Lie Groups
Michaelmas 2007
Question Sheet 2

Campbell-Baker-Hausdorff formula; Lie algebras of linear groups.

1. Use the Campbell-Baker-Haussdorff formula to prove

   i. $\exp X \exp Y \exp(-X) = \exp(Y + [X, Y] + \dots);$

   ii. $\exp X \exp Y \exp(-X) \exp(-Y) = \exp([X, Y] + \dots);$

   where ... indicates a summand involving three or more factors.

2. Let $\phi : G \to H$ be a differential homomorphism of linear groups. Show that for all $X \in L(G)$,

$$\phi(\exp X) = \exp\left(\frac{d}{dt}\phi(\exp tX)|_{t=0}\right).$$

   [Hint: Show that $\phi(\exp(tX))$ satisfies the differential equation of $\exp$.] Give an interpretation of this identity in terms of the 'bigger picture'.

3. Determine the Lie algebra $L(G)$ and its dimension where G is in turn the special orthogonal group SO_n, the orthogonal group O_n, the unitary group U_n, the special unitary group SU_n, the special real linear group $SL_n(\mathbb{R})$, the special complex linear group $SL_n(\mathbb{C})$, and the group of real upper triangular non-sigular $n \times n$ matrices.

4. Let $\{E_1, E_2, E_3\}$ be the basis for $L(SO_3)$ given by the three matrices below, and let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. Define $\phi : L(SO_3) \to \mathbb{R}^3$ by $\phi(E_i) = e_i$. Show ϕ is an isomorphism of Lie algebras where the bracket in $\mathbb{R}^3$ is given by the vector cross product. Determine all Lie subalgebras of $L(SO_3)$.

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5. Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a matrix group and assume its Lie algebra $L(G)$ is a complex vector subspace of $M_n(\mathbb{C})$. Prove that $L(G)$ is a complex Lie subalgebra.