ON THE FARRELL COHOMOLOGY OF THE MAPPING CLASS GROUP OF NON-ORIENTABLE SURFACES

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1. Introduction

Because of their close relation to moduli spaces of Riemann surfaces, the mapping class groups of orientable surfaces have been the attention of much mathematical research for a long time. Less well studied is the mapping class group of non-orientable surfaces. But recently, the study of mapping class groups has also been extended to the non-orientable case. This paper contributes to this programme. While Wahl [W] proved the analogue of Harer's (co)homology stability to the non-oriented case, we concentrate here on the unstable part of the cohomology. In particular, we study the question of p-periodicity.

Recall that a group G of finite virtual cohomological dimension (*vcd*) is said to be p-periodic if the p-primary component of its Farrell cohomology ring, $\hat{H}^*(G, \mathbf{Z})_{(p)}$, contains an invertible element of positive degree. Farrell cohomology extends Tate cohomology of finite groups to groups of finite *vcd*. In degrees above the *vcd* it agrees with the ordinary cohomology of the group. For the mapping class group in the oriented case, the question of p-periodicity has been examined by Xia [X] and by Glover, Mislin and Xia [GMX]. Here we determine exactly for which genus and prime p the non-orientable mapping class groups are p-periodic. In the process we also establish that these groups are of finite cohomological dimension and present a classification theorem for finite group actions on non-orientable surfaces.

Let N_g be a non-orientable surface of genus g, i.e. the connected sum of g projective planes. The associated mapping class group $\mathcal{N}_g$ is defined to be the group of connected components of the group of homeomorphisms of N_g. The mapping class groups of the projective plane and the Klein bottle are well known to be trivial and the Klein 4-group respectively,

$$\mathcal{N}_1 = \{e\} \quad \text{and} \quad \mathcal{N}_2 = C_2 \times C_2.$$

Throughout this paper we may therefore *assume that* $g \geq 3$. Our main result can now be stated as follows.

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Theorem 1.1. *$\mathcal{N}_g$ is p-periodic unless one of the two following conditions holds:*

(1) *$p = 2$;*
(2) *p is odd, $g = lp + 2$ for some $l > 0$, and for $0 \leq t < p$ with $l \equiv -t$ (mod p) we have $l + t + 2p > tp$.*

In particular, $\mathcal{N}_g$ is p-periodic whenever p is odd and g is not equal to 2 mod p. On the other hand, for a fixed odd p, there are only finitely many g with g equal to 2 mod p for which $\mathcal{N}_g$ is p-periodic.

In outline, we will first show that the mapping class group $\mathcal{N}_g$ of a non-orientable surface of genus g is a subgroup of the mapping class group Γ_{g-1} of an orientable surface of genus $g-1$. Many properties of Γ_{g-1} are thus inherited by $\mathcal{N}_g$. In particular it follows that $\mathcal{N}_g$ is of finite virtual cohomological dimension and its Farrell cohomology is well-defined. We then recall that a group G is not p-periodic precisely when G has a subgroup isomorphic to $C_p \times C_p$, the product of two cyclic groups of order p. Motivated by this we prove a classification theorem for actions of finite groups on non-orientable surfaces. From this it is straightforward to deduce necessary and sufficient conditions for $C_p \times C_p$ to act on N_g. Finally, we discuss some open questions.

2. Preliminaries

Let Σ_{g-1} be a closed orientable surface of genus $g-1$, embedded in $\mathbb{R}^3$ such that Σ_{g-1} is invariant under reflections in the xy-, yz-, and xz-planes. Define an (orientation-reversing) homeomorphism $J : \Sigma_{g-1} \to \Sigma_{g-1}$ by

$$J(x, y, z) = (-x, -y, -z).$$

J is reflection in the origin. Under the action of J on Σ_{g-1}, the orbit space is homeomorphic to a non-orientable surface N_g of genus g with associated orientation double cover

$$p : \Sigma_{g-1} \longrightarrow N_g.$$

Let $\Gamma^{\pm}_{g-1}$ denote the extended mapping class group, i.e. the group of connected components of the homeomorphisms of Σ_{g-1}, not necessarily orientation-preserving. Γ_{g-1} as usual will denote its index 2 subgroup corresponding to the orientation preserving homeomorphisms.

Birman and Chillingworth [BC] give the following description of the mapping class group $\mathcal{N}_g$. Let $C\langle J\rangle \subset \Gamma^{\pm}_{g-1}$ be the group of connected components of

$$S(J) := \{\varphi \in \mathrm{Homeo}(\Sigma_{g-1}) |\ \exists\, \tilde{\varphi} \text{ isotopic to } \varphi \text{ such that } \tilde{\varphi} J = J \tilde{\varphi}\},$$

the subgroup of homeomorphisms that commute with J up to isotopy. By definition, J generates a normal subgroup of $C\langle J\rangle$. Birman and Chillingworth identify the quotient group with the mapping class group of the orbit space $N_g = \Sigma_{g-1}/\langle J\rangle$,

$$\mathcal{N}_g \cong \frac{C\langle J\rangle}{\langle J\rangle}.$$

The following result has proved very useful as many properties of Γ_{g-1} are inherited by $\mathcal{N}_g$.

Key-Lemma 2.1. *$\mathcal{N}_g$ is isomorphic to a subgroup of Γ_{g-1}.*

Proof. Consider the projection

$$\pi : C\langle J\rangle \longrightarrow \frac{C\langle J\rangle}{\langle J\rangle} \cong \mathcal{N}_g.$$

For a subgroup G of $\mathcal{N}_g$ write

$$\pi^{-1}(G) = G^+ \cup G^- \subset C\langle J\rangle,$$

where

$$G^+ := \pi^{-1}(G) \cap \Gamma_{g-1} \quad \text{and} \quad G^- := \pi^{-1}(G) \cap (\Gamma^{\pm}_{g-1} \setminus \Gamma_{g-1}).$$

Note, $G^- = JG^+$. We claim that $\pi|_{G^+} : G^+ \to G$ is an isomorphism. Indeed, injectivity follows as the only non-zero element J in the kernel of π is not an element of G^+. Surjectivity is also immediate as every element in G has exactly two pre-images under π which differ by J. Thus exactly one of them is an element in the orientable mapping class group Γ_{g-1}, that is an element of G^+. $\square$

Recall, Farrell cohomology is defined only for groups of finite virtual cohomological dimension.

Corollary 2.2. *The non-orientable mapping class group $\mathcal{N}_g$ has finite virtual cohomological dimension with*

$$vcd\,\mathcal{N}_g \leq 4g - 9.$$

Proof. The mapping class group Γ_{g-1} is virtually torsion free. Furthermore, from Harer [H], we know that Γ_{g-1} is of finite virtual cohomological dimension $4(g-1)-5$. Hence every subgroup of Γ_{g-1} will also have finite virtual cohomological dimension with *vcd* less or equal to $4g - 9$, cf. [B] Exercise 1, p. 229. The corollary now follows by the Key-Lemma. $\square$

3. Classifying finite group actions on N_g

The purpose of this section is to give necessary and sufficient criterions for when a finite group is isomorphic to a subgroup of $\mathcal{N}_g$. For the purpose of this paper we are only interested in groups of odd order.

Theorem 3.1. *Let N_g denote a non-orientable surface of genus g, and let A be a finite group of odd order. Then A is isomorphic to a subgroup of* $\mathrm{Homeo}(\mathrm{N_g})$ *if and only if A has partial presentation*

$$\langle c_1, \ldots, c_h, y_1, \ldots, y_t | \ldots\rangle$$

such that

(1) *$h \geq 1$;*
(2) $\prod_{j=1}^{h} c_j^2 \prod_{i=1}^{t} y_i = 1$;
(3) *the order of y_i in A is m_i;*
(4) *the Riemann-Hurwitz equation holds:*

$$g - 2 = |A|(h-2) + |A| \sum_{i=1}^{t} (1 - \frac{1}{m_i}).$$

The proof of the theorem is an application of the theory of covering spaces. Different versions of the theorem can be found in the literature, see for example [T]. For completeness and convenience for the reader we include a proof.

Proof. Assume A has a partial presentation of the form described in the theorem, and let N_h be a non-orientable surface of genus $h \geq 1$. Represent N_h as a $2h$-sided polygon with sides to be identified in pairs, where the polygon is bounded by the cycle $c_1c_1c_2c_2 \ldots c_hc_h$. At a vertex add t (non-intersecting) loops $y_1, \ldots, y_t$ so that the resulting 2-cells bounded by $y_1, \ldots, y_t$ are mutually disjoint and are contained in the polygon, see Figure 1. Choose a direction for each of the loops $y_1, \ldots, y_t$ and call the resulting one-vertex graph G. Note that we can give N_h the structure of a CW-complex so that G is cellularly embedded in N_h.

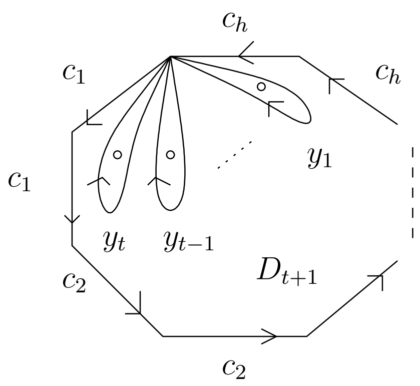

FIGURE 1. A non-orientable surface of genus h.

A covering graph $\tilde{G}$ is obtained from G as follows. Its vertex set and edge set are A and $E \times A$ respectively, where E is the edge set of the graph G. If e is an edge of G, then the edge (e, a) of $\tilde{G}$ runs from the vertex a to the vertex ae. The forgetful map of graphs $p : \tilde{G} \to G$ is a covering map which we now extend to a branched covering map $p : S \to N_h$ of surfaces as follows.

Label the regions of N_h as $D_1, D_2, \ldots, D_t, D_{t+1}$, where $D_1, D_2, \ldots, D_t$ are bounded by the loops $y_1, y_2, \ldots, y_t$, and where D_{t+1} is the remaining region. For each cycle C in G, $p^{-1}(C)$ is a collection of cycles in $\tilde{G}$. The cycles y_i have $\frac{|A|}{m_i}$ corresponding cycles in $\tilde{G}$, for each $i \in \{1, \dots, t\}$. Finally, the cycle $c_1 c_1 \dots c_h c_h y_1 \dots y_t$, bounding D_{t+1}, has $|A|$ cycles above it in $\tilde{G}$, because the order of $\prod_{j=1}^{h} c_j^2 \prod_{i=1}^{t} y_i$ in A is 1.

To each of these cycles in $\tilde{G}$ attach a 2-cell. Then extend p by mapping the interior of each 2-cell onto the interior of the 2-cell D_n by using the maps $z \to z^d$, where $d = m_i$ for $n \in \{1, \dots, t\}$, and $d = 1$ for $n = t + 1$. We obtain a surface S which admits a CW-structure with $\tilde{G}$ cellularly embedded in S.

We now argue by contradiction that S is non-orientable. Suppose that S is an orientable surface, and let $A^0 \subset A$ be the subgroup of homeomorphisms which preserve the orientation. Now $A^0 \neq A$ since N_g is non-orientable. So, A^0 is a subgroup of index 2 in A which contradicts our assumption that A is of odd order. So S is non-orientable. Finally, its genus g is determined by the Riemann-Hurwitz formula, condition (4).

Vice versa, assume A acts on the non-orientable surface $S = N_g$. As A is of odd order, A acts without reflections and its singular set is discrete. Thus the quotient map $p : S \to S/A$ is a branched covering, and S/A is a non-orientable surface of genus $h \geq 1$. Represent S/A as a $2h$-sided polygon with sides $c_1, c_1, c_2, c_2, \dots, c_h, c_h$ to be identified in pairs, and in which the branch points of p are in the interior of the polygon. Now add mutually disjoint loops $y_1, \dots, y_t$ around each branch point, all starting at the same vertex as indicated in Figure 1. Let us call the resulting one-vertex graph G. Its inverse image $p^{-1}(G)$ is a Cayley graph for the group A: vertices correspond to the elements of A and at each vertex there are $2(h + t)$ directions corresponding to generators c_i and y_i. The three conditions for the partial presentations are easily verified. First note that h is positive as S/A is non-orientable. As $\prod_{j=1}^{h} c_j^2 \prod_{i=1}^{t} y_i = 1$ is a closed curve in S/A, so it is in S and hence must represent the identity in A. The order m_i of y_i is precisely the branch number of the singular point that y_i encircles. Thus the formula in condition (4) follows from the Riemann-Hurwitz equation. □

As we are interested in subgroups of the mapping class group we state the following result which is well-known at least for orientable surfaces.

Theorem 3.2. *A finite group G is a subgroup of $\mathcal{N}_g$ if and only if it is a subgroup of* $\mathrm{Homeo(N_g)}$.

Proof. If G is a finite subgroup of $\mathcal{N}_g$ then it follows by the Nielsen realisation problem for non-orientable surfaces [K] that G lifts to a subgroup of $\mathrm{Homeo(N_g)}$. Vice versa, let G be a finite subgroup of $\mathrm{Homeo(N_g)}$. An application of the Lefschetz Fixed Point Formula shows that for all $g \geq 3$, any element of finite order in $\mathrm{Homeo}(\mathrm{N}_g)$ cannot be homotopic to the identity. Hence the kernel of the canonical projection $\mathrm{Homeo}(\mathrm{N}_g) \to \mathcal{N}_g$ when restricted to a finite subgroup $G \in \mathrm{Homeo}(\mathrm{N}_g)$ must be trivial. □

Theorem 3.1 and Theorem 3.2 together imply that a finite group A of odd order is a subgroup of the mapping class group $\mathcal{N}_g$ if and only if it has partial presentation such that conditions (1) to (4) in Theorem 3.1 hold.

4. The p-periodicity of $\mathcal{N}_g$

Using the result of the previous section we can now prove our main result. Theorem 1.1 is equivalent to the following three lemmata. Recall, cf. [B] Theorem 6.7, that a group of finite *vcd* is p-periodic if and only if it does not contain an elementary abelian subgroup of rank two.

Lemma 4.1. *$\mathcal{N}_g$ is not 2-periodic.*

Proof. It will suffice to exhibit a subgroup of $\mathcal{N}_g$ isomorphic to $C_2 \times C_2$. Let R_1 and R_2 be homeomorphisms of Σ_{g-1} (embedded in $\mathbb{R}^3$ as before,) which are rotations by π, given by the formulae

$$R_1(x,y,z) = (-x,-y,z);$$

$$R_2(x,y,z) = (x,-y,-z).$$

Clearly, J, R_1 and R_2 are all involutions. For $g \geq 3$ the induced actions on the first homology groups $H_1(\Sigma_{g-1}$ are non-trival and all different, they define non-trivial, distinct elements of order two in $\Gamma^{\pm}_{g-1}$. From their defining formulas it is clear that they commute with each other. Hence, they generate a subgroup

$$H = C_2 \times C_2 \times C_2 \subset C\langle J\rangle \subset \Gamma^{\pm}_{g-1},$$

and thus

$$\pi(H) \cong C_2 \times C_2 \subset \frac{C\langle J\rangle}{\langle J\rangle} \cong \mathcal{N}_g.$$

Thus $\mathcal{N}_g$ is never 2-periodic, □

Lemma 4.2. *Assume p is odd, $g = lp + 2$ for some $l > 0$, and for $0 \leq t < p$ with $l \equiv -t$ (mod p) we have $l + t + 2p > tp$. Then $\mathcal{N}_g$ is not p-periodic.*

Proof. We will now use Theorem 3.1 (and Theorem 3.2) to exhibit subgroups $C_p \times C_p \subset \mathcal{N}_g$. We distinguish three cases depending on the value of t.

Case 1.: $t = 0$.
Write $l = kp$ for some $k \geq 1$, and let $h = k + 2 \geq 3$. A presentation of $A = C_p \times C_p = \langle c_1\rangle \times \langle c_2\rangle$ can now be given as follows:

$$A = \langle c_1, \dots, c_h | c_3 = c_1^{p-1}c_2^{p-1}, c_4 = \dots = c_h = 1, c_1c_2 = c_2c_1, c_1^p = c_2^p = 1\rangle.$$

One checks that the four conditions of Theorem 3.1 are satisfied; here

$$g - 2 = p^2(h - 2).$$

Case 2.: $t = 1$.

Write $l = kp - 1$ for some $k \geq 1$, and let $h = k + 1 \geq 2$. A presentation of $A = C_p \times C_p = \langle c_1 \rangle \times \langle c_2 \rangle$ is given by:

$$A = \langle c_1, \ldots, c_h, y_1 | \, y_1 = c_1^{p-2} c_2^{p-2}, c_3 = \ldots = c_h = 1, \, c_1 c_2 = c_2 c_1, \, c_1^p = c_2^p = 1 \rangle.$$

Again one easily checks that the four conditions of Theorem 3.1 are satisfied; in this case

$$g - 2 = p^2(h-2) + p^2(1 - \frac{1}{p}).$$

Case 3.: $t \geq 2$ and $l + t + 2p > tp$.
Write $l = kp - t$ for some $k \geq 1$. As $l + t + 2p > tp$ and both sides are divisible by p, we can find an integer $h \geq 1$ such that $l + t + 2p = tp + hp$, and hence $l = p(h - 2 + t) - t$. A presentation of $A = C_p \times C_p = \langle y_1 \rangle \times \langle y_2 \rangle$ is now given by:

$$A = \langle c_1, \ldots, c_h, y_1, y_2, \ldots, y_t | \, c_1 = y_1^{\frac{p-1}{2}} y_2^{\frac{p-1}{2}} \ldots y_t^{\frac{p-1}{2}},$$

$$y_2 = y_3 = \ldots = y_t, \, c_2 = c_3 = \ldots = c_h = 1, \, y_1 y_2 = y_2 y_1, \, y_1^p = y_2^p = 1 \rangle.$$

This presentation satisfies the conditions of Theorem 3.1 with

$$g - 2 = p^2(h-2) + p^2 t(1 - \frac{1}{p}).$$

Hence in all these three cases, i.e. whenever condition (2) of Theorem 1.1 holds, the mapping class group $\mathcal{N}_g$ is not p-periodic. □

Lemma 4.3. *If p is odd and g does not satisfy the condition of Lemma 4.2, then $\mathcal{N}_g$ is p-periodic.*

Proof. Let p be odd and suppose that there exists a subgroup $A = C_p \times C_p$ contained in $\mathcal{N}_g$. Then by Theorem 3.1 (and Theorem 3.2), A acts on N_g and the Riemann-Hurwitz Formula must be satisfied for some $h \geq 1$ where h is the genus of the quotient surface N_g/A. Let s be the number of singular points of the action of A on N_g, and let a be an element in the stabiliser of some singular point x. By the Key-Lemma 2.1, a lifts to an element of Γ_{g-1} and by the Nielsen realization problem to a homeomorphism, also denoted by a, of Σ_{g-1}. The singular point x lifts to two points in Σ_{g-1}, and under the action of a these form two separate orbits as the group A and hence the element a are of odd order. So a is in the stabiliser of these two points, and therefore must act freely on the tangent planes at these points (for otherwise a would be homotopic to a homeomorphism that fixes a whole disk; but all such homeomorphisms are well-known to give rise to elements of infinite order in the mapping class group). This also implies that the action of a on the tangent plane at x in N_g is free. It follows that the stabiliser of each singular point is isomorphic to C_p as these are the only non-trivial subgroups of A that are also subgroups of $\mathrm{GL}_2(\mathbb{R})$. So, for some $h \geq 1$,

$$g - 2 = p^2(h-2) + ps(p-1).$$

From this it follows that $g = lp + 2$ for some $l \geq 1$, and furthermore, that $l = p(h - 2 + s) - s$. Note that $l = -s \pmod{p}$. Now write $s = qp + t$ for some $q \geq 0$

and $0 \leq t < p$. Then $l = p(\tilde{h} - 2 + t - q) - t$ for $\tilde{h} = h + q(p-1) \geq 1$. Thus we are in the situation of Lemma 4.2, and hence Lemma 4.3 follows. □

Remark 4.4. A group is p-periodic if and only if it does not contain a subgroup isomorphic to $C_p \times C_p$. Therefore, any subgroup of a p-periodic group is p-periodic. Hence by the Key-Lemma 2.1, the p-periodicity of any Γ_{g-1} implies the p-periodicity of $\mathcal{N}_g$. (In particular, as for odd p and g not equal to 2 mod p, Γ_{g-1} is always p periodic, so is $\mathcal{N}_g$.) However, comparing our results with those of Xia [X], we note here that the converse is false. For example, when $p = 5$ and $g = 7$, Γ_6 *is not* p-periodic but $\mathcal{N}_7$ *is*. However, for a fixed p there are at most finitely many such g where Γ_{g-1} is not p-periodic but $\mathcal{N}_g$ is.

5. The p-period and other open questions

We will briefly discuss three questions that arise from our study.

5.1. The p-period. Recall that the p-period d of a p-periodic group G is the least positive degree of an invertible element in its Farrell cohomology group $\hat{H}^*(G, \mathbf{Z})_{(p)}$. The question thus arises what the p-period of $\mathcal{N}_g$ is when $\mathcal{N}_g$ is p-periodic.

For any group G of finite *vcd*, an invertible element in $\hat{H}^*(G, \mathbf{Z})_{(p)}$ restricts to an invertible element in the Farrell cohomology of any subgroup of G. Thus the p-period of a subgroup divides the p-period of G.

The main result of [GMX] is that for all g such that Γ_{g-1} is p-periodic, the p-period divides $2(p-1)$. Hence for all such g, the p-period of $\mathcal{N}_g$ also divides $2(p-1)$. However, as we noted above, there are pairs p and g for which $\mathcal{N}_g$ is p-periodic but Γ_{g-1} is not. We expect that the methods of [GMX] can be pushed to cover also these cases. It remains also to find lower bounds for the p-period.

5.2. Punctured mapping class groups. In the oriented case Lu [L1], [L2] has studied the p-periodicity of the mapping class groups with marked points, and proved that they are all p-periodic of period 2. One might expect a similar result should hold for the mapping class group of non-orientable surfaces with marked points.

5.3. The virtual cohomological dimension. We have established in Corollary 2.2 that $\mathcal{N}_g$ has finite virtual cohomological dimension and that this dimension is less than or equal to $4g - 9$. It seems an interesting project to determine the *vcd* of $\mathcal{N}_g$.

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