Topological Quantum Field Theories in Homotopy Theory II

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Leitmotif = Understanding Manifolds

1. Classical Cobordism Theory (Thom, ...)

2. Topological Field Theory (Witten, Atiyah, Segal, ...)

3. Cobordism Hypothesis (Baez-Dolan, Lurie, ...)

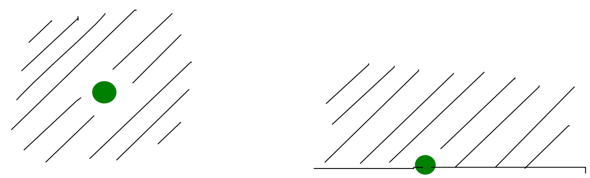
4. Classifying spaces of cobordism categories
—— classifying space of cobordism categories
. (Galatius–Madsen–Tillmann–Weiss)
—— the cobordism hypothesis for invertible TQFTs
—— filtration of the classical theory

5. Extracting information on Diff(M)
—— Mumford conjecture (Madsen–Weiss, ...)
—— higher dimensional analogues
. (Galatius–Randal-Williams, ...)

Manifolds

W is a manifold of dimension d if locally it is diffeomorphic to

$$\mathbb{R}^d \quad \text{or} \quad \mathbb{R}^d_{\geq 0}.$$



W is closed if it is compact and has no boundary.

Fundamental problem:
– classify compact smooth manifolds W of dim d
– understand their groups of diffeomorphisms Diff(W)
– understand the topology of $\mathcal{M}^{top}(W) = B\text{Diff}(W)$
– understand characteristic classes $H^*(\mathcal{M}^{top}(W))$

Topological moduli space

W closed of dimension d

$$\mathcal{M}^{top}(W) := \mathsf{Emb}(W, \mathbb{R}^\infty)/\mathsf{Diff}(W)$$

By Whitney, $\mathsf{Emb}(W, \mathbb{R}^\infty) \simeq *$ and hence

$$\mathcal{M}^{top}(W) \simeq B\mathsf{Diff}(W)$$

• when W is oriented, the diffeomorphisms preserve the orientation.
• when $\partial W \neq \emptyset$, the diffeomorphisms fix a collar of the boundary.

Example 0: $d = 0$, $W = n$ pts

$$\mathcal{M}^{top}(W) = \mathrm{Emb}(n \text{ pts}, \mathbb{R}^\infty)/\Sigma_n \simeq B\Sigma_n$$

This is the configuration space of n unordered points in $\mathbb{R}^\infty$.

Example 1: $d = 1$, $W = S^1$

Note that $\mathrm{Diff}^+(S^1) \simeq S^1$ so that

$$\mathcal{M}^{top}(S^1) \simeq BS^1 \simeq \mathbb{C}P^\infty$$

Example 2: $d = 2$

Compare with the moduli spaces of Riemann surfaces:

$$\mathcal{M}_g = \mathcal{T}_g/\Gamma_g = \mathbb{R}^{6g-6}/\Gamma_g$$

Note, for $g > 1$,

$$\mathrm{Diff}^+(F_g) \xrightarrow{\simeq} \pi_0(\mathrm{Diff}^+(F_g) =: \Gamma_g$$

Hence, the forgetful map

$$\mathcal{M}^{top}(F_g) \longrightarrow \mathcal{M}_g$$

is a rational equivalence and

$$H^*(\mathcal{M}^{top}(F_g);\mathbb{Q}) = H^*(\mathcal{M}_g;\mathbb{Q})$$

Characteristic classes

There is a universal W-bundle on $\mathcal{M}^{top}(W)$, and any family E of manifolds diffeomorphic to W and parametrized by a space X is given by a map

$$\phi_E : X \to \mathcal{M}^{top}(W)$$

Hence, the characteristic classes for W bundles are given by

$$H^*(\mathcal{M}^{top}(W)) = H^*(B\mathrm{Diff}(W))$$

Construction of characteristic classes

Given a smooth orientable W-bundle

$$W^d \to E \xrightarrow{\pi} X$$

consider the vertical tangent bundle

$$\mathbb{R}^d \to T^v E := \coprod_{x \in X} TE_x \to E$$

It is classified by a map

$$\phi_{T^v E} : E \to BSO(d)$$

For every $c \in H^n(BSO(d))$ consider

$$\kappa_c(E) := \pi_!(\phi_E(c)) \in H^{n-d}(X)$$

Here $\pi_!$ is the Gysin map. So

$$\kappa_c \in H^{n-d}(\mathcal{M}(W)) = H^{n-d}(B\text{Diff}(W))$$

Example 2: $d = 2$

$$H^*(BSO(2)) = H^*(\mathbb{C}P^\infty) = \mathbb{Z}[e]$$

For $c = e^{i+1} \in H^{2i+2}(CP^\infty)$, κ_c corresponds to the ith Morita-Mumford-Miller class

$$\kappa_i := \kappa_{e^{i+1}} \in H^{2i}(\mathcal{M}_g; \mathbb{Q})$$

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Homology stability [Harer, Ivanov, Boldsen, RW]
For $* < 2g/3$, $H_*(B\Gamma_{g,n})$ is independent of g and n.

Mumford conjecture: For $* \leq (2g-2)/3$

$$H^*(\mathcal{M}_g; \mathbb{Q}) \simeq \mathbb{Q}[\kappa_1, \kappa_2, \ldots]$$

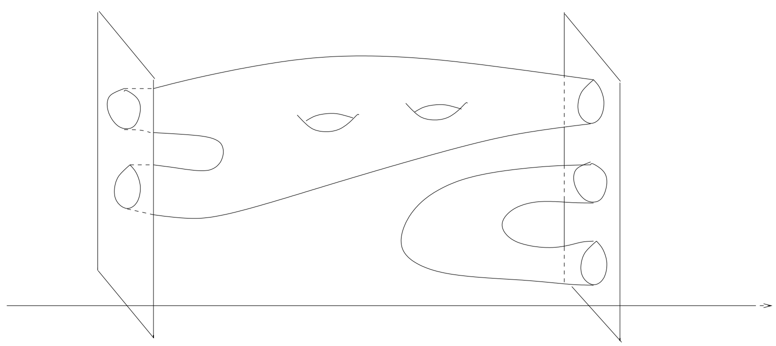
. **early 1980s**

4. Classifying space of cobordism categories

The **enriched cobordism category** $\mathcal{C}ob_d$:

Objects: $\mathbb{R} \times \coprod_{M^{d-1}} \mathcal{M}(M) \simeq \coprod_{M^{d-1}} B\mathrm{Diff}(M)$

Morphisms:
$mor_{\mathcal{C}ob_d}(M_0, M_1) \simeq \coprod_{W^d} \mathcal{M}(W) \simeq \coprod_{W^d} B\mathrm{Diff}(W; \partial)$



$$M_0 \longrightarrow W \longleftarrow M_1$$

Example: $d = 0$

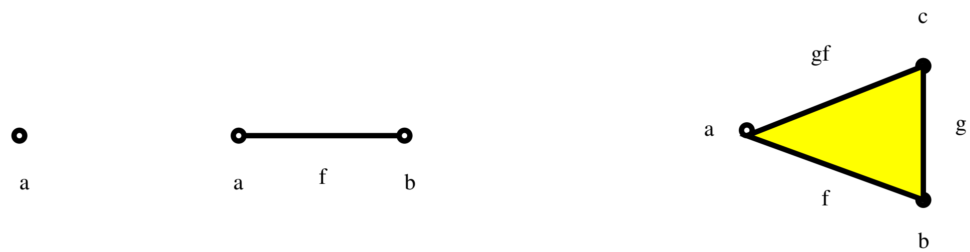
Objects: $\mathbb{R}$ as $M^{-1} = \emptyset$

Morphisms:

$$\mathit{mor}_{\mathcal{C}ob_d}(a_0, a_1) = \coprod_n Emb(n \text{ pts}, [a_0, a_1] \times \mathbb{R}^\infty)/\Sigma_n$$

Classifying space functor

B : Topological Categories $\longrightarrow$ Spaces, $\qquad \mathcal{C} \mapsto B\mathcal{C}$



$$B\mathcal{C} := \left(\coprod_{k \geq 0} \triangle^k \times \{(f_1, \ldots, f_k) | \text{composable morphs}\} \right) / \sim$$

For every object a in $\mathcal{C}$, the adjoint of

$$\triangle^1 \times \mathit{mor}_{\mathcal{C}}(a, a) \longrightarrow B\mathcal{C}$$

gives rise to a characteristic map

$$\alpha : \mathit{mor}_{\mathcal{C}}(a, a) \longrightarrow \text{maps}([0,1], \partial; B\mathcal{C}, a) = \Omega B\mathcal{C}$$

Theorem [Galatius, Madsen, T., Weiss]

$$\Omega B(\mathcal{C}ob_d^+) \simeq \Omega^\infty \mathbf{MTSO}(d) = \lim_{n\to\infty} \Omega^{d+n}((U_{d,n}^\perp)^c)$$

where $U_{d,n}^\perp$ is the orthogonal complement of the universal bundle $U_{d,n} \to Gr^+(d,n)$.

Note: $Gr^+(d,n) \hookrightarrow Gr^+(d,n+1)$ and $U_{d,n}^\perp \times \mathbb{R} \hookrightarrow U_{d,n+1}^\perp$
Hence $\Sigma(U_{d,n}^\perp)^c \to (U_{d,n+1}^\perp)^c$

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Examples:

$$\Omega B(\mathcal{C}ob_0) = \Omega^\infty S^\infty$$

Theorem [Galatius, Madsen, T., Weiss]

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Examples:

$$\Omega B(\mathcal{C}ob_0) = \Omega^\infty S^\infty$$

$$\Omega B(\mathcal{C}ob_0^+) = \Omega^\infty S^\infty \times \Omega^\infty S^\infty$$

$$\Omega B(\mathcal{C}ob_1^+) = \Omega^{\infty+1} S^\infty$$

A similar theorem holds for any tangential structure θ!

Excursion: tangential structures

$$\text{Vect}_d(W) = [W, BO(d)] = [W, Gr(d, \infty)]$$

$$E \leftrightarrow \phi_E$$

Definition: Let $\theta : \mathcal{X} \to BO(d)$ be a fiber bundle. A θ-structure on W^d is a lift of $\phi_{TW} : W \to BO(d)$ to $\mathcal{X}$

Oriented $+$: $\mathbb{Z}/2\mathbb{Z} \to BSO(d) \to BO(d)$
Framed fr: $O(d) \to EO(d) \to BO(d)$
Background X: $X \to BO(d) \times X \to BO(d)$
Connected $< k >$: $BO(d) < k > \longrightarrow BO(d)$

$$\Omega^\infty \mathbf{MT}\theta = \lim_{n\to\infty} \Omega^{d+n}(\theta^*(U_{d,n}^\perp)^c)$$

Example: $\theta : BO(d) \times X \to BO(d)$

$$\Omega B(\mathcal{C}ob_d^X) \simeq \Omega^\infty \mathbf{MTSO}(d) \wedge X_+ = \lim_{n\to\infty} \Omega^{d+n}((U_{d,n}^\perp)^c \wedge X_+)$$

A θ-structure on W is a map $f : W \to X$.

For $d = 2$, this gives a topological version of **Gromov-Witten theory**

Note: the cobordism category of manifolds in a background space X gives rise to a generalised cohomology!

Lemma: $H^*(\Omega_0^\infty \mathbf{MTSO}(d), \mathbb{Q}) \simeq \mathbf{\Lambda}^*(H^{*>0}(BSO(d);\mathbb{Q})[-d])$

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Proof: For fixed $* > 0$ and large n,

$$\begin{aligned}
\pi_*(\Omega_0^\infty \mathbf{MTSO(d)}) \otimes \mathbb{Q} &= \pi_*(\lim_{n\to\infty} \text{maps}_*(S^{d+n}, (U_{d,n}^\perp)^c)) \otimes \mathbb{Q} \\
&= \pi_*(\text{maps}_*(S^{d+n}, (U_{d,n}^\perp)^c) \otimes \mathbb{Q} \\
&= \pi_{*+d+n}((U_{d,n})^\perp)^c \otimes \mathbb{Q} \\
&= H_{*+d+n}((U_{d,n}^\perp)^c) \otimes \mathbb{Q} \quad \text{by Serre} \\
&= H_{*+d}(Gr^+(d,n)) \otimes \mathbb{Q} \quad \text{by Thom} \\
&= H_{*+d}(BSO(d)) \otimes \mathbb{Q}
\end{aligned}$$

Now apply a theorem by Milnor-Moore on structure of commutative Hopf algebras and take duals.

Lemma: $H^*(\Omega_0^\infty \mathbf{MTSO}(d), \mathbb{Q}) \simeq \mathbf{\Lambda}^*(H^{*>0}(BSO(d); \mathbb{Q})[-d])$

For any closed d-dimensional manifold W

$\alpha : \mathcal{M}^{top}(W) \subset mor_{\mathcal{C}ob_d}(\emptyset, \emptyset) \to \Omega B\mathcal{C}ob_d \simeq \Omega^\infty \mathbf{MTSO}(d)$

For every $c \in H^{*+d}(BSO(d))$

$$\alpha^*(c) = \kappa_c \in H^*(\mathcal{M}^{top}(W); \mathbb{Q})$$

Example: For $d = 2$,

$$H^*(BSO(2)) = H^*(\mathbb{C}P^\infty) = \mathbb{Z}[e], \quad \deg(e) = 2$$

So

$$H^*(\Omega_0^\infty \mathbf{MTSO}(2), \mathbb{Q}) = \mathbb{Q}[\kappa_1, \kappa_2, \ldots], \quad \deg(\kappa_i) = 2i$$

Fibration sequence

$$\Omega^\infty \mathbf{MTSO}(d) \xrightarrow{\omega} \Omega^\infty\Sigma^\infty(BSO(d)_+) \longrightarrow \Omega^\infty\mathbf{MTSO}(d-1).$$

ω is induced by

$U_{d,n}^{\perp} \to U_{d,n} \times_{Gr^+(d,n)} U_{d,n}^{\perp} \simeq Gr^+(d,n) \times \mathbb{R}^{d+n}$

Genauer proves that this corresponds to natural maps of cobordism categories:

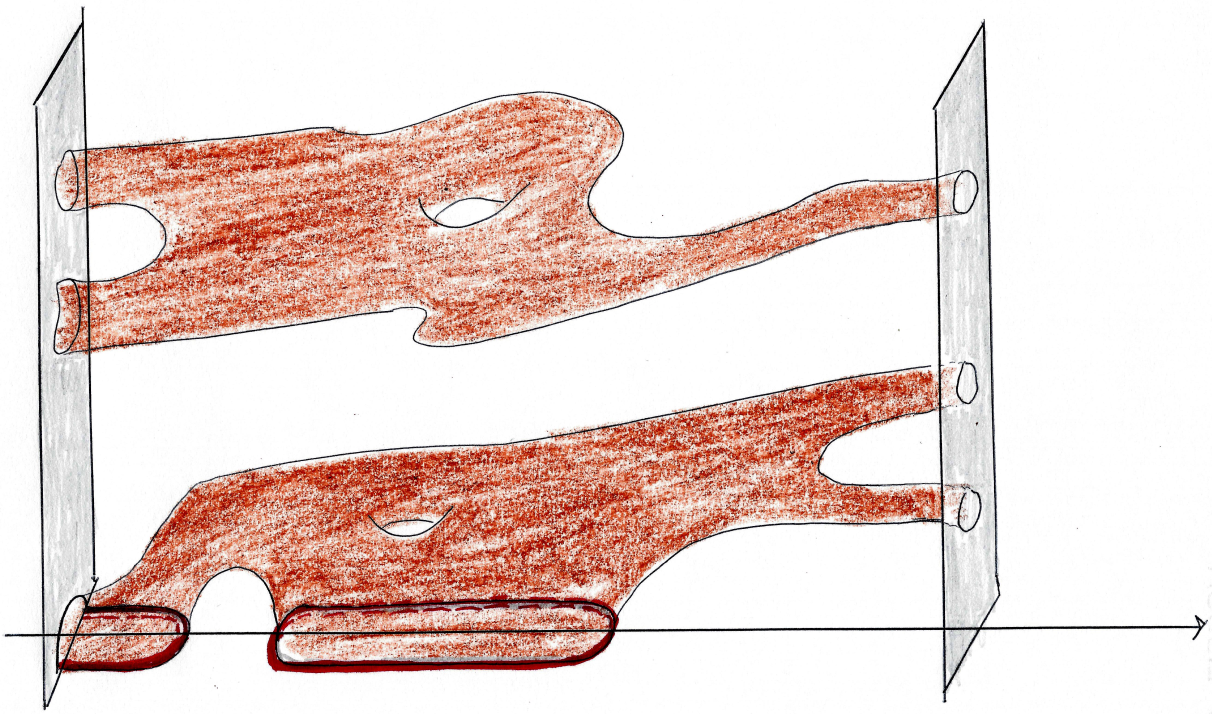
$\mathcal{C}ob_d$: d-dim cobordisms in $[a_0, a_1] \times \mathbb{R}^{d+n-1} \times (0, \infty)$

$\bigcap$

$\partial - \mathcal{C}ob_d$: d-dim cobordisms in $[a_0, a_1] \times \mathbb{R}^{d+n-1} \times [0, \infty)$

$\downarrow$

$\mathcal{C}ob_{d-1}$: $d-1$-dim cobordisms in $[a_0, a_1] \times \mathbb{R}^{d-1+n} \times \{0\}$



Application

Suppose W^d is an oriented manifold with boundary. Then restriction defines a map

$$r : B\text{Diff}(W) \to B\text{Diff}(\partial W)$$

Let

$$c = p_1^{k_1} \dots p_r^{k_r} e^s \in H^*(BSO(d); \mathbb{Z}) \quad r = d/2 \text{ or } (d+1)/2$$

Corollary [Giansiracusa-T.]
$r^*(\kappa_c) = 0$ when d even, or when d odd and s even.

In particular, κ_{2i+1} goes to zero when restricted to the handlebody subgroup $\mathcal{H}_g \subset \Gamma_g$.

Sketch of proof

Starting with Madsen-Weiss, the proof has been continuously simplified and the theorem generalised [Galatius–Madsen–T.–Weiss, Galatius, Galatius–Randal-Williams].

$\Phi_{d,n}$: space of all embedded d-manifolds without boundary which are closed as a subset in $\mathbb{R}^{d+n}$; base point $\varnothing$; topologized so that manifolds can disappear at infinity.

$\Phi^k_{d,n}$: subspace of manifolds embedded in $\mathbb{R}^k \times (0,1)^{n+d-k}$.

$$\Phi_{d,n} = \Phi^{d+n}_{d,n} \supset \cdots \supset \Phi^k_{d,n} \supset \cdots \supset \Phi^0_{d,n} \simeq \coprod_{W,\, \partial = \varnothing} B\mathrm{Diff}(W)$$

$\mathcal{C}ob^k_{d,n}$: k-fold cobordisms category of d-manifolds embedded in $\mathbb{R}^{d+n}$

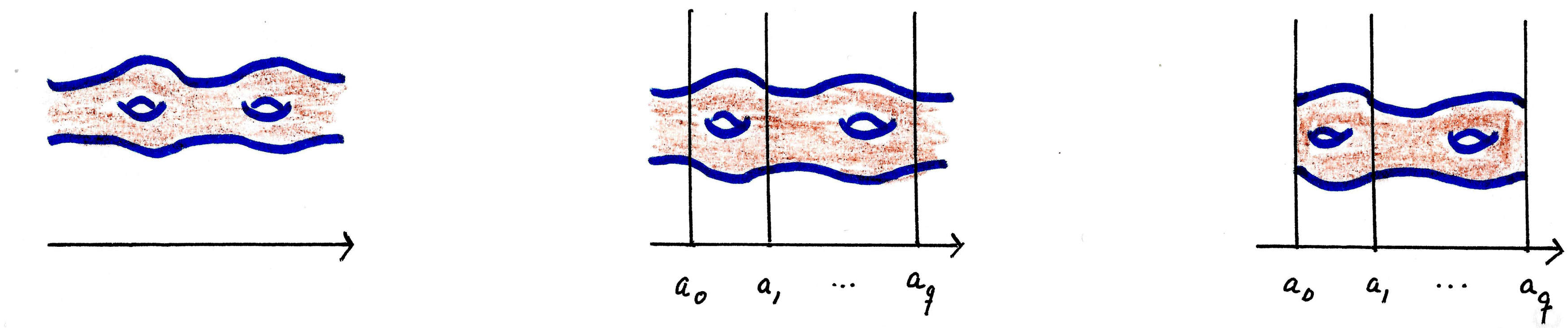
$\lim_{n\to\infty} \mathcal{C}ob^1_{d,n} = \mathcal{C}ob_d$ and $\lim_{n\to\infty} \mathcal{C}ob^d_{d,n} = ex\mathcal{C}ob_d$.

Step 1: $(U^{\perp}_{d,n})^c \simeq \Phi^{d+n}_{d,n}$
tangential information: $(P, v) \mapsto v - P$.

Step 2: $\Phi^k_{d,n} \xrightarrow{\simeq} \Omega\Phi^{k+1}_{d,n}$ for $k > 0$
adjoint of $\mathbb{R}\times\Phi^k_{d,n} \to \Phi^{k+1}_{d,n}, \quad (t, W) \mapsto W+(0, \dots, t, \dots, 0)$

Step 3: $B(\mathcal{C}ob^k_{d,n}) \simeq \Phi^k_{d,n}$

$$\Phi^1_{d,n} \simeq B(\mathcal{C}ob^1_{d,n})$$



$$. \qquad \Phi^1_{d,n} \xleftarrow{\simeq} N_q(\mathcal{P}) \xrightarrow{\simeq} N_q(\mathcal{C}ob_{d,n})$$

The fiber of the left arrow at W is the poset of regular values of the projection onto $\mathbb{R}$; its nerve is contractible.

Step 1: $(U_{d,n}^{\perp})^c \simeq \Phi_{d,n}^{d+n}$
tangential information: $(P, v) \mapsto v - P$.

Step 2: $\Phi_{d,n}^{k} \xrightarrow{\simeq} \Omega\Phi_{d,n}^{k+1}$ for $k > 0$
adjoint of $\mathbb{R}\times\Phi_{d,n}^{k} \to \Phi_{d,n}^{k+1}$, $\quad (t, W) \mapsto W + (0, \ldots, t, \ldots, 0)$

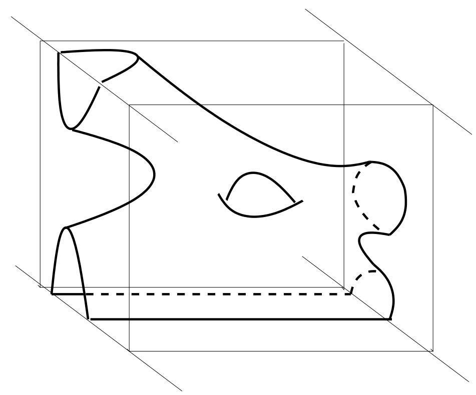
Step 3: $B(\mathcal{C}ob_{d,n}^{k}) \simeq \Phi_{d,n}^{k}$

$\Longrightarrow B(\mathcal{C}ob_{d,n}^{k}) \simeq \Phi_{d,n}^{k} \simeq \cdots \simeq \Omega^{d+n-k}\Phi_{d,n}^{d+n} \simeq \Omega^{d+n-k}(U_{d,n}^{\perp})^c$

for $k = 1$ and $n \to \infty$, $B(\mathcal{C}ob_d) \simeq \Omega^{\infty-1}\mathbf{MTSO}(d)$
for $k = d + n$ and $n \to \infty$, $B(ex\mathcal{C}ob_d) \simeq \Omega^{\infty-d}\mathbf{MTSO}(d)$

An even finer filtration

$$\begin{aligned}\Omega^{\infty}\mathbf{MSO} &\simeq \lim_{n\to\infty}\lim_{d\to\infty}\Omega^n(U_{n,d})^c \\ &\simeq \lim_{d\to\infty}\lim_{n\to\infty}\Omega^n(U_{d,n}^{\perp})^c \\ &\simeq \lim_{d\to\infty}\lim_{n\to\infty} B(\mathcal{C}ob_{d,n}^d)\end{aligned}$$



A 2-morphism in $\mathcal{C}ob^2_{2,1}$.

5. Unstable information

Question: Given W^d with θ-structure, how close to an isomorphism in homology is α?

$$\alpha : \mathcal{M}^{top}(W) = B\mathsf{Diff}(W) \longrightarrow \Omega^\infty \mathbf{MT}\theta(d)$$

5. Unstable information

Question: Given W^d with θ-structure, how close to an isomorphism in homology is α?

$$\alpha : \mathcal{M}^{top}(W) = B\mathrm{Diff}(W) \longrightarrow \Omega_0^\infty \mathbf{MT}\theta(d)$$

Theorem [Barrett-Priddy, Quillen, Segal]
For $d = 0$: $B\Sigma_n \xrightarrow{\alpha} \Omega^\infty \mathbf{MTO}(0) \simeq \Omega^\infty S^\infty$ is a homology isomorphism in degrees $* \leq n/2$.

Theorem [Galatius, Madsen, T., Weiss]

$$\Omega B(\mathcal{C}ob_d^\partial) \simeq \Omega B(\mathcal{C}ob_d) \simeq \Omega^\infty \mathbf{MTSO}(d)$$

$\mathcal{C}ob_d^\partial$ contains only cobordisms W with $\pi_0(W) \to \pi_0(M_1)$ surjective

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$\mathcal{C}ob_d^\partial$ contains only cobordisms W with $\pi_0(W) \to \pi_0(M_1)$ surjective

Previous result:

Theorem [T.] $\quad \Omega B(\mathcal{C}ob_2^\partial) \simeq \mathbb{Z} \times B\Gamma_\infty^+$

By homology stability, for $* \leq (2g-2)/3$

$$H_*(B\Gamma_\infty^+) = H_*(B\Gamma_\infty) = H_*(B\Gamma_g)$$

Together these theorems imply

Theorem [Madsen-Weiss]
For $d = 2$: $B\text{Diff}(F_g) \xrightarrow{\alpha} \Omega_0^\infty \mathbf{MTSO}(2)$ is a homology isomorphism in degrees $* \leq (2g-2)/3$.

$\Longrightarrow$ **Mumford's Conjecture**:

$$H^*(\mathcal{M}_g; \mathbb{Q})) \simeq H^*(B\text{Diff}(F_g), \mathbb{Q}) \sim \mathbb{Q}[\kappa_1, \kappa_2, \ldots]$$

Let $W_g = \sharp_g S^d \times S^d$ and $d > 2$. As W_g is $(d-1)$-connected, it has a θ structure for

$$\theta : BO(2d) < d > \longrightarrow BO(d)$$

Theorem [Galatius, Randal-Williams]
For $* < (g-4)/2$,

$$H_*(B\mathrm{Diff}(W_g; D^{2d})) \simeq H_*(B\mathrm{Diff}(W_{g+1}; D^{2d})$$

Theorem [Galatius, Randal-Williams]
$\alpha : \mathrm{hocolim}_{g\to\infty} B\mathrm{Diff}(W_g; D^{2d}) \to \Omega_0^\infty \mathbf{MT}\theta(d)$
is a homology equivalence

Similar results hold for more general simply connected manifolds of **even dimension**.

There are problems in **odd dimensions**!

Note

For $d = 1$: $B\mathrm{Diff}(S^1) \simeq \mathbb{C}P^\infty \xrightarrow{\alpha} \Omega_0^\infty \mathbf{MTSO}(1) \simeq \Omega^{\infty+1} S^\infty$ is trivial in rational homology fro any W^3.

There are problems in **odd dimensions**!

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For $d = 1$: $B\mathrm{Diff}(S^1) \simeq \mathbb{C}P^\infty \xrightarrow{\alpha} \Omega_0^\infty \mathbf{MTSO}(1) \simeq \Omega^{\infty+1} S^\infty$ is trivial in rational homology fro any W^3.

Theorem [Ebert]
For $d = 3$: $B\mathrm{Diff}(W^3) \xrightarrow{\alpha} \Omega^\infty \mathbf{MTSO}(3)$ is trivial in rational homology.

More generally, there are rational cohomology classes in the cohomology of $\Omega^\infty \mathbf{MTSO}(2n+1)$ that are not detected by any $2n+1$-dimensional manifold.

$Q_g = \sharp_g S^d \times S^{d+1}$

Theorem **Perlmutter**

For $* \leq (g-3)/2$,

$$H_*(B\mathrm{Diff}(Q_g; D^{2d+1}) \simeq H_*(B\mathrm{Diff}(Q_{g+1}; D^{2d-1})$$

Hebestreit-Perlmutter

construct a variant of the cobordism category that has a good chance to play the role of $\mathcal{C}ob_d$ in the odd dimensional case.

Also true for other product of spheres.

General scheme of proof

1. Homology stability with respect to connected sum with a suitable basic manifolds, e.g. $S^d \times S^d$

2. Group completion argument to identify the homology of the limit with the classifying space of a subcatgory of an appropriate cobordism category, e.g. $\lim_{g\to\infty} H_* B\mathrm{Diff}(F_g, D^2) \simeq H_* \Omega B\mathcal{C}ob_2^\partial$

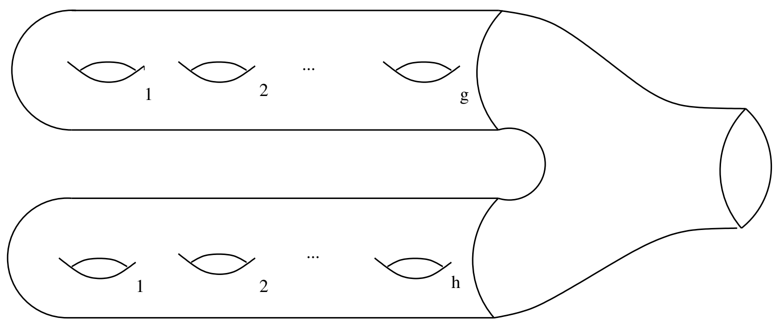
3. Parametrised surgery to show that the subcategory has a classifying space that is homotopic to the whole category, e.g. $B\mathcal{C}ob_d^\partial \simeq B\mathcal{C}ob_d$

4. Identify the homotopy type of the classifying space of the whole category, e.g. $\Omega B\mathcal{C}ob_d \simeq \Omega^\infty \mathbf{MTSO}(d)$

Multiplicative structure

$\Sigma_n \times \Sigma_m \longrightarrow \Sigma_{n+m}$

$\mathrm{Diff}(F_{g,1}) \times \mathrm{Diff}(F_{h,1}) \longrightarrow \mathrm{Diff}(F_{g+h,1})$



$\mathrm{Aut}F_n \times \mathrm{Aut}F_m \longrightarrow \mathrm{Aut}F_{n+m}$

. $\mathrm{Aut}F_n \simeq \mathrm{HtEq}(V_n S^1)$

Products

$$G_n \times G_m \longrightarrow G_{n+m}$$

induce a monoid structure on

$$M := \coprod_{n \geq 0} BG_n$$

If products are commutative upto conjugation by an element in G_{n+m},

then the induced product on $H_*(M)$ is commutative

Group completion

Algebraic: $M \longrightarrow \mathcal{G}(M) =$ Grothendieck group of M
Example: $\mathbb{N} \longrightarrow \mathcal{G}(\mathbb{N}) = \mathbb{Z}$

Homotopy theoretic:
$M \longrightarrow \Omega BM = \mathrm{map}_*(S^1, BM) =$ loop space of BM

- $M = G$ a group $\Longrightarrow \Omega BG \simeq G$
- M discrete $\Longrightarrow \Omega BM \simeq \mathcal{G}(M)$

Group Completion Theorem:
Let $M = \coprod_{n \geq 0} M_n$ be a topological monoid such that the multiplication on $H_*(M)$ is commutative. Then

$$H_*(\Omega BM) = \mathbb{Z} \times \lim_{n \to \infty} H_*(M_n) = \mathbb{Z} \times H_*(M_\infty)$$

W_n	H_*-stab.	$\Omega B(\sqcup_n \mathcal{M}(W_n))$	
n pts	n/2	$\Omega^\infty S^\infty$	Barratt-Eccles-Priddy
$S_{g,1}$	2g/3	$\Omega^\infty \mathbf{MTSO(2)}$	Madsen-Weiss
$N_{g,1}$	g/3	$\Omega^\infty \mathbf{MTO(2)}$	Madsen-Weiss & Wahl
$V_n S^1$	n/2	$\Omega^\infty S^\infty$	Galatius
$\#_g(S^d \times S^d)_1$	(g-4)/2	$\Omega^\infty \mathbf{MTSO(2d)}\langle d \rangle$	Galatius-Randal-Williams
$(S^d \times D^{d+1})_1$	(g-4)/2	$Q(BO(2d+1)\langle d \rangle)$	Botvinnik-Perlmutter
discrete	(g-4)/2	$\Omega^\infty X^{-\gamma}$	Nariman
$(S^d \times S^{d+1})_1$	(g-3)/2	candidate	Perlmutter