INFINITE LOOP SPACES AND NILPOTENT K-THEORY

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Abstract. Using a construction derived from the descending central series of the free groups first considered in [1], we produce filtrations by infinite loop spaces for the classical infinite loop spaces BSU, BU, BSO, BO, BSp, $BGL_\infty(R)^+$ and $Q_0(\mathbb{S}^0)$. We show that these infinite loop spaces are the zero spaces of E_∞-ring spectra. We introduce the notion of q-nilpotent K-theory of a CW-complex X for any $q \geq 2$, which extends the notion of commutative K-theory as defined in [2], and show that it is represented by $\mathbb{Z} \times B(q, U)$, were $B(q, U)$ is the q-th term of the aforementioned filtration of BU.

For the proof we introduce an alternative way of associating an infinite loop space to a commutative $\mathbb{I}$-monoid and give criteria when it can be identified with the plus construction on the associated limit space. Furthermore, we introduce the notion of a commutative $\mathbb{I}$-rig and show that they give rise to E_∞-ring spectra.

1. Introduction

Let G denote a locally compact, Hausdorff topological group such that $1_G \in G$ is a non-degenerate base point. It is well-known that we can obtain a model for the classifying space BG as the geometric realization of the classical bar construction B_*G. Now fix an integer $q \geq 2$ and let Γ_n^q be the q-th stage of the descending central series of the free group on n-letters F_n, with the convention $\Gamma_n^1 = F_n$. Consider the set of homomorphisms $B_n(q, G) := \mathrm{Hom}(F_n/\Gamma_n^q, G)$. If $e_1, \dots, e_n$ are generators of F_n, then evaluation on the classes corresponding to $e_1, \dots, e_n$ provides a natural inclusion $B_n(q, G) \subset G^n$. Using this inclusion we can give $B_n(q, G)$ the subspace topology. Therefore $B_n(q, G)$ is precisely the space of ordered n-tuples in G generating a subgroup of G with nilpotence class less than q. For each fixed $q \geq 2$ the collection $\{B_n(q, G)\}_{n\geq 0}$ forms a simplicial space with face and degeneracy maps induced by those in the bar construction. The geometric realization of this simplicial space is denoted by $B(q, G)$. These spaces were first introduced in [1], where many of their basic properties were established. They give rise to a natural filtration of the classifying space

$$B(2, G) \subset B(3, G) \subset \cdots \subset B(q, G) \subset B(q + 1, G) \subset \cdots \subset BG.$$

For $q = 2$ we obtain $B_{com}G := B(2, G)$ which is constructed by assembling the different spaces of ordered commuting n-tuples in the group G. In [2] it was shown that for Lie groups this space plays the role of a classifying space for commutativity. More generally $B(q, G)$ is a classifying space for G-bundles of transitional nilpotency class less than q.

For the infinite unitary group $U = \mathrm{colim}_{n\to\infty} U(n)$, it is well known that BU is the infinite loop space underlying an E_∞-ring spectrum, i.e. it is a so called E_∞-ring space. A basic question is whether the above gives rise to a filtration of BU by E_∞-ring spaces. The main purpose of this paper is to show that indeed this is the case, not only for U but also for other linear groups.

Theorem 1.1. *The spaces $B(q, SU)$, $B(q, U)$, $B(q, SO)$ $B(q, O)$ and $B(q, Sp)$ provide a filtration by E_∞-ring spaces of the classical infinite loop spaces BSU, BU, BSO, BO and BSp respectively.*

The q-nilpotent K-theory of a space X is defined using isomorphism classes of bundles on X whose transition functions generate subgroups of nilpotence class less than q. We show that $K_{q-nil}(X) \cong [X, \mathbb{Z} \times B(q, U)]$, from which we obtain

Corollary 1.2. *$K_{q-nil}(-)$ is the zero-th term of a generalized multiplicative cohomology theory.*

In particular we obtain a sequence of multiplicative cohomology theories

$$K_{com}(X) = K_{2-nil}(X) \to K_{3-nil}(X) \to \cdots \to K_{q-nil}(X) \to \cdots \to K(X).$$

We also show that $B(q, U) \to BU$ splits as a map of infinite loop spaces, whence we see that topological K-theory is a direct summand in K_{q-nil}.

The infinite loop space structure on $B(q, G)$ for $G = U, SU, SO, O, Sp$ is obtained by using the machinery of commutative $\mathbb{I}$-monoids first introduced by Bökstedt and developed in [18], [16], [9]. Here $\mathbb{I}$ is the category of finite sets and injections. In addition to the usual construction, we associate an infinite loop space to a commutative $\mathbb{I}$-monoid by restricting the usual homotopy colimit construction to the subcategory $\mathbb{P}$ of finite sets and isomorphisms. This allows us to identify the homotopy type of the homotopy colimit under certain conditions. Another addition to infinite loop space theory is the introduction of the notion of a commutative $\mathbb{I}$-rig which we show to give rise to a bipermutative category and hence an E_∞-ring spectrum.

Our main examples above all arise from commutative $\mathbb{I}$-rigs where we can identify the infinite loop space as the plus construction of the associated limit space. A more complicated situation arises for $Q_0(\mathbb{S}^0) \simeq B\Sigma_\infty^+$ and $BGL_\infty(R)^+$. Our methods give rise to natural filtrations but the terms are not easy to describe.

The outline of this article is as follows. In Section 2 we use the machinery of commutative $\mathbb{I}$-monoids ($\mathbb{I}$-rigs) to produce two associated infinite loop spaces (E_∞-ring spaces). In Section 3 we show that these are homotopy equivalent and identify them under suitable assumptions. Then in Section 4 we apply these results to prove Theorem 1.1 and show that the spaces $B(q, U)$ for $q \geq 2$ are infinite loop spaces and that BU splits off. Finally, in Section 5 we introduce the notion of q-nilpotent K-theory and show that it is represented by the infinite loop spaces $\mathbb{Z} \times B(q, U)$, answering the question raised for commutative K-theory in [2].

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2. Commutative $\mathbb{I}$-monoids and infinite loop spaces

The standard construction of the infinite loop space structure on BU from the permutative category of complex vector spaces and their isomorphisms does not restrict to give an infinite loop space structure on $B(q, U)$. Instead we are going to use certain constructions on commutative $\mathbb{I}$-monoids. More precisely, we will give two constructions of permutative categories from commutative $\mathbb{I}$-monoids. For the case of interest the permutative categories are actually bipermutative and hence give rise to E_∞-ring spectra. We start by setting up some notations and basic definitions following [18], [16], and [9]. We will use [5] as a reference for bipermutative categories and the associated multiplicative infinite loop space machinery.

2.1. The category $\mathbb{I}$ and its subcategories $\mathbb{P}$ and $\mathbb{N}$. These three categories are skeletons of the category of finite sets and injections, the category of finite sets and isomorphisms, and the translation category associated to the monoid of natural numbers. We will use the following notation.

For every integer $n \geq 0$ let $\mathbf{n}$ denote the set $\{1, 2, \ldots, n\}$. When $n = 0$ we use the convention $\mathbf{0} := \emptyset$. Let $\mathbb{I}$ denote the category whose objects are the elements of the form $\mathbf{n}$ for all integers $n \geq 0$ with morphisms given by all injective maps. Note that in particular $\mathbf{0}$ is an initial object in the category $\mathbb{I}$. $\mathbb{I}$ is a symmetric monoidal category under concatenation $\mathbf{m} \sqcup \mathbf{n} := \{1, 2, \ldots, m + n\}$ with the symmetry morphism given by the (m, n)-shuffle map

$$\tau_{\mathbf{m},\mathbf{n}} : \mathbf{m} \sqcup \mathbf{n} \to \mathbf{n} \sqcup \mathbf{m}.$$

It is also symmetric monoidal under Cartesian product

$$\mathbf{m} \times \mathbf{n} := \{1 = (1, 1), 2 = (1, 2), \ldots, n + 1 = (2, 1), \ldots, mn = (m, n)\}$$

given by lexicographic ordering. By definition $\mathbf{0} \times \mathbf{n} = \mathbf{0} = \mathbf{n} \times \mathbf{0}$. The associated symmetry morphism is given by a permutation

$$\tau^{\times}_{\mathbf{mn}} : \mathbf{m} \times \mathbf{n} \to \mathbf{n} \times \mathbf{m}.$$

The latter monoidal product is distributive over the former. More precisely left distributivity

$$\delta^{l}_{\mathbf{m},\mathbf{n},\mathbf{k}} : \mathbf{m} \times \mathbf{k} \sqcup \mathbf{n} \times \mathbf{k} \to (\mathbf{m} \sqcup \mathbf{n}) \times \mathbf{k}$$

is given by the identity and right distributivity is given by a permutation

$$\delta^{r}_{\mathbf{m},\mathbf{n},\mathbf{k}} : \mathbf{m} \times \mathbf{n} \sqcup \mathbf{m} \times \mathbf{k} \to \mathbf{m} \times (\mathbf{n} \sqcup \mathbf{k}).$$

These two structures make $\mathbb{I}$ into a bipermutative category in the sense of [5, Def. 3.6].

$\mathbb{I}$ has two natural subcategories. Let $\mathbb{P}$ be the totally disconnected subcategory containing all objects and all isomorphisms $\sigma : \mathbf{n} \to \mathbf{n}$ but no other morphisms, and let $\mathbb{N}$ denote the connected subcategory containing all objects, their identities and only the canonical inclusions $j : \mathbf{n} \to \mathbf{m}$. While $\mathbb{P}$ is a bipermutative subcategory, $\mathbb{N}$ does not inherit any monoidal structure from $\mathbb{I}$.

2.2. Definitions of commutative $\mathbb{I}$-monoids and $\mathbb{I}$-rigs. An $\mathbb{I}$-space is a functor $X : \mathbb{I} \to \mathrm{Top}$. Every morphism in $\mathbb{I}$ can be factored as a composition of a canonical inclusion $j : \mathbf{n} \hookrightarrow \mathbf{m}$ and a permutation $\sigma : \mathbf{m} \to \mathbf{m}$. Therefore an $\mathbb{I}$-space $X : \mathbb{I} \to \mathrm{Top}$ determines a sequence of spaces $X(\mathbf{n})$ together with an induced action of the symmetric group Σ_n for $n \geq 0$, and structural maps $j_n : X(\mathbf{n}) \to X(\mathbf{n+1})$ that are equivariant in the sense that $j_n(\sigma \cdot x) = \sigma \cdot j_n(x)$ for every $\sigma \in \Sigma_n$ and $x \in X(\mathbf{n})$. On the right hand side we see σ as element in Σ_{n+1} via the canonical inclusion $\Sigma_n \hookrightarrow \Sigma_{n+1}$. Vice versa, given such a sequence of Σ_n-spaces $X(\mathbf{n})$ and compatible structure maps j_n, they give rise to an $\mathbb{I}$-space if and only if for any two elements $\sigma, \sigma' \in \Sigma_m$ with identical restrictions to $\mathbf{n}$ we have $\sigma(x) = \sigma'(x)$ for all $x \in j(X(\mathbf{n}))$. We note that this condition is not satisfied by the sequence $X(\mathbf{n}) = \Sigma_n$

with the left or right multiplication action, but *is* satisfied by the sequence $X(\mathbf{n}) = \mathbf{n}$ with the natural permutation action.

We say that an $\mathbb{I}$-space is an $\mathbb{I}$-monoid if it comes equipped with a natural transformation

$$\mu_{\mathbf{m},\mathbf{n}} : X(\mathbf{m}) \times X(\mathbf{n}) \to X(\mathbf{m} \sqcup \mathbf{n})$$

of functors defined on $\mathbb{I} \times \mathbb{I}$ and a natural transformation

$$\eta_{\mathbf{n}} \colon * \to X(\mathbf{n})$$

from the constant $\mathbb{I}$-space $*(\mathbf{n}) = *$ to X satisfying associativity and unit axioms for $* \in X(\mathbf{0})$. We say that X is a commutative $\mathbb{I}$-monoid if μ is commutative, meaning that the diagram

$$\begin{array}{ccc} X(\mathbf{m}) \times X(\mathbf{n}) & \xrightarrow{\mu_{\mathbf{m},\mathbf{n}}} & X(\mathbf{m} \sqcup \mathbf{n}) \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \tau_{\mathbf{m},\mathbf{n}}} \\ X(\mathbf{n}) \times X(\mathbf{m}) & \xrightarrow{\mu_{\mathbf{n},\mathbf{m}}} & X(\mathbf{n} \sqcup \mathbf{m}) \end{array}$$

commutes, where $\tau(x, y) = (y, x)$.

An $\mathbb{I}$-rig is a commutative $\mathbb{I}$-monoid equipped with natural transformations

$$\pi_{\mathbf{m},\mathbf{n}} : X(\mathbf{m}) \times X(\mathbf{n}) \to X(\mathbf{m} \times \mathbf{n})$$

of functors defined on $\mathbb{I} \times \mathbb{I}$ satisfying associativity and unit axioms for $* \in X(\mathbf{1})$ as well as left distributivity, i.e. that the diagram

$$\begin{array}{ccc} (X(\mathbf{m}) \times X(\mathbf{n})) \times X(\mathbf{k}) & \xrightarrow{\pi_{\mathbf{m}\sqcup\mathbf{n},\mathbf{k}} \circ (\mu_{\mathbf{m},\mathbf{n}} \times 1)} & X((\mathbf{m} \sqcup \mathbf{n}) \times \mathbf{k}) \\ {\scriptstyle (1\times\tau\times 1)\circ(1\times 1\times\triangle)}\downarrow & & \uparrow{\scriptstyle \delta^{l}_{\mathbf{m},\mathbf{n},\mathbf{k}}} \\ X(\mathbf{m}) \times X(\mathbf{k}) \times X(\mathbf{n}) \times X(\mathbf{k}) & \xrightarrow{\mu_{\mathbf{m}\times\mathbf{k},\mathbf{n}\times\mathbf{k}} \circ (\pi_{\mathbf{m},\mathbf{k}} \times \pi_{\mathbf{n},\mathbf{k}})} & X(\mathbf{m} \times \mathbf{k} \sqcup \mathbf{n} \times \mathbf{k}) \end{array}$$

commutes, and right distributivity, which is given by an analogous commutative diagram. Here $\triangle$ is the diagonal map. A commutative $\mathbb{I}$-rig is an $\mathbb{I}$-rig in which π is commutative in the sense that the diagram

$$\begin{array}{ccc} X(\mathbf{m}) \times X(\mathbf{n}) & \xrightarrow{\pi_{\mathbf{m},\mathbf{n}}} & X(\mathbf{m} \times \mathbf{n}) \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \tau^{\times}_{\mathbf{m},\mathbf{n}}} \\ X(\mathbf{n}) \times X(\mathbf{m}) & \xrightarrow{\pi_{\mathbf{n},\mathbf{m}}} & X(\mathbf{n} \times \mathbf{m}) \end{array}$$

commutes.

2.3. Associated (bi)permutative translation categories. We will use the following notation for translation categories. If $Y \colon \mathcal{C} \to \mathrm{Top}$ is a functor from a category $\mathcal{C}$ to the category of topological spaces, we let $\mathcal{C} \ltimes Y$ denote the translation category on Y. The translation category, also known as the Grothendieck construction, is a topological category whose objects are pairs (c, x) consisting of an object c of $\mathcal{C}$ and a point $x \in Y(c)$. A morphism in $\mathcal{C} \ltimes Y$ from (c, x) to (c', x') is a morphism $\alpha \colon c \to c'$ in $\mathcal{C}$ satisfying the equation $Y(\alpha)(x) = x'$. For example, if $\mathcal{C} = G$ is a group, thought of as a one object category, then the translation category $G \ltimes Y$ is the action groupoid for the G-space Y and its classifying space is the homotopy orbit space $B(G \ltimes Y) = EG \times_G Y$. In general, the classifying space $B(\mathcal{C} \ltimes Y)$ is homeomorphic to the homotopy colimit $\operatorname{hocolim}_{\mathcal{C}} Y$ of Y over $\mathcal{C}$ defined using the bar construction.

Suppose now that X is a commutative $\mathbb{I}$-monoid. Then the translation category $\mathbb{I} \ltimes X$ is a permutative category, as we now explain. The monoidal structure $\oplus$ is defined on objects $(\mathbf{m}, x)$ and $(\mathbf{n}, y)$ by

$$(\mathbf{m}, x) \oplus (\mathbf{n}, y) = (\mathbf{m} \sqcup \mathbf{n}, \mu_{\mathbf{m},\mathbf{n}}(x, y)),$$

and on morphisms $\alpha\colon (\mathbf{m}, x) \to (\mathbf{m}', x')$ and $\beta\colon (\mathbf{n}, y) \to (\mathbf{n}', y')$ by letting

$$\alpha \oplus \beta\colon (\mathbf{m}, x) \oplus (\mathbf{n}, y) \to (\mathbf{m}', x') \oplus (\mathbf{n}', y')$$

be determined by the morphism

$$\alpha \sqcup \beta\colon \mathbf{m} \sqcup \mathbf{n} \to \mathbf{m}' \sqcup \mathbf{n}'$$

in the category $\mathbb{I}$. Notice that $X(\alpha \sqcup \beta)(\mu_{\mathbf{m},\mathbf{n}}(x, y)) = \mu_{\mathbf{m}',\mathbf{n}'}(x', y')$ by the naturallity of μ, so that this is well-defined. The associativity and unit conditions for X imply that $\mathbb{I} \ltimes X$ is a strict monoidal category with strict unit object $(\mathbf{0}, *)$ determined by the unit η of the $\mathbb{I}$-monoid X. The commutativity of X implies that $\mathbb{I} \ltimes X$ is a permutative category, see for example [5, Def 3.1].

Suppose now that X is a commutative $\mathbb{I}$-rig. Then by the same reasoning as above, π induces a permutative category structure on $\mathbb{I} \ltimes X$ with product $\otimes$ induced by π and strict unit object $(\mathbf{1}, *)$. The distributivity axioms for X translate to distributivity axioms for bipermutative categories, [5, Def. 3.6].

Furthermore, a natural transformation T between two $\mathbb{I}$-spaces X and Y induces a functor $\mathbb{I} \ltimes X \to \mathbb{I} \ltimes Y$. If X and Y are commutative $\mathbb{I}$-monoids ($\mathbb{I}$-rigs) and the natural transformation commutes with the μ (and π) in the sense that $T \circ \mu_{\mathbf{m},\mathbf{n}} = \mu_{\mathbf{m},\mathbf{n}} \circ T \times T$, then the induced functor of translation categories is a functor of (bi)permutative categories.

We note that the permutative or bipermutative structure on $\mathbb{I} \ltimes X$ restricts to the subcategory $\mathbb{P} \ltimes X$.

2.4. Construction of two infinite loop spaces. Let X be a commutative $\mathbb{I}$-monoid. As explained in [11], the classifying space of a permutative category is an E_∞-space structured by an action of the Barratt-Eccles operad. Therefore, the homotopy colimit $\operatorname{hocolim}_{\mathbb{I}} X = B(\mathbb{I} \ltimes X)$ is an E_∞-space. Without further assumptions on X, we do not know if this E_∞-space is group-like, meaning that the monoid $\pi_0(\operatorname{hocolim}_{\mathbb{I}} X)$ is a group. However, we can always form the group completion $\Omega B(\operatorname{hocolim}_{\mathbb{I}} X)$ to get the associated infinite loop space.

Furthermore, a bipermutative category gives rise to an E_∞-ring space, compare [5, Cor. 3.9]. We can thus summarize with the following theorem.

Theorem 2.1. *Suppose that $X : \mathbb{I} \to \mathrm{Top}$ is a commutative $\mathbb{I}$-monoid. Then the homotopy colimit $\operatorname{hocolim}_{\mathbb{I}} X$ is an E_∞-space. In particular, if each $X(\mathbf{n})$ is path-connected, then so is $\operatorname{hocolim}_{\mathbb{I}} X$ and it is an infinite loop space. If X is furthermore a commutative $\mathbb{I}$-rig then $\operatorname{hocolim}_{\mathbb{I}} X$ is an E_∞-ring space.*

Remarks. In [18], Schlichtkrull defined a different infinite loop space associated to X, using the language of Γ-spaces. Schlichtkrull's construction is the same as May's construction [12] of a Γ-space applied to the permutative category $\mathbb{I} \ltimes X$. By the uniqueness result of [12], the infinite loop space $\Omega B(\operatorname{hocolim}_{\mathbb{I}} X)$ is equivalent to that defined by Schlichtkrull.

The infinite loop space $\operatorname{hocolim}_{\mathbb{I}} X$ provided in the previous theorem is also equivalent to an infinite loop space structure on the homotopy colimit of X over a larger category of linear isometries. This approach and its relation to the linear isometries operad is studied in [9].

We now give a different construction of an infinite loop space associated to X. To start note the decomposition of categories

$$\mathbb{P} \ltimes X = \bigsqcup_{n \geq 0} \Sigma_n \ltimes X(\mathbf{n}),$$

where Σ_n is seen as a category with one object. Thus $\mathbb{P} \ltimes X$ is a topological category with classifying space

$$M := \operatorname{hocolim}_{\mathbb{P}} X = B(\mathbb{P} \ltimes X) \simeq \bigsqcup_{n \geq 0} E\Sigma_n \times_{\Sigma_n} X(\mathbf{n}).$$

As $\mathbb{P} \ltimes X$ is a permutative category, $M = B(\mathbb{P} \ltimes X)$ is an E_∞-space and thus its group completion, ΩBM, is an infinite loop space. The reduction maps $X(\mathbf{n}) \to *$ define a map of permutative categories $\mathbb{P} \ltimes X \to \mathbb{P} \ltimes *$ and hence a map of infinite loop spaces

$$\rho^X : \Omega B(\operatorname{hocolim}_{\mathbb{P}} X) \longrightarrow \Omega B(\operatorname{hocolim}_{\mathbb{P}} *).$$

If X is a commutative $\mathbb{I}$-rig then $\mathbb{P} \ltimes X \to \mathbb{P} \ltimes *$ is a map of bipermutative categories. We thus have proved

Theorem 2.2. *For any commutative $\mathbb{I}$-monoid X the homotopy fiber,* $\operatorname{hofib} \rho^X$, *of ρ^X is an infinite loop space. If furthermore X is a commutative $\mathbb{I}$-rig, then* $\operatorname{hofib} \rho^X$ *is an E_∞-ring space.*

2.5. The main example. For any group G, conjugation by G or action by any other automorphism of G induces a well-defined action on $B_n(q, G) = \operatorname{Hom}(F_n/\Gamma_n^q, G)$ by post-composition. The action is also compatible with the simplicial face and degeneracy maps in the bar construction and hence induces an action on $B(q, G)$.

For every $q \geq 2$ we define an $\mathbb{I}$-space $B(q, U(-))$ by setting $\mathbf{n} \mapsto B(q, U(n))$ with morphisms induced by the natural inclusions and the action of Σ_n on $B(q, U(n))$ given by conjugation through permutation matrices. Being induced by the natural action of Σ_n on $\mathbf{n}$, it can be checked that this compatible sequence defines indeed an $\mathbb{I}$-space.

We give $B(q, U(-))$ the structure of an $\mathbb{I}$-monoid by defining the unit map $\eta_{\mathbf{n}} \colon * \to B(q, U(n))$ to be the inclusion of the base-point and defining the monoid structure map

$$\mu_{\mathbf{n},\mathbf{m}} : B(q, U(n)) \times B(q, U(m)) \to B(q, U(n+m))$$

to be induced by the block sum of matrices. To see that $\mu_{\mathbf{n},\mathbf{m}}$ is well-defined note that block sum defines a group homomorphism $U(n) \times U(m) \to U(n+m)$. When taking elements of the symmetric groups to permutation matrices, the disjoint union of sets corresponds to block sum of matrices. Thus μ defines a natural transformation of functors defined on $\mathbb{I} \times \mathbb{I}$. One checks compatibility with τ and hence $B(q, U(-))$ is a commutative $\mathbb{I}$-monoid.

Next we note that tensor product of matrices induces a well-defined map

$$\pi_{\mathbf{n},\mathbf{m}} : B(q, U(n)) \times B(q, U(m)) \longrightarrow B(q, U(nm)).$$

To see this note that tensor product commutes with matrix multiplication and hence induces a homomorphism $U(n) \times U(m) \to U(nm)$. The map is equivariant for the symmetric group actions because the permutation matrix associated to the product of two permutations is the same as the tensor product of the corresponding permutation matrices. π is compatible with τ and induces a second permutative structure on $B(q, U(-))$. Distributivity of block sum and tensor product of matrices induces distributivity maps for the two permutative structures in $\mathbb{I} \ltimes B(q, U(-))$. We have shown

Theorem 2.3. *$B(q, U(-))$ is a commutative $\mathbb{I}$-rig.*

As a consequence, we may apply Theorem 2.1 and Theorem 2.2 to get a pair of E_∞-ring spaces. In the next section, we will show that these two E_∞-ring spaces are equivalent.

3. Identifying and comparing the infinite loop spaces

Let X be a commutative $\mathbb{I}$-monoid. We will first identify $\operatorname{hofib} \rho^X$ under certain assumptions and then show it is homotopy equivalent as an infinite loop space to $\operatorname{hocolim}_{\mathbb{I}} X$.

Consider the space

$$X_\infty := \operatorname{hocolim}_n X(\mathbf{n}) = \operatorname{hocolim}_{\mathbb{N}} X.$$

Note that $X_\infty \simeq \operatorname{colim}_n X(\mathbf{n})$ if the structural maps $j_n : X(\mathbf{n}) \to X(\mathbf{n+1})$ are cofibrations. In our applications this will always be the case. Let X_∞^+ denote Quillen's plus construction applied with respect to the maximal perfect subgroup of $\pi_1(X_\infty)$. Also recall that a space Z is abelian if $\pi_1(Z)$ is abelian and acts trivially on homotopy groups $\pi_*(Z)$. It is well-known that H-spaces are abelian.

Theorem 3.1. *Suppose that $X : \mathbb{I} \to \mathrm{Top}$ is a commutative $\mathbb{I}$-monoid. Assume that $\pi_0(X(\mathbf{n})) = 0$ and the action of Σ_n on $X(\mathbf{n})$ is homologically trivial for every $n \geq 0$. Assume further that the commutator subgroup of $\pi_1(X_\infty)$ is perfect and that X_∞^+ is abelian. Then $\operatorname{hofib} \rho^X \simeq X_\infty^+$, and in particular X_∞^+ is an infinite loop space.*

Proof. Let $M = \operatorname{hocolim}_{\mathbb{P}} X = B(\mathbb{P} \ltimes X)$ and m be the point corresponding to the base point in $X(\mathbf{1})$. Then

$$\operatorname{Tel}(M \overset{\cdot m}{\to} M \overset{\cdot m}{\to} M \overset{\cdot m}{\to} \cdots) \simeq \mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty).$$

As $\mathbb{P} \ltimes X$ is a symmetric monoidal category, its classifying space M is a homotopy commutative topological monoid. Thus the group completion theorem [13], [15] can be applied to conclude that there is a map

$$f : \mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty) \to \Omega BM$$

which induces an isomorphism on homology with all systems of local coefficients on ΩBM. Furthermore, the fundamental group of $E\Sigma_\infty \times_{\Sigma_\infty} X_\infty$ has a perfect commutator subgroup by [15], and f extends to a homology equivalence between abelian spaces

$$f^+ : \mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty)^+ \to \Omega BM,$$

which is thus a homotopy equivalence. This shows in particular that $\mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty)^+$ is an infinite loop space as ΩBM is the group completion of an E_∞-space.

Consider now the fibration sequence

$$X_\infty \to E\Sigma_\infty \times_{\Sigma_\infty} X_\infty \xrightarrow{p} B\Sigma_\infty. \tag{1}$$

and the associated map of plus constructions

$$p^+ : \mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty)^+ \to \mathbb{Z} \times B\Sigma_\infty^+.$$

Since f^+ is a homotopy equivalence and $\Omega B(\mathrm{hocolim}_{\mathbb{P}} *) \simeq \mathbb{Z} \times B\Sigma_\infty^+$, we can identify the homotopy fiber of p^+ with $\mathrm{hofib}\,\rho^X$. By assumption Σ_n acts trivially on $H_*(X(\mathbf{n}))$ and hence the action of Σ_∞ on X_∞ is homologically trivial. Also, we are assuming that X_∞^+ is abelian and in particular nilpotent. Under these conditions the fibre sequence (1) remains a fibre sequence after passing to plus constructions, see [4, Theorem 1.1]. Thus we have a homotopy fibration

$$X_\infty^+ \to \mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty)^+ \to \mathbb{Z} \times B\Sigma_\infty^+.$$

This shows that the homotopy fibre of p^+ is X_∞^+ and so $X_\infty^+ \simeq \mathrm{hofib}\,\rho^X$. □

Remark 3.2. For any commutative $\mathbb{I}$-monoid X the multiplication on $M_X := \coprod_{n\geq 0} X(\mathbf{n})$ is commutative up to the action of the shuffle maps $\tau_{\mathbf{m},\mathbf{n}}$. These are induced by the action of the symmetric group. So, assuming that these actions are trivial in homology, it follows that the Pontrjagin product is commutative on the level of homology. In particular $\pi_0(M_X)$ is in the centre of the Pontrjagin ring $H_*(M_X)$. Thus by the group completion theorem [13], the map

$$\mathbb{Z} \times X_\infty \longrightarrow \Omega B(M_X)$$

is a homology isomorphism. In recent work [7], Gritschacher has shown that without any further assumption, the commutator subgroup of $\pi_1(X_\infty)$ is always perfect and that X_∞^+ is always an abelian space. In other words, the assumptions in Theorem 3.1 on $\pi_1(X_\infty)$ and X_∞^+ are actually consequences. [1]

In contrast, the condition that the symmetric groups act homologically trivial is necessary. To see this consider the commutative $\mathbb{I}$-space X with $X(\mathbf{n}) := Z^n$ for some pointed connected space Z. Then, by the parametrized version of the Barratt-Priddy-Quillen theorem (see for example [11], [19]),

$$\Omega B(\mathrm{hocolim}_{\mathbb{P}} X) \simeq Q(Z_+)$$

and thus $\mathrm{hofib}\,\rho^X \simeq \mathrm{hofib}\,p^+ \simeq Q(Z)$ while $X_\infty \simeq \mathrm{hocolim}_n Z^n$. Here $Q = \Omega^\infty\Sigma^\infty$ and Z_+ denotes the space Z with an additional base point.

Remark 3.3. We will need a slight extension of Theorem 3.1 in our applications where the condition on $\pi_0(X(\mathbf{n}))$ is relaxed to $\pi_0(X(\mathbf{n})) = \pi_0(X_\infty) = A$ being an abelian group with multiplication compatible with the Pontrjagin product and in the centre of the Pontrjagin ring. More precisely, Theorem 3.1 also holds if instead of $\pi_0(X(\mathbf{n})) = 0$ for all n, we assume that $\pi_0(M)$ is isomorphic to $A \times \mathbb{N}$ as a monoid and that it is in the centre of $H_*(M)$. The

[1] As we do not know whether M_X is homotopy commutative, the results of [15] cannot be applied directly to conclude that the induced map $\mathbb{Z} \times X_\infty^+ \to \Omega B(M_X)$ is a homotopy equivalence.

proof above goes through verbatim. (Indeed, it is enough to have the same assumptions on $\pi_0(M)$ as needed for the group completion theorem [13], [15], that is that $H_*(M)[\pi_0^{-1}]$ can be constructed by right fractions.)

We now turn to the question of comparing the infinite loop spaces $\operatorname{hofib}\rho^X$ and $\operatorname{hocolim}_{\mathbb{I}} X$. Suppose that X is a commutative $\mathbb{I}$-monoid with $X(\mathbf{n})$ path-connected. Consider the following commutative diagram of strict functors between permutative categories

$$\begin{array}{ccc} \mathbb{P} \ltimes X & \xrightarrow{\alpha_X} & \mathbb{I} \ltimes X \\ {\scriptstyle \rho^X}\downarrow & & \downarrow{\scriptstyle \rho_1^X} \\ \mathbb{P} \ltimes * & \xrightarrow{\alpha_*} & \mathbb{I} \ltimes *. \end{array}$$

The horizontal maps are induced by the inclusion $\mathbb{P} \to \mathbb{I}$. In the above diagram $*$ is the trivial commutative $\mathbb{I}$-monoid and the vertical maps ρ^X and ρ_1^X are induced by the projection maps to a point. Passing to the level of classifying spaces and applying group completion we obtain a commutative diagram of infinite loop spaces

$$\begin{array}{ccc} \Omega B(\operatorname{hocolim}_{\mathbb{P}} X) & \xrightarrow{\alpha_X} & \operatorname{hocolim}_{\mathbb{I}} X \\ {\scriptstyle \rho^X}\downarrow & & \downarrow{\scriptstyle \rho_1^X} \\ \Omega B(\operatorname{hocolim}_{\mathbb{P}} *) & \xrightarrow{\alpha_*} & \operatorname{hocolim}_{\mathbb{I}} * \simeq *. \end{array}$$

Note that the empty set is an initial object for $\mathbb{I}$ and hence $\operatorname{hocolim}_{\mathbb{I}} * = B\mathbb{I} \simeq *$. We are assuming that each $X(\mathbf{n})$ is path-connected and thus $\operatorname{hocolim}_{\mathbb{I}} X$ is group-like since it is also path-connected. The above diagram induces an infinite loop map between the homotopy fibers of the maps ρ^X and ρ_1^X. By definition the homotopy fiber on the left is the space $\operatorname{hofib}\rho^X$. Also, since $\operatorname{hocolim}_{\mathbb{I}} *$ is contractible, the homotopy fiber on the right can be identified with $\operatorname{hocolim}_{\mathbb{I}} X$. This shows that we have a map of infinite loop spaces

$$\operatorname{hofib}\rho^X \xrightarrow{g} \operatorname{hocolim}_{\mathbb{I}} X.$$

If X is a commutative $\mathbb{I}$-rig then g is a map of E_∞-ring spaces. Note that ρ^X has a canonical splitting of (bi-)permutative categories induced by the unit of the $\mathbb{I}$-monoid X. Thus it follows from the following theorem that g is a homotopy equivalence whenever the stated conditions on X are satisfied.

Theorem 3.4. *Let X be a commutative $\mathbb{I}$-monoid such that all maps $j : X(\mathbf{n}) \to X(\mathbf{m})$ induced by inclusions $j : [\mathbf{n}] \to [\mathbf{m}]$ are cofibrations. Furthermore, assume that $\mu_{\mathbf{n},\mathbf{m}}(x, y)$ is in the image of a strict inclusion if and only if x or y is. Then*

$$\alpha_X \times \rho^X : \Omega B(\operatorname{hocolim}_{\mathbb{P}} X) \longrightarrow \Omega B(\operatorname{hocolim}_{\mathbb{I}} X) \times \Omega B(\operatorname{hocolim}_{\mathbb{P}} *)$$

is a homotopy equivalence of infinite loop spaces.

If X is a commutative $\mathbb{I}$-rig and furthermore $\pi_{\mathbf{n},\mathbf{m}}(x, y)$ is in the image of a strict inclusion if and only if x or y is, then $\alpha_X \times \rho^X$ is a homotopy equivalence of E_∞-ring spaces.

A version of the theorem was proved by Fiedorowicz and Ogle [6] in the setting of simplicial sets. See also Gritschacher [7, Section 4]. For convenience of the reader we sketch a streamlined argument following [7].

Proof. Given $x \in X(\mathbf{n})$ we can write it as $x = j_x(\bar{x})$ where $\bar{x} \in X(\bar{\mathbf{n}})$, $j_x : [\bar{\mathbf{n}}] \to [\mathbf{n}]$ is an order preserving inclusion, and $\bar{n}$ is minimal. We call $\bar{x}$ reduced. By our assumption on the structure maps, $\bar{x}$ and j_x are uniquely determined. Denote by $\bar{X}(\mathbf{n})$ the set of reduced elements in $X(\mathbf{n})$. The assignment $\mathbf{n} \mapsto \bar{X}(\mathbf{n})$ defines a $\mathbb{P}$-diagram.

For X discrete, the assignment $(\mathbf{n}, x) \mapsto (\bar{\mathbf{n}}, \bar{x})$ on objects extends to define a (bi-)permutative functor

$$R_X : \mathbb{I} \ltimes X \longrightarrow \mathbb{P} \ltimes \bar{X}.$$

It has a right inverse given by the inclusion $\iota_X : \mathbb{P} \ltimes \bar{X} \to \mathbb{I} \ltimes X$. Furthermore, the maps j_x define a natural transformation from $R_X \circ \iota_X$ to the identity on $\mathbb{I} \ltimes X$.

The inclusions $\mathbb{P} \ltimes \bar{X} \to \mathbb{P} \ltimes X$ and $\mathbb{P} \to \mathbb{P} \ltimes X$ combine via the monoidal product functor to a functor

$$(\mathbb{P} \ltimes \bar{X}) \times \mathbb{P} \longrightarrow \mathbb{P} \ltimes X$$

that maps the object $((\bar{\mathbf{n}}, \bar{x}), \mathbf{n})$ to $(\bar{\mathbf{n}} + \mathbf{n}, j(\bar{x}))$, where j is the canonical inclusion $[\bar{\mathbf{n}}] \hookrightarrow [\bar{\mathbf{n}} + \mathbf{n}]$. An analysis of the effect of permutations on reduced points shows that the functor is bijective on automorphism groups of objects. As both source and target categories are groupoids and every isomorphism class of the target category has a representative in the image, this is an equivalence of categories.

If $f : X \to Y$ is a morphism of $\mathbb{I}$-monoids we can define a morphism of $\mathbb{P}$-diagrams $\bar{f} : \bar{X} \to \bar{Y}$ by setting $\bar{f} := R_Y \circ f \circ \iota_X$. By the uniqueness of the reductions this is functorial. Applying this to boundary and face maps allows us to extend the above arguments to $\mathbb{I}$-diagrams in simplicial sets. More precisely, for any commutative $\mathbb{I}$-diagram X in simplicial sets we have a sequence of functors of simplicial categories

$$\mathbb{P} \longrightarrow \mathbb{P} \ltimes X \longrightarrow \mathbb{I} \ltimes X \xrightarrow{\simeq} \mathbb{P} \ltimes \bar{X}$$

which on classifying spaces give a homotopy fiber sequence at each simplicial level (with constant fibers) and hence on total spaces. On taking group completions this yields the homotopy fiber sequence

$$\Omega B(B\mathbb{P}) \longrightarrow \Omega B(B(\mathbb{P} \ltimes X)) \longrightarrow \Omega B(B(\mathbb{I} \ltimes X)).$$

This is a sequence of maps of infinite loop spaces (E_∞-ring spaces) as they are induced by functors of (bi-)permutative categories.

Finally, any $\mathbb{I}$-diagram X in topological spaces gives rise to an $\mathbb{I}$-diagram in simplicial sets by replacing every space by its singular simplicial set. As a space is homotopy equivalent to the realisation of its singular simplicial set, the theorem follows. □

Example 3.5. Consider the commutative $\mathbb{I}$-space X with $X(\mathbf{n}) := Z^n$, where Z is a well-pointed connected space. Note that in this case Σ_n does not act trivially on $H_*(Z^n)$ and hence Theorems 3.1 does not apply. As before, by the parametrized version of the Barratt-Priddy-Quillen theorem,

$$\Omega B(\mathrm{hocolim}_{\mathbb{P}} X) \simeq Q(Z_+) \simeq Q(\mathbb{S}^0) \times Q(Z)$$

and hence hofib $\rho^X \simeq Q(Z)$. Thus, by the above theorem, we also have $\operatorname{hocolim}_{\mathbb{I}} X \simeq Q(Z)$, which is in agreement with a result of Schlichtkrull [17].

4. Constructing filtrations by infinite loop spaces

In this section we use the results obtained in the previous sections to produce filtrations of classical infinite loop spaces by sequences of infinite loop spaces arising from the descending central series of the free groups.

Theorem 4.1. *The spaces $B(q,U)$, $B(q,SU)$, $B(q,SO)$ $B(q,O)$ and $B(q,Sp)$ provide a filtration by E_∞-ring spaces of the classical E_∞-ring spaces BU, BSU, BSO, BO and BSp respectively.*

Proof. By our main example in Section 2, each $\mathbf{n} \mapsto B(q,U(n))$ for $q \geq 2$ is a commutative $\mathbb{I}$-rig, and hence the hofib $\rho^{B(q,U(-))}$ provide a filtration by E_∞-ring spaces of BU. The same argument can be applied to prove the analogous result for the other Lie groups. It remains to identify their homotopy type. For this we check the conditions of Theorem 3.1 (and Theorem 3.4).

The conjugation action of Σ_n on $B(q,U(n))$ is homologically trivial because this action factors through the conjugation action of $U(n)$ which is trivial, up to homotopy, since $U(n)$ is path-connected. Next, we argue that the space $B(q,U)$ is abelian. To see this note that in the underlying simplicial space defining $B(q,U(n))$ we have $B_0(q,U(n)) = *$ and $B_1(q,U(n)) = U(n)$ which is path-connected. It follows that $B(q,U(n))$ is simply connected, see for example [10, Theorem 11.12]. Thus $B(q,U) = \operatorname{colim}_{n\to\infty} B(q,U(n))$ is also simply connected and hence trivially abelian. Using Theorem 3.1 we conclude that hofib $\rho^{B(q,U(-))} \simeq B(q,U)$ is an infinite loop space for every $q \geq 2$.

The very same arguments can be used to prove analogous statements for the commutative $\mathbb{I}$-rig $\mathbf{n} \mapsto B(q,SU(n))$ and $\mathbf{n} \mapsto B(q,Sp(n))$ for any $q \geq 2$.

In case of the commutative $\mathbb{I}$-rig $\mathbf{n} \mapsto B(q,SO(n))$ we note that Σ_n is not a subgroup of $SO(n)$. Nevertheless, the alternating group A_n is contained in $SO(n)$ and by the same argument as above acts therefore trivially on the homology of $B(q,SO(n))$. Furthermore, any odd permutation is represented by a matrix with determinant equal to -1 and can be path-connected to the diagonal matrix $-I$ with constant entry -1. As $-I$ is in the center of $O(n)$ it acts trivially by conjugation on $B(q,SO(n))$ and hence also on its homology. But then so does any odd permutation.

Finally, in case of the commutative $\mathbb{I}$-rig $\mathbf{n} \mapsto B(q,O(n))$, the fundamental group of $B(q,O)$ is $\mathbb{Z}/2\mathbb{Z}$ and not trivial. To see that $B(q,O)$ is still abelian note that it is an H-space with multiplication induced by block sum. □

As remarked in [1], the natural map $\Omega B(q,G) \to \Omega BG$ admits a splitting up to homotopy. It is given by a factorization of the usual homotopy equivalence $G \to \Omega BG$. Indeed we have that $\Sigma G = F_1 B(q,G) = F_1 BG$, where F_1 denotes the first layer in the usual filtration of the geometric realization of these simplicial spaces. Hence, the adjoint of $\Sigma G \to BG$ factors through $\Omega B(q,G)$. Note that this splitting does not in general admit a delooping; see [1, Section 6] for a counter-example. Nevertheless, we have the following theorem. Here $E(q,G)$

denotes the pull-back of the universal G-bundle EG over BG. It is homotopy equivalent to the homotopy fiber of the inclusion $B(q,G)\to BG$.

Theorem 4.2. *For all $q\geq 2$, and $G=U,SU,SO,O$ and Sp, we have a splitting of E_∞-ring spaces*

$$B(q,G)\simeq BG\times E(q,G).$$

Proof. Let G_n denote one of the groups $U(n)$, $SU(n)$, $SO(n)$, or $Sp(n)$ (but not $O(n)$) so that $G_\infty=\operatorname{colim}_n G_n$ denotes the group U, SU, SO or Sp respectively. For each fixed $q\geq 2$, the assignment $\mathbf{n}\mapsto\Omega B(q,G_n)$ defines a commutative $\mathbb{I}$-rig with μ given by block sum and π given by tensor product of matrices. In the same way the assignment $\mathbf{n}\mapsto\Omega BG_n$ also defines a commutative $\mathbb{I}$-rig and the inclusion map $\Omega B(q,G_n)\to\Omega BG_n$ defines a morphism of commutative $\mathbb{I}$-rigs. We claim that the commutative $\mathbb{I}$-rigs G_-, $\Omega B(q,G_-)$ and ΩBG_- satisfy the hypotheses of Theorem 3.1. Indeed, for every $n\geq 0$, $G_n\simeq\Omega BG_n$ is path-connected and, as $\pi_0(\Omega B(q,G_n))\cong\pi_1(B(q,G_n))$ is trivial, so is $\Omega B(q,G_n)$. Also, the action of Σ_n is homologically trivial because this action factors through the conjugation action of G_n which is trivial, up to homotopy, since G_n is path-connected. In addition, the commutator group of $\pi_1(\Omega B(q,G_n))\cong\pi_2(B(q,G_n))$ is trivial as this group is abelian. Finally, it is easy to see that $\Omega B(q,G_\infty)$ is abelian since it is an H-space with multiplication once again induced by block sum. By Theorem 3.1 we obtain maps of infinite loop spaces

$$G_\infty\to\Omega B(q,G_\infty)\to\Omega BG_\infty$$

whose composition is a homotopy equivalence. On the other hand, for every $q\geq 2$ we have a homotopy fibration sequence $E(q,G_\infty)\to B(q,G_\infty)\to BG_\infty$ as $EG_\infty\simeq *$. So the $E(q,G_\infty)$ are also E_∞-ring spaces and we have proved the splitting theorem.

When $G_n=O(n)$ we need to adjust the above argument. In order to identify the infinite loop space associated to the three commutative $\mathbb{I}$-rigs we simply note that these $\mathbb{I}$-rigs satisfy the extension of Theorem 3.1 as described in Remark 3.3. ☐

We have concentrated so far on compact groups such as $O(n)$ and $U(n)$, although the methods clearly extend to other linear groups. Using some results by Pettet-Souto [14] and Bergeron [3] we can prove the following theorem.

Theorem 4.3. *Suppose that G is the group of complex or real points in a reductive linear algebraic group (defined over $\mathbb{R}$ in the the real case). Let $K\subset G$ be a maximal compact subgroup. Then the inclusion map $i:B(q,K)\to B(q,G)$ is a homotopy equivalence for every $q\geq 2$.*

Proof. By [3, Theorem I] it follows that the inclusion map $i_n:B_n(q,K)\to B_n(q,G)$ is a homotopy equivalence for all $q\geq 2$ and all $n\geq 0$. Thus the inclusion map induces a simplicial map $i_*:B_*(q,K)\to B_*(q,G)$ that is a level-wise homotopy equivalence. Since G is assumed to be the group of complex or real points in a reductive linear algebraic group (defined over $\mathbb{R}$ in the the real case), then we can identify G with a Zariski closed subgroup of $SL_N(\mathbb{C})$ for some $N\geq 0$. Also, for every $n\geq 0$ we can see the space $B_n(q,G)$ as an algebraic variety since it is defined in terms of iterated commutators of elements in G and such equations can be defined in terms of polynomial functions. Moreover, the subspace $S^1_n(q,G)\subset B_n(q,G)$ consisting of all n-tuples in $B_n(q,G)$ for which at least one of the coordinates is equal to

1_G is an algebraic subvariety of $B_n(q,G)$. By the semi-algebraic triangulation theorem (see [8, Section 1]) it follows that $B_n(q,G)$ has the structure of a CW-complex in such a way that $S^1_n(q,G)$ is a subcomplex. In particular, it follows that the pair $(B_n(q,G), S^1_n(q,G))$ is a strong NDR-pair. This proves that $B_*(q,G)$ is a proper simplicial space. The same is true for $B_*(q,K)$. Using the glueing lemma, for example see [11, Theorem A.4], we obtain the result of the theorem. □

Our tools can also be used to obtain a similar filtration for the infinite loop space defining algebraic K-theory for any discrete ring R. Indeed, suppose that R is a discrete ring with unit and let $q \geq 2$. Consider the commutative $\mathbb{I}$-rig $B(q, GL_-(R))$ defined by $\mathbf{n} \mapsto B(q, GL_n(R))$. As before the morphisms are induced by the natural inclusions and the conjugation action of Σ_n on $B(q, GL_n(R))$. The multiplication map

$$\mu_{\mathbf{n},\mathbf{m}} : B(q, GL_n(R)) \times B(q, GL_m(R)) \to B(q, GL_{n+m}(R))$$

is also given by the block sum and π by tensor product of matrices. Note that Theorem 3.4 applies to give

$$\operatorname{hocolim}_{\mathbb{I}} B(q, GL_-(R)) \simeq \operatorname{hofib} \rho^{B(q,GL_-(R))}.$$

By Theorem 2.1 or 2.2, this space has the structure of an E_∞-ring space. This way we obtain a filtration by E_∞-ring spaces:

$$\operatorname{hofib} \rho^{B(2,GL_-(R))} \subset \cdots \subset \operatorname{hofib} \rho^{B(q,GL_-(R))} \subset \cdots \subset \operatorname{hofib} \rho^{BGL_-(R)}.$$

As is well known, the conjugation action of Σ_n on $BGL_n(R)$ is homologically trivial. It follows from Theorems 3.1 and 3.4 that we have an equivalence of E_∞-ring spaces

$$BGL_\infty(R)^+ \simeq \operatorname{hofib} \rho^{BGL_-(R)} \simeq \operatorname{hocolim}_{\mathbb{I}} BGL_-(R).$$

Thus the above gives a filtration by E_∞-ring spaces of the algebraic K-theory of R. However, unlike the case of $BGL_n(R)$, we do not know whether the conjugation action of Σ_n on $B(q, GL_n(R))$ is homologically trivial, and we expect that the natural map

$$B(q, GL_\infty(R)) \to \operatorname{hocolim}_{\mathbb{I}} B(q, GL_-(R))$$

is not a homology isomorphism.

In a similar way we can obtain a filtration by E_∞-ring spaces of $Q(\mathbb{S}^0)$. For this note that the conjugation action of Σ_n on $B\Sigma_n$ is homologically trivial. Therefore, by the Barratt-Priddy-Quillen theorem, the level zero component of $Q(\mathbb{S}^0)$ is equivalent to the homotopy colimit over $\mathbb{I}$ of the classifying spaces of the symmetric groups:

$$Q_0(\mathbb{S}^0) \simeq (B\Sigma_\infty)^+ \simeq \operatorname{hofib} \rho^{B\Sigma_-} \simeq \operatorname{hocolim}_{\mathbb{I}} B\Sigma_-.$$

Consider the commutative $\mathbb{I}$-monoid $B(q, \Sigma_-)$ defined by $\mathbf{n} \mapsto B(q, \Sigma_n)$. The structural maps are given by conjugation of Σ_n and inclusions in an analogous way as above. Then by Theorem 2.2 we have a filtration by E_∞-ring spaces

$$\operatorname{hofib} \rho^{B(2,\Sigma_-)} \subset \cdots \subset \operatorname{hofib} \rho^{B(q,\Sigma_-)} \subset \cdots \subset \operatorname{hofib} \rho^{B\Sigma_-} \simeq Q_0(\mathbb{S}^0).$$

As in the case of $B(q, GL_n(R))$, the conjugation action of Σ_n on $B(q, \Sigma_n)$ may fail to be homologically trivial (for example this is the case for the conjugation action of Σ_3 on $B(2, \Sigma_3)$, see [1]). The conditions of Theorem 3.4 are satisfied but the homotopy types of the spaces $\operatorname{hocolim}_{\mathbb{I}} B(q,\Sigma_-) \simeq \operatorname{hofib} \rho^{B(q,\Sigma_-)}$ remain to be determined.

Corollary 4.4. *The spaces*

$$\operatorname{hocolim}_{\mathbb{I}} B(q, GL_-(R)) \simeq \operatorname{hofib} \rho^{B(q,GL_-(R))} \quad \text{and} \quad \operatorname{hocolim}_{\mathbb{I}} B(q,\Sigma_-) \simeq \operatorname{hofib} \rho^{B(q,\Sigma_-)}$$

provide filtrations by E_∞-ring spaces of the classical E_∞-ring spaces $BGL_\infty(R)^+$ and $Q_0(\mathbb{S}^0)$.

5. Transitional nilpotence, bundles and K-theory

In this section we extend the notions of transitionally commutative bundles and commutative K-theory as defined in [2] to more general q-nilpotent notions for $q \geq 2$, reflecting the filtration induced by the descending central series of the free groups. We will show that these geometrically defined theories are represented by the E_∞-ring spaces $\mathbb{Z} \times B(q,U)$.

Definition 5.1. For a CW-complex X a principal G-bundle $\pi : E \to X$ is said to have *transitional nilpotency class* at most q if there exists an open cover $\{U_i\}_{i\in I}$ of X such that the bundle $\pi : E \to X$ is trivial over each U_i and for every $x \in X$ the group generated by the collection $\{\rho_{i,j}(x)\}_{i,j}$ is a group of nilpotency class at most q. Here $\rho_{i,j} : U_i \cap U_j \to G$ denotes the transition functions and i, j run through all indices in I for which $x \in U_i \cap U_j$. The minimum of all such numbers q is said to be transitional nilpotency class of $\pi : E \to X$.

The principal G-bundle $p_q : E(q,G) \to B(q,G)$ is universal for all principal G-bundles with transitional nilpotency class less than q.

Theorem 5.2. *Assume that G is an algebraic subgroup of $GL_N(\mathbb{C})$ for some $N \geq 0$, X is a finite CW-complex and that $\pi : E \to X$ is a principal G-bundle over X. Then, for any $q \geq 2$, the classifying map $f : X \to BG$ of π factors through $B(q,G)$ (up to homotopy) if and only if π has transitional nilpotency class less than q.*

Proof. The case $q = 2$ was treated in [2, Theorem 2.2] and in fact this theorem is true for any Lie group in this case. The proof goes through verbatim also for $q > 2$ using the fact that when G is an algebraic subgroup of $GL_N(\mathbb{C})$, then the simplicial space $B_*(q,G)$ is proper as was pointed out in the proof of Theorem 4.3. □

As $[\Sigma X, BG] = [X, \Omega BG]$ and the canonical map $\Omega B(q,G) \to \Omega BG$ always admits a splitting up to homotopy, any principal G-bundle on a suspension ΣX has transitional nilpotency class less than q. However, the nilpotency structure is not unique in general, not even up to isomorphism in the sense of the following definition.

Definition 5.3. Let $\pi_0 : E_0 \to X$ and $\pi_1 : E_1 \to X$ be two principal G-bundles with transitional nilpotency class less than q. We say that these bundles are q-transitionally isomorphic if there exists a principal G-bundle $p : E \to X \times [0,1]$ with transitional nilpotency class less than q such that $\pi_0 = p_{|p^{-1}(X\times\{0\})}$ and $\pi_1 = p_{|p^{-1}(X\times\{1\})}$.

A complex vector bundle $\pi : E \to X$ is said to have transitional nilpotency class less than q if the corresponding frame bundle, under a fixed Hermitian metric on E, has transitional nilpotency class less than q. Theorem 4.2 can then be interpreted to say that any vector bundle is stably of transitional nilpotency class less than q for all $q \geq 2$, and there is a functorial choice of such a structure. The set $Vect_{q-nil}(X)$ of q-transitionally isomorphism classes of complex vector bundles over X with transitional nilpotency class less than q is a monoid under the direct sum of vector bundles. The q-nilpotent K-theory of X is defined as the associated Grothendieck group.

Definition 5.4. $K_{q-nil}(X) := Gr(Vect_{q-nil}(X))$.

Proposition 5.5. *For any finite CW-complex X there is a natural isomorphism of rings*

$$K_{q-nil}(X) \cong [X, \mathbb{Z} \times B(q,U)].$$

Hence, it is the zeroth term of a multiplicative generalized cohomology theory.

Proof. This follow immediately from the fact that the group completion of a (discrete) monoid can be realised as the fundamental group of its classifying space. □

This answers the question raised in [2] for $q = 2$. Moreover, we have a sequence of cohomology theories and maps between them

$$K_{com}(X) = K_{2-nil}(X) \to K_{3-nil}(X) \to \cdots \to K_{q-nil}(X) \to \cdots \to K(X).$$

By Theorem 4.2 topological K-theory splits off q-nilpotent K-theory for all $q \geq 2$. These theories are not well understood and would seem to warrant further attention. For example in [2] it was shown that $K_{com}(\mathbb{S}^i) \cong K(\mathbb{S}^i)$ for $0 \leq i \leq 3$, but that $K_{com}(\mathbb{S}^4) \neq K(\mathbb{S}^4)$.

We leave it to the reader to formulate q-nilpotent versions of real and hermitian K-theory.

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