Algebraic Topology
Michaelmas 2013
Question Sheet 4

Mayer-Vietoris sequence, degree.

1. Use the Mayer-Vietoris theorem to find the homology groups of the Klein bottle. [*Hint: Decompose it into two copies of* $S^1 \times I$.]

2. Prove that:

   (i) If $f : S^n \to S^n$ is induced by an orthogonal transformation $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ then $\deg(f) = \det(A)$. [*Hint: Every orthogonal matrix is a product of reflections.*]

   (ii) If $f : S^n \to S^n$ has no fixed points then $\deg(f) = (-1)^{n+1}$.

3. Identify the torus T as the quotient group $\mathbb{R}^2/\mathbb{Z}^2$ with the quotient topology: $(x, y) \sim (x', y')$ if and only if there exist $(n, m) \in \mathbb{Z}^2$ such that $(x, y) = (n + x', m + y')$. Let A be a 2×2 matrix with integer coefficients. Prove that A induces a map of T to itself and compute the induced map A_* on $H_2T = \mathbb{Z}$ and $H_1T = \mathbb{Z}^2$ when

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}.$$

   Make a conjecture what A_* is in general.

4. Brouwer Fixed Point Theorem: Any continuous map $f : D^n \to D^n$ has a fixed point.

5. Let $G_1, \ldots, G_n$ be any finite sequence of finitely generated abelian groups. Show that there is a path-connected space K, with $H_i(K) \cong G_i$ for $1 \leq i \leq n$ and $H_i(X) = 0$ for $i > n$.

6. Let $\pi : \tilde{X} \to X$ be a continuous map such that for each $x \in X$ there exist an open neighborhood U_x with the property that $\pi^{-1}(U_x)$ is the disjoint union of open neighborhoods V_{x_i} of the points $x_i \in \pi^{-1}(x)$, $i = 1, \ldots, d$. Construct a wrong way map $f_*^! : H_n(X) \to H_n(\tilde{X})$ such that composition with f_* is the multiplication by d map. [*Hint: consider singular homology local with respect to the cover* $\{U_x\}_{x \in X}$.]

   Consider the action of the cyclic group C_d on S^{2n+1} given by diagonal multiplication by λ_d of $(z_0, \ldots, z_n) \in S^{2n+1} \subset \mathbb{C}^n$, where λ_d is a primitive root of unity of degree d. Prove that the orbit space $L(d, 2n+1)$ has at most torsion homology in degrees $0 < * < 2n$. What else can be said about the homology groups in these degrees?