSO}$ which respects the additive and multiplicative structure

All Thom classes are in degree zero!

An even finer filtration

$$\begin{aligned}\Omega^\infty \mathbf{MSO} &\simeq \lim_{n\to\infty} \lim_{d\to\infty} \Omega^n (U_{n,d})^c \\ &\simeq \lim_{d\to\infty} \lim_{n\to\infty} \Omega^n (U^\perp_{d,n})^c \\ &\simeq \lim_{d\to\infty} \lim_{n\to\infty} B(\mathcal{C}ob^d_{d,n})\end{aligned}$$



A 2-morphism in $\mathcal{C}ob^2_{2,1}$.

For the framed theory, this is the constant filtration

$$B(ex\mathcal{C}ob_1^{fr}) \simeq \cdots \simeq B(ex\mathcal{C}ob_d^{fr}) \simeq \cdots \simeq \Omega^\infty S^\infty$$

Compare:

Classically, framed cobordism theory is isomorphic to stable homotopy theory.

For the framed theory, this is the constant filtration

$$B(ex\mathcal{C}ob_1^{fr}) \simeq \cdots \simeq B(ex\mathcal{C}ob_d^{fr}) \simeq \cdots \simeq \Omega^\infty S^\infty$$

Proof sketch:

$$B(ex\mathcal{C}ob_d^{fr}) = \lim_{n\to\infty} \Omega^n (U_{d,n}^{fr,\perp})^c \simeq \Omega^\infty S^\infty$$

For the framed theory, this is the constant filtration

$$B(ex\mathcal{C}ob_1^{fr}) \simeq \cdots \simeq B(ex\mathcal{C}ob_d^{fr}) \simeq \cdots \simeq \Omega^\infty S^\infty$$

Proof sketch:

$$B(ex\mathcal{C}ob_d^{fr}) = \lim_{n\to\infty} \Omega^n (U_{d,n}^{fr,\perp})^c \simeq \Omega^\infty S^\infty$$

fr is defined by $EO(d) \to BO(n)$ and

$$EO(d) = \lim_{n\to\infty} U_{d,n} \simeq *$$

$U_{d,n}^{fr}$ is the universal bundle over the Stiefel manifold of framed d-planes in $\mathbb{R}^{d+n}$ so that

$$(U_{d,n}^{fr,\perp})^c \text{ is approximately } S^n$$

Cobordism Hypothesis for invertible theories

An extended framed TFT

$$\mathcal{F} : ex\mathcal{C}ob_d^{fr} \longrightarrow \mathcal{V}_d$$

induces a map of infinite loop spaces

$$B\mathcal{F} : B(ex\mathcal{C}ob_d^{fr}) \simeq \Omega^\infty S^\infty \longrightarrow B(\mathcal{V}_d).$$

$\Omega^\infty S^\infty$ is the free infinite loop space on one point
$\Longrightarrow B\mathcal{F}$ is determined by its value on that point, $B\mathcal{F}(*)$.

If $\mathcal{F}$ is invertible (in the sense that the images of all morphisms are invertible) it factors through $B\mathcal{F}$.

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