

# MAPPING CONFIGURATION SPACES TO MODULI SPACES

GRAEME SEGAL AND ULRIKE TILLMANN

*Dedicated to Shigeyuki Morita on the occasion of his 60th birthday*

## 1. INTRODUCTION

Let  $\mathcal{C}_n$  be the space of unordered  $n$ -tuples of distinct points in the interior of the unit disc  $D = \{z \in \mathbb{C} : |z| \leq 1\}$ , and let  $\mathcal{M}_{g,2}$  denote the moduli space of connected Riemann surfaces of genus  $g$  with two ordered and parametrised boundary components. There is a natural map

$$\Phi : \mathcal{C}_{2g+2} \longrightarrow \mathcal{M}_{g,2};$$

it takes a subset  $\underline{a} = \{a_1, \dots, a_{2g+2}\} \subset D$  to the part of the Riemann surface  $\Sigma_{\underline{a}}$  of the function

$$f_{\underline{a}}(z) = ((z - a_1) \dots (z - a_{2g+2}))^{1/2}$$

which lies over the disc  $D$ .

The purpose of this note is to describe this map in topological terms. On passing to the fundamental groups it gives a geometric definition of a well-known algebraic homomorphism  $\phi : \mathfrak{B}\mathfrak{r}_{2g+2} \rightarrow \Gamma_{g,2}$  from the braid group on  $2g + 2$  strings to the mapping class group which is described below. The main result is that  $\Phi$  is compatible with naturally defined actions of the framed little 2-discs operad on configuration spaces and moduli spaces. As immediate consequences we deduce that  $\Phi$  is trivial in homology with field coefficients, and trivial on stable homology with any constant coefficient system. In particular, this proves an unstable version of a conjecture by Harer, and simplifies the proof of the triviality of the map in stable homology given in [ST], to which we refer for more background on this problem.

In an appendix we discuss the map in homology given by the inclusion of the hyper-elliptic involution into the mapping class group.

## 2. THE INDUCED MAP ON FUNDAMENTAL GROUPS

Let  $F_{g,2}$  be a surface of genus  $g$  with two boundary circles. The moduli space  $\mathcal{M}_{g,2}$  is homotopic to the classifying space of the group of homeomorphisms of  $F_{g,2}$  which leave the boundary circles pointwise fixed:

$$\mathcal{M}_{g,2} \simeq B\text{Homeo}^+(F_{g,2}; \partial) \simeq B\Gamma_{g,2};$$

---

*Date:* July 2007.

here  $\Gamma_{g,2}$  is the associated mapping class group. Similarly, the configuration space  $\mathcal{C}_{2g+2}$  is homotopic to the classifying space of the group of homeomorphisms of  $D \setminus \underline{a}$  — the disc with  $2g + 2$  interior points removed — which fix the boundary of the disc pointwise but are allowed to permute the points of  $\underline{a}$ :

$$\mathcal{C}_{2g+2} \simeq B\text{Homeo}^+(D \setminus \underline{a}) \simeq B\mathfrak{B}\mathfrak{r}_{2g+2};$$

the associated mapping class group is the braid group  $\mathfrak{B}\mathfrak{r}_{2g+2}$  on  $2g + 2$  strings. Thought of as the fundamental group of  $\mathcal{C}_{2g+2}$  based at  $\underline{a}$ , the group  $\mathfrak{B}\mathfrak{r}_{2g+2}$  is generated by the braids  $\sigma_i$ ,  $i = 1, \dots, 2g + 1$ , where  $\sigma_i$  interchanges the points  $a_i$  and  $a_{i+1}$ .

We shall now determine the map that  $\Phi$  induces on fundamental groups. Write  $\pi : \Sigma_{\underline{a}} \rightarrow D$  for the double covering map branched at  $a_1, \dots, a_{2g+2}$ . Let  $D_i$  be a subdisc in the interior of  $D$  containing the two points  $a_i$  and  $a_{i+1}$ . Then  $\pi^{-1}(D_i)$  is an annulus contained in  $\Sigma_{\underline{a}}$ . To see this, note that  $\pi^{-1}(D_i)$  has two boundary components and that its Euler characteristic is given by

$$\chi(\pi^{-1}(D_i)) = 2\chi(D_i) - 2 = 0.$$

**Proposition 2.1.** *On fundamental groups, the map  $\Phi$  takes  $\sigma_i$  to the Dehn twist corresponding to the annulus  $\pi^{-1}(D_i)$  in  $\Sigma_{\underline{a}}$ .*

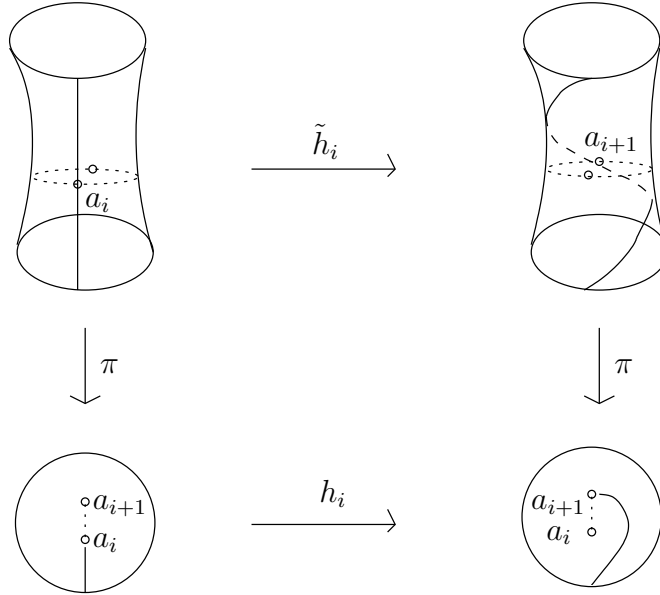


FIGURE 1. Lifting  $h_i$

*Proof.* Homeomorphisms of the base of a cyclic branched covering of a surface that permute the branch points can be lifted, and this lift is unique up to covering transformations, cf. Lemma 5.1 of [BH]. The braid  $\sigma_i$  corresponds to a homeomorphism

$h_i$  of  $D$  supported in the subdisc  $D_i$ , and hence lifts to a homeomorphism  $\tilde{h}_i$  of  $\Sigma_a$  supported in the annulus  $\pi^{-1}(D_i)$ . There is only one non-trivial covering transformation for  $\pi$ , given by the hyper-elliptic involution  $J$ , which interchanges the boundary components of the annulus  $\pi^{-1}(D_i)$ . Thus the lift  $\tilde{h}_i$  is unique if we ask for the boundary circles to be fixed.

The mapping class group of an annulus is generated by the Dehn twist around its waist. By analysing the effect of  $\tilde{h}_i$  on a path connecting the two boundary components of  $\pi^{-1}(D_i)$ , one checks the claim. See Figure 1.  $\square$

Thus, on fundamental groups,  $\Phi$  induces the group homomorphism

$$\phi : \mathfrak{B}\mathfrak{r}_{2g+2} \rightarrow \Gamma_{g,2} \rightarrow \Gamma_g$$

where the generator  $\sigma_i$  is mapped to the Dehn twist around the curve  $\alpha_i$  in Figure 2. This is the group homomorphism studied in [ST].

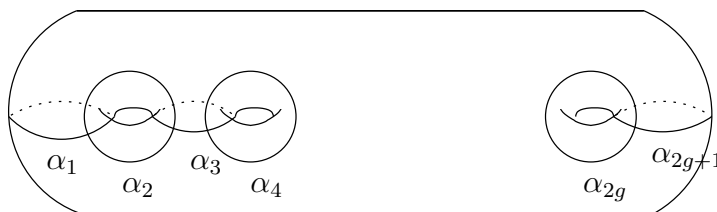


FIGURE 2. The image of  $\phi$

### 3. ACTION OF THE FRAMED LITTLE 2-DISCS OPERAD

Let  $\mathcal{D}_k$  be the space of smooth embeddings of  $k$  disjoint, ordered copies of the disc  $D$  into its interior which restrict on each disc to the composition of a translation and a multiplication by an element of  $\mathbb{C}^\times$ . The spaces  $\{\mathcal{D}_k\}$  form an operad  $\mathcal{D}$ , the framed little 2-discs operad, with structure maps

$$\gamma : \mathcal{D}_k \times (\mathcal{D}_{m_1} \times \cdots \times \mathcal{D}_{m_k}) \longrightarrow \mathcal{D}_{\Sigma_{m_i}}$$

given by composition of embeddings. The operad  $\mathcal{D}$  acts naturally on the configuration spaces  $\mathcal{X} = \coprod_{m \geq 0} \mathcal{X}_m$  with  $\mathcal{X}_m = \mathcal{C}_{2m}$ . This action is defined by maps

$$\gamma_{\mathcal{X}} : \mathcal{D}_k \times (\mathcal{X}_{m_1} \times \cdots \times \mathcal{X}_{m_k}) \longrightarrow \mathcal{X}_{\Sigma_{m_i}}$$

which take a point  $(f; \underline{a}_1, \dots, \underline{a}_k)$  to the image of the union  $\underline{a}_1 \cup \cdots \cup \underline{a}_k$  under the embedding  $f$ .

Now put  $\mathcal{Y}_m = \mathcal{M}_{m-1,2}$  for  $m \geq 1$ , and  $\mathcal{Y}_0 = \mathcal{M}_{0,1} \sqcup \mathcal{M}_{0,1}$ . Each surface  $\Sigma \in \mathcal{Y}_m$  has two ordered parametrised boundary circles  $\partial_1 \Sigma$  and  $\partial_2 \Sigma$ . For  $f \in \mathcal{D}_k$ , let  $P_f = D \setminus f(D \cup \cdots \cup D)$ . We think of  $P_f$  as a Riemann surface with  $k+1$  parametrised boundary components. There is an operad action

$$\gamma_{\mathcal{Y}} : \mathcal{D}_k \times (\mathcal{Y}_{m_1} \times \cdots \times \mathcal{Y}_{m_k}) \longrightarrow \mathcal{Y}_{\Sigma_{m_i}}$$

defined by

$$(f; \Sigma_1 \cup \cdots \cup \Sigma_k) \mapsto (P_f \cup P_f \cup \Sigma_1 \cup \cdots \cup \Sigma_k) / \cong$$

where the union of the first boundary components  $\cup_i \partial_1 \Sigma_i$  is identified with the interior boundary circles of the first  $P_f$ , and  $\cup_i \partial_2 \Sigma_i$  is identified with the interior boundary circles of the second  $P_f$  using the parametrisations, cf. Figure 3. The resulting surface has a well-defined complex structure that restricts to the given complex structures on the subsurfaces, cf. [S].

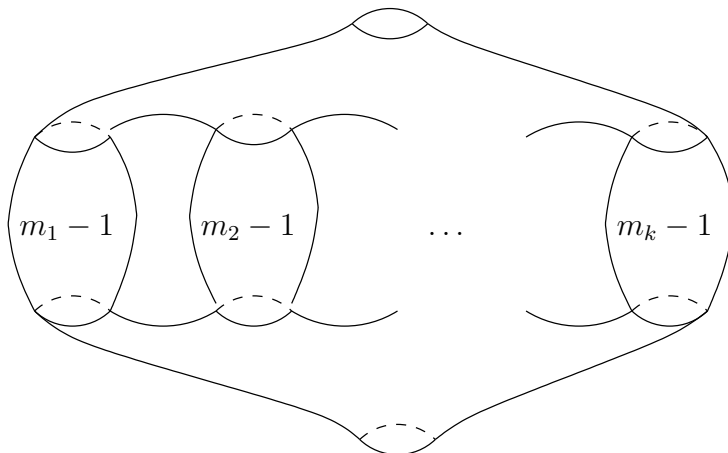


FIGURE 3.  $\mathcal{D}$  acting on moduli spaces

**Proposition 3.1.** *The map  $\Phi$  defines a map  $\mathcal{X} \rightarrow \mathcal{Y}$  of algebras over the framed little 2-discs operad.*

*Proof.* The index  $m$  was chosen so that  $\Phi$  takes  $\mathcal{X}_m$  to  $\mathcal{Y}_m$ . Both  $\gamma_{\mathcal{X}}$  and  $\gamma_{\mathcal{Y}}$  are defined by gluing of surfaces. The composed surfaces has a unique conformal structure that restricts to the given conformal structures on the subsurfaces. Furthermore, the covering map  $\pi : \Sigma_{\underline{a}} \rightarrow D$  is conformal away from the set  $\underline{a}$ . It therefore follows that the assignment  $\underline{a} \mapsto \Sigma_{\underline{a}}$  commutes with the gluing operations of  $\gamma_{\mathcal{X}}$  and  $\gamma_{\mathcal{Y}}$ .  $\square$

**Remark 3.2.** We can also work on the level of mapping class groups and their classifying spaces. For this, replace  $\mathcal{D}_k$  with the homotopy equivalent space  $B\Gamma_{0,k+1}$ , the classifying space of the mapping class group of  $D$  with  $k$  interior discs removed. Similarly, we also replace  $\mathcal{X}_m$  with  $B\mathfrak{Br}_{2m}$  and  $\mathcal{Y}_m$  with  $B\Gamma_{m-1,2}$ , for  $m > 0$ . The actions of  $\gamma_{\mathcal{X}}$  and  $\gamma_{\mathcal{Y}}$ , reinterpreted in these terms, are now induced by maps of the underlying mapping class groups. By definition they commute with  $\Phi$ , or more precisely with  $B\phi$ . This corrects the statement and proof of 2.6 in [C].

## 4. THE INDUCED MAP IN HOMOLOGY

Proposition 3.1 has immediate consequences for the map that  $\Phi$  induces on homology. Let  $p$  be a prime. The idea for the following is due to F. Cohen [C]. The case  $p = 2$  is the (stronger) unstable version of Harer's conjecture, cf. [ST].

**Corollary 4.1.** *For  $g > 2$  and  $* > 0$ ,*

$$\Phi_* : H_*(\mathcal{C}_{2g+2}; \mathbb{Z}/p\mathbb{Z}) \rightarrow H_*(\mathcal{M}_{g,2}; \mathbb{Z}/p\mathbb{Z}) \text{ is zero .}$$

*Proof.* Recall that the first homology group of the mapping class group  $\Gamma_{g,2}$ , and hence that of  $\mathcal{M}_{g,2}$  is zero for  $g > 2$ . On the other hand,  $H_*(\mathcal{C}_{2g+2}, \mathbb{Z}/p\mathbb{Z})$  is generated by the first homology group under products and the operations induced from the action of  $\mathcal{D}$ , cf. [CLM]. The result now follows by Proposition 3.1.  $\square$

We also deduce the main result of [ST]. Maps of algebras over the little framed 2-discs operad group-complete to maps of double loop spaces. In particular,  $\Phi$  induces a map of double loop spaces from the group-completion of  $\mathcal{X}$  to that of  $\mathcal{Y}$ ,

$$\Phi^+ : 2\mathbb{Z} \times B\mathfrak{B}\tau_\infty^+ \longrightarrow \mathbb{Z} \times B\Gamma_\infty^+;$$

here  $X^+$  denotes Quillen's plus construction on  $X$ , while  $\mathfrak{B}\tau_\infty := \lim_{k \rightarrow \infty} \mathfrak{B}\tau_k$  and  $\Gamma_\infty := \lim_{g \rightarrow \infty} \Gamma_{g,2}$  are the infinite braid and stable mapping class groups. Note that the zero component in  $\mathbb{Z} \times B\Gamma_\infty^+$  corresponds to surfaces of Euler characteristic 2.

**Corollary 4.2.**  *$\phi_* : H_*(\mathfrak{B}\tau_\infty; k) \rightarrow H_*(\Gamma_\infty; k)$  is zero for  $* > 0$ , and any constant coefficient system  $k$ .*

*Proof.* As the plus construction does not alter the homology, this follows because on connected components the map  $\Phi^+$  is homotopic to the constant map. Indeed, any double loop space map from  $B\mathfrak{B}\tau_\infty^+ \simeq \Omega^2 S^3$  is determined by its restriction to  $S^1 \subset \Omega^2 \Sigma^2(S^1)$ . For  $\Phi^+$  this restriction has to be homotopic to the constant map as  $B\Gamma_\infty^+$  is simply connected.  $\square$

Recall that by the Harer-Ivanov homology stability theorem, the homology of the mapping class group is independent of the genus  $g$  and the number of boundary components in dimensions  $* < g/2$ . Thus Corollary 4.2 can be rephrased as follows.

**Corollary 4.3.** *For  $g > 2* > 0$ ,*

$$\Phi_* : H_*(\mathcal{C}_{2g+2}; k) \rightarrow H_*(\mathcal{M}_{g,2}; k) \text{ is zero .}$$

## 5. APPENDIX: RELATION TO HYPER-ELLIPTIC MAPPING CLASS GROUPS

By definition  $\Phi$  factors through the subspace of hyper-elliptic curves in  $\mathcal{M}_{g,2}$ . Note that the hyper-elliptic involution  $J$  interchanges the boundary components of  $F_{g,2}$  and is only an element of the extension  $\Gamma_{g,(2)}$  of  $\Gamma_{g,2}$ , the mapping class group associated to homeomorphisms that may exchange the boundary components as long as the parametrisations are preserved. The hyper-elliptic mapping class groups

$\Delta_{g,(2)}$  and  $\Delta_g$  are the commutants of  $J$  in  $\Gamma_{g,(2)}$  and  $\Gamma_g$ . They are extensions of the braid groups of the disc  $D$  and the sphere  $S^2$  respectively:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \langle J \rangle & \longrightarrow & \Delta_{g,(2)} & \longrightarrow & \mathfrak{Br}_{2g+2} \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \langle J \rangle & \longrightarrow & \Delta_g & \longrightarrow & \mathfrak{Br}_{2g+2}(S^2) \longrightarrow 0. \end{array}$$

The kernel of the middle and right vertical maps is an extension of the free group on  $2g+1$  generators by  $\mathbb{Z}$  for  $g > 0$ . The homomorphism  $\phi$  provides a splitting of the top row. As  $J$  maps the curves  $\alpha_i$ ,  $i = 1, \dots, 2g+1$  in Figure 2 to themselves,  $J$  commutes with the corresponding Dehn twists, cf. Lemma 4.6.7 of [B]. Hence

$$\Delta_{g,(2)} \simeq \mathfrak{Br}_{2g+2} \times \langle J \rangle.$$

At the conference, R. Hain raised the question whether Corollary 4.2 can be extended to the hyper-elliptic mapping class group. This is not the case, as we shall explain now.

Consider the universal surface bundle  $E$  over  $B\Gamma_g \simeq B\text{Homeo}^+(F_g)$  and the associated pull-back bundle  $\text{incl}^*(E)$  over  $B(\langle J \rangle) = BC_2$ . The vertical tangent bundle on  $E$  is classified by a map  $\omega : E \rightarrow \mathbb{C}P^\infty$ . Consider the composition

$$\theta : BC_2 \xrightarrow{\text{incl}} B\Gamma_g \xrightarrow{\text{tr}} Q(E_+) \xrightarrow{\omega} Q(\mathbb{C}P_+^\infty) = Q(\mathbb{C}P^\infty) \times Q(S^0);$$

here  $\text{tr}$  denotes the Becker-Gottlieb transfer map of the bundle  $E$ , while  $X_+$  stands for the union of  $X$  with a disjoint base point, and  $Q = \lim_{n \rightarrow \infty} \Omega^n S^n$  is the free infinite loop space functor. A straightforward computation of transfer maps, cf. Lemma 5.2 of [GMT], gives

$$\theta \simeq (2g+2)\psi + (-2g)\hat{t},$$

where

$$\psi : BC_2 \longrightarrow \mathbb{C}P^\infty \longrightarrow Q(\mathbb{C}P^\infty)$$

is induced by the inclusion  $C_2 \rightarrow S^1$ , and  $\hat{t}$  is the transfer associated to the universal bundle  $EC_2 \rightarrow BC_2$ . The map  $\psi$  is non-trivial in homology with  $\mathbb{Z}/2\mathbb{Z}$ -coefficients, and hence so is  $\text{incl}$ . We have proved

**Proposition 5.1.** *The inclusion  $\text{incl} : \langle J \rangle = C_2 \rightarrow \Gamma_g$  is non-trivial on stable homology with  $\mathbb{Z}/2\mathbb{Z}$ -coefficients.*

## References.

- [B] J. Birman: Braids, Links, and Mapping Class Groups. Ann. of Math. Studies **82**, PUP (1975)
- [BH] J. Birman, H. Hilden: On isotopies of homeomorphisms of Riemann surfaces. Ann. of Math. (2) **97**, 424-439 (1973)
- [C] F. R. Cohen: Homology of mapping class groups for surfaces of low genus, The Lefschetz Centennial conference, Part II, Contemp. Math. **58** II, 21-30, Amer. Math. Soc., 1987

- [CLM] F. R. Cohen, T. J. Lada, J. P. May: The homology of iterated loop spaces. Lec. Notes in Math. **533**, Springer-Verlag, 1976
- [GMT] S. Galatius, I. Madsen, U. Tillmann: Divisibility of the stable Miller-Morita-Mumford classes. Journal AMS **19**, 759-779 (2006)
- [S] G. Segal: The Definition of Conformal Field Theorie. In Topology, Geometry and Quantum Field Theory ed. U. Tillmann, LMS Lecture Notes Series **308**, CUP, 2004
- [ST] Y. Song, U. Tillmann: Braids, mapping class groups, and categorical delooping. Math. Annalen, (2007)

GRAEME SEGAL, ALL SOULS COLLEGE, OXFORD UNIVERSITY, OXFORD OX1 4AL, UK,  
*E-mail address:* graeme.segal@all-souls.ox.ac.uk

ULRIKE TILLMANN, MATHEMATICAL INSTITUTE, OXFORD UNIVERSITY, OXFORD OX1 3LB, UK  
*E-mail address:* tillmann@maths.ox.ac.uk