Lie Groups
Michaelmas 2009
Question Sheet 6

Representations and characters.

1. Let G be a compact abelian linear group, and (V, ρ) be an irreducible complex representation. Prove that $\rho(G) \subset S^1 \subset Aut(V)$. What can be said when (V, ρ) is an irreducible real representation?

2. Prove that the functional defined for real valued functions f on S^1 by
$$\int_{S^1} f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\,\theta}) d\theta$$
is the bi-invariant normalized integral for S^1.

3. Let V and W be two finite dimensional vector spaces. Show that $\alpha : V^* \otimes W \to$ Hom (V, W) defined by $\alpha(v^* \otimes w)(v) := v^*(v)w$ defines a linear isomorphism; here V^* denotes the dual of V. Now let $V = W$. Prove that
$$tr(\alpha(v^* \otimes v)) = v^*(v).$$

4. Let $\alpha : V \to V$ and $\beta : W \to W$ be linear transformations with matrix representations (a_{ij}) and (b_{ij}) respectively. Find a matrix representation for $\alpha \otimes \beta$. Prove that $tr(\alpha \otimes \beta) = tr(\alpha)tr(\beta)$. Assume now that V and W are representations of a linear group G. Deduce that $\chi_{V \otimes W} = \chi_V \chi_W$.

5. Let G be a compact linear group and (V, π) and (W, ρ) be two irreducible unitary representations. By considering the G-map associated to $T = E_{jl}$, prove that

   i . If $\pi \nsim \rho$ then $\int_G \pi_{ij}(g)\overline{\rho_{kl}(g)} = 0$.

   ii. If $\pi = \rho$ then $\int_G \pi_{ij}(g)\overline{\rho_{kl}(g)} = \frac{1}{\dim V} \delta_{ik}\delta_{jl}$.

6. Let the standard maximal torus T of SO_{2n} act on the complexified Lie algebra
$$L(SO_{2n}) \otimes \mathbb{C} \simeq L(SO_{2n}\mathbb{C})$$
via the adjoint action Ad . Find its decompositions into irreducible T-representations. Describe the associated character. What can you say about the character of Ad for the whole group SO_{2n}? [*You may assume here that the roots of $SO_{2n}\mathbb{C}$ are $\pm(\lambda_j \pm \lambda_k)$ for $j < k$, and that the associated eigenvectors are $E_{jk} - E_{n+k,n+j}$, $E_{j,n+k} - E_{k,n+j}$ and $E_{n+j,k} - E_{n+k,j}$.*]