Homology stability for symmetric diffeomorphism and mapping class groups

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Abstract

For any smooth compact manifold W of dimension of at least two we prove that the classifying spaces of its group of diffeomorphisms which fix a set of k points or k embedded disks (up to permutation) satisfy homology stability. The same is true for so-called symmetric diffeomorphisms of W connected sum with k copies of an arbitrary compact smooth manifold Q of the same dimension. The analogues for mapping class groups as well as other generalisations will also be proved.

1 Introduction

Compact smooth manifolds and their diffeomorphism groups are fundamental objects in geometry and topology. If one wants to understand diffeomorphic families of a given compact manifold W one is led to study the associated topological moduli space

$$\mathcal{M}(W) := \lim_{N \to \infty} \operatorname{Emb}(W; \mathbb{R}^N) / \operatorname{Diff}(W),$$

the orbit space of the group of diffeomorphisms of W acting on the space of smooth embeddings of W into larger and larger Euclidean space. This is also a model for the classifying space of $\operatorname{Diff}(W)$ and hence its cohomology is the set of characteristic classes for smooth W-bundles. To understand the topology of $\mathcal{M}(W) \simeq B\operatorname{Diff}(W)$ is a difficult problem and only a few special cases are fully understood.

Stabilisations:

To simplify the problem one might consider to study an associated stabilised problem. One way to stabilise that has played an important role in the past is to thicken the manifold by taking the cross product with the unit interval $I = [0, 1]$ and to consider the pseudo-isotopies $P(W)$, which are diffeomorphisms of $W \times I$ which fix $W \times \{0\}$ and $\partial W \times I$ point-wise. Repeating the process, the stable pseudo-isotopy group is then defined as the limit space $\mathcal{P}(W) := \lim_{k\to\infty} P(W \times I^k)$. It can be studied via K-theory. Indeed, in the late 1970s Waldhausen [W] proved that his K-theory $A(W)$ is a double deloop of $\mathcal{P}(W)$ times the well-understood free infinite loop space on W. That is to say, he proved that

$$A(W) \simeq Wh(W) \times \Omega^\infty\Sigma^\infty(W_+)$$

where $Wh(W)$ is the smooth Whitehead space of W and $\Omega^2 Wh(W) \simeq \mathcal{P}(W)$. The crucial fact that allows one to deduce in principle something for $P(M \times I^k)$ from the K-theory $A(W)$ is

Igusa's stability theorem [I]. It says that the maps $P(M \times I^k) \to P(M \times I^{k+1})$ are c-connected for $\dim W + k \geq \max(2c+7, 3c+4)$. Though Waldhausen K-theory is well studied, it remains however difficult to deduce concrete information for specific manifolds.

More recently, another stabilisation process has been considered and its study has proved very fruitful. Rather than increasing the dimension of the manifold we increase its complexity in the following sense. Let Q be a manifold of the same dimension as W and consider the connected sum $W\sharp_k Q$ of W with k copies of Q. So far the most important example is when W is the sphere and Q is the two-dimensional torus in which case $W\sharp_k Q$ is a surface of genus k. As k goes to infinity, the cohomology of the classifying space of its diffeomorphism group is understood by the Madsen-Weiss theorem [MW], see also [GMTW]. In this case it is Harer's homology stability theorem for the mapping class group of surfaces that allows us to deduce information for a compact surface. These theorems have recently been generalised by Galatius and Randal-Williams [GRW1], [GRW2] to manifolds W of even dimensions $2d$ for $d > 2$ and with $Q = S^d \times S^d$, and by Perlmutter [Pe] in cases which include also some odd-dimension manifolds.

Homology stability:

Motivated by the second approach to stabilisation, we look at the question of homology stability for diffeomorphisms groups of manifolds more generally. Until the recent paper [GRW2], homology stability for (classifying spaces of) diffeomorphism groups was only known in the case of surfaces, and here only through the homology stability of the associated discrete groups, the mapping class groups – the mapping class groups are homotopy equivalent to the diffeomorphism groups for surfaces of negative Euler characteristic. The purpose of this paper is to show that homology stability holds quite broadly for certain diffeomorphisms groups involving arbitrary manifolds W and Q of both odd and even dimensions. The point of view taken is that the homology stability of the symmetric groups, and more generally of configuration spaces, is fundamental. The stability theorems we prove here are derived from this basic example.

Main results:

Throughout this paper, let W be a smooth, compact, path-connected manifold of dimension $d \geq 2$ with a (parametrised) closed $(d-1)$-dimensional disk ∂_0 in its boundary ∂W. Let

$$\mathrm{Diff}(W) := \mathrm{Diff}(W; \mathrm{rel}\, \partial_0)$$

denote its group of smooth diffeomorphisms that fix ∂_0 point-wise. More precisely, we will assume that any diffeomorphism ψ fixes a collar of a small neighbourhood of ∂_0 in ∂W and furthermore that restricted to a fixed collar of ∂W it is of the form $1 \times \psi|_{\partial W}$. **Note**, as they fix ∂_0 point-wise, the diffeomorphisms will automatically be orientation preserving if W is oriented!

Let W^k denote the manifold W with k distinct marked points $w_1, \ldots, w_k$ in its interior. Consider the subgroup $\mathrm{Diff}^k(W)$ of $\mathrm{Diff}(W)$ consisting of those diffeomorphisms that permute the marked points.

Theorem 1.1. *There are maps $H_*(B\,\mathrm{Diff}^k(W)) \to H_*(B\,\mathrm{Diff}^{k+1}(W))$ which are split injections for all degrees $*$ and isomorphisms in degrees $* \leq k/2$.*

Now fix k closed disjoint disks in $W \setminus \partial W$ and parametrisations $\phi_1, \ldots, \phi_k : D^k \to \partial W \setminus \partial W$ which are compatible with the orientation of W if it is oriented. Then consider the group $\mathrm{Diff}_k(W)$ of diffeomorphisms $\psi \in \mathrm{Diff}(W)$ which commute with these parametrisations in the sense that for some permutation $\sigma \in \Sigma_k$ depending on ψ we have

$$\psi \circ \phi_i = \phi_{\sigma(i)} \quad \text{for all } i = 1, \dots, k.$$

Theorem 1.2. *There are maps $H_*(B\,\mathrm{Diff}_k(W)) \to H_*(B\,\mathrm{Diff}_{k+1}(W))$ which are split injections for all degrees $*$ and isomorphisms in degrees $* \leq k/2$.*

Denote by W_k the manifold W with the interior of the k parametrised disks removed. Let Q be another smooth, connected manifold of dimension d. Remove an open disk and glue k copies of it to W_k to form the connected sum $W\sharp_k Q$. We will be interested in those diffeomorphisms that map W_k (and hence the disjoint union of the k copies of Q_1) onto themselves. More precisely, let $G \subset O(d) \subset \mathrm{Diff}(S^{d-1})$ be a closed subgroup. When W is orientable we assume that $G \subset SO(d)$. Let $\Sigma_G \,\mathrm{Diff}(W\sharp_k Q)$ denote the diffeomorphisms of $W\sharp_k Q$ that map W_k onto itself and restrict to an element of the wreath product $\Sigma_k \wr G$ on the k boundary spheres.

Theorem 1.3. *There are maps $H_*(B\Sigma_G \,\mathrm{Diff}(W\sharp_k Q)) \to H_*(B\Sigma_G \,\mathrm{Diff}(W\sharp_{k+1} Q))$ which are split injections for all degrees $*$ and isomorphisms in degrees $* \leq k/2$.*

We will prove a more general result, Theorem 3.4, which may be thought of as a homology stability theorem for diffeomorphisms of W_k with decorations given by some subgroup of the diffeomorphisms of Q or indeed any other group. For all three theorems we will also prove the stronger statement that the maps of the underlying spaces are stable split injections (in the sense of stable homotopy theory). This uses the results in [MT]. The other basic ingredients are a generalisation of what is known in dimension two as the Birman exact sequence and repeated spectral sequence arguments building on the homology stability of configuration spaces with labels in a fibre bundle. After collecting some preliminary results in section 2, the three theorems above are proved sequentially in section 3.

There are also analogues of these results for mapping class groups, see Theorems 5.1, 5.2, and 5.3. Theorem 5.1 had previously been proved by Hatcher and Wahl [HW]. Theorem 5.2 seems to be new even in the case of surfaces. After establishing the homology stability of the fundamental group of configuration spaces with twisted labels in section 4, the mapping class group analogues of the above theorems are stated and proved in section 5.

In section 6 we collect some variations, generalisations and consequences of our results.

2 Preliminary results

In this section we will recall the homology stability and the stable splitting theorems for configuration spaces with twisted labels. We will also recall some basic results relating to the diffeomorphism groups and describe the spectral sequence argument that we will use repeatedly.

2.1 Homology stability and stable splittings for configuration spaces

Let W be a smooth, compact, path-connected manifold with non-empty boundary ∂W of dimension $d \geq 2$. We fix a collar of ∂W and a $(d-1)$-dimensional disk ∂_0 in ∂W. Furthermore, let $\pi : E \to W$ be a fibre bundle and assume that for each $w \in W$, the fibre E_w over w is path-connected. We define the configuration space of k ordered particles in W with twisted labels in π as

$$\widetilde{C}_k(W;\pi) := \{(\boldsymbol{w},\boldsymbol{x}) \in \mathrm{int}(W)^k \times E^k : \pi(x_i) = w_i, w_i \neq w_j \text{ for } i \neq j, 1 \leq i, j, \leq k\}.$$

and denote by $C_k(W;\pi)$ its orbit space under the natural action by the symmetric group Σ_k by permutations. Here $\mathrm{int}(W)$ denotes the interior of W. When $E = W \times X$ for some space X and π is the projection onto W, this is the more familiar space of configurations of W with labels in X which we also denote by $C_k(W;X)$. When X is a point we use the notation $C_k(W)$. The twisted labels we will primarily be interested in are the tangent bundle $\tau : TW \to W$ and its associated frame bundle Fr.

We now define inclusion maps that allow us to think of the configurations of k points as a subspace of the configuration space of $k+1$ points. Define $W^+ := W \cup \partial W \times [0,1]$ to be W with a collar glued to its boundary, and extend π to a bundle $\pi^+ : E^+ \to W^+$ by pull back along the natural projection $W^+ \to W$. Fix points $w_0 \in \mathrm{int}(\partial_0 \times [0,1])$ and $x_0 \in E^+_{w_0}$. We define a stabilisation map

$$C_k(W;\pi) \xrightarrow{\sigma} C_{k+1}(W^+;\pi^+) \longrightarrow C_{k+1}(W;\pi) \tag{2.1}$$

where the first map adds the particle (w_0, x_0) to any configuration and the second map is a homeomorphism induced by a map that isotopes W^+ into W mapping $\partial W \times [0,1]$ into a collar of ∂W in W while at the same time leaving the complement of the collar in W fixed. We will generally suppress this map and identify W^+ with W.

Let $\mathrm{Diff}(W;\pi)$ denote the group of automorphims of π that cover a diffeomorphism $\beta \in \mathrm{Diff}(W)$ with $\beta^*(\pi)$ and π isomorphic as bundles over W. When π is a trivial bundle or the tangent bundle of W then $\mathrm{Diff}(W;\pi)$ contains $\mathrm{Diff}(W)$. Note that because of our assumption on the restriction of a diffeomorphism to a collar of ∂W there is a natural extension map $\mathrm{Diff}(W;\pi) \to \mathrm{Diff}(W^+;\pi^+)$ mapping ψ to ψ^+ which is defined on the extension by $\psi^+|_{\partial W \times [0,1]} = \psi_{\partial W} \times 1$. In particular, ψ^+ fixes $\pi^+|_{\partial_0 \times [0,1]}$. With these definitions it is easy to check that the inclusion σ is $\mathrm{Diff}(W;\pi)$-equivariant.

We will need the following two theorems.

Theorem 2.1. *The stabilisation map $\sigma : C_k(W;\pi) \to C_{k+1}(W;\pi)$ is $\mathrm{Diff}(W;\pi)$-equivariantly stably split injective.* □

This is well-known in the non-equivariant setting by the Snaith splitting theorem when W is Euclidean space and by a theorem of Bödigheimer [B] for general W. That these splittings are $\mathrm{Diff}(W;\pi)$-equivariant was proved in [MT].

The following is a generalisation of the standard homology stability for configurations spaces in a manifold, see [Se], [McD], [RW1], to configuration spaces with labels in a bundle. Indeed, the proof in [RW1] can be generalised to this case. For details of this proof and alternative proofs we refer to [CP] and [KM].

Theorem 2.2. *The stabilisation map $\sigma : C_k(W;\pi) \to C_{k+1}(W;\pi)$ induces an isomorphism on homology in degrees $* \leq k/2$.* □

2.2 Some lemmata concerning diffeomorphism groups

We will need the following well-known transitivity properties of the diffeomorphism groups.

Lemma 2.3. *(i) Given any two sets $\{x_1, \dots, x_k\}$ and $\{w_1, \dots, w_k\}$ of k points in the interior of the connected manifold W there is a diffeomorphism $\psi \in \mathrm{Diff}(W; \partial_0)$ such that $\psi(x_i) = w_i$ for all $i = 1, \dots, k$.*

(ii) Furthermore, when W is not orientable, given any linear maps $L_i : T_{x_i}W \to T_{w_i}W$, ψ may be chosen so that $D_{x_i}\psi = L_i$. When W is orientable the same is true as long as the L_i are all orientation preserving.

Proof. For (i) choose paths connecting x_i to w_i and consider the associated slides. As $d \geq 2$, the paths and slides can be chosen so as to avoid any of the other points x_j and w_j for $j \neq i$. Performing one slide after the other defines a diffeomorphism ψ with the required properties.

For (ii) first note that by part (i) we may assume that $x_i = w_i$. Next note that if W is non-orientable we can choose a path connecting x_i to itself along which the slide induces an orientation reversing map of tangent spaces. This corresponds to an element in $\pi_1(W; x_i)$ mapping to the non-trivial element in $\mathbb{Z}/2\mathbb{Z} = \pi_1(BGL_d(\mathbb{R}))$ via the map induced by the classifying map of the tangent bundle. Thus, it is enough to construct for any orientation preserving linear map L_i from $T_{x_i}W$ to itself, a diffeomorphism α_i with $D_{x_i}\alpha_i = L_i$ that is the identity outside a small neighbourhood of x_i. [1]

To do so, choose a metric on W and $\epsilon > 0$ small enough so that the exponential map $\exp : T_{x_i}W \to W$ is injective on the ϵ-ball around zero and the closure of its image does not contain x_j for $i \neq j$. Define $\alpha_i : W \to W$ to be the identity outside this image and otherwise by

$$\alpha_i(x) := (\exp \circ \Phi \circ \exp^{-1})(x)$$

for some diffeomorphisms Φ of $T_{x_i}W$ satisfying $D_0\Phi = L_i$ and $\Phi(v) = v$ for $|v| \geq \epsilon$. Then in particular $D_{x_i}\alpha_i = L_i$. It remains to prove Φ exists, and we now sketch the construction of such a Φ.

As L_i is orientable, we may choose a smooth path $A(t)$ in $\mathrm{GL}_d^+(\mathbb{R})$ from L_i to the identity which is defined for $t \in [\epsilon/2, \epsilon]$ and constant near the boundary. For $|v| \in [\epsilon/2, \epsilon]$ define

$$\Phi(v) := A(|v|)(v)(|v|/|A(|v|)(v)|).$$

The multiplication by the scalar $|v|/|A(|v|)(v)|$ ensures that the sphere of radius $|v|$ is mapped to itself. Thus Φ is a diffeomorphism on its domain of definition and we can extend it by the identity for v with $|v| \geq \epsilon$. For $|v| \leq \epsilon/2$ define

$$\Phi(v) = (\rho \circ L_i)(v)$$

where ρ is a suitable diffeomorphisms of $T_{x_i}W$ that maps the convex image under L_i of the $\epsilon/2$-ball back to itself, for example by using a radial contraction/expansion map which is constant around 0. Then Φ is also a diffeomorphism on the $\epsilon/2$-ball and satisfies $D_0\Phi = L_i$. □

The group $\mathrm{Diff}_1(W)$ can be identified with the group $\mathrm{Diff}(W; D \coprod \partial_0)$ of diffeomorphisms of W that fix an embedded closed disk D point-wise in addition to ∂_0, and $\mathrm{Diff}^1(W)$ can be identified with the group $\mathrm{Diff}(W; w_1 \cup \partial_0)$ of diffeomorphisms that fix the additional point w_1. Let $\mathrm{Diff}(W; w_1 \cup \partial_0; T_{w_1}W)$ denote the group of diffeomorphisms which fix the point w_1 and its tangent space. With these identifications there are canonical injective homomorphisms

$$\mathrm{Diff}_1(W) \xrightarrow{\iota} \mathrm{Diff}(W; w_1 \cup \partial_0; T_{w_1}W) \longrightarrow \mathrm{Diff}^1(W)$$

[1] The existence of such maps can also be deduced from (2.2) below with $W = M$ and K the union of k closed disks centered at $x_1, \dots, x_k$.

connecting these three groups, each forgetting some of the structure.

Lemma 2.4. *The map* $\mathrm{Diff}_1(W) \xrightarrow{\iota} \mathrm{Diff}(W; w_1 \cup \partial_0; T_{w_1}W)$ *is a homotopy equivalence.*

In the proof we will repeatedly use the following result, see [Pal]. Let K be a compact submanifold of W and M be another smooth manifold. Then the restriction map from the space of smooth embeddings of W into M to the space of smooth embeddings of K into M is a locally trivial fibre bundle:

$$\mathrm{Emb}\ (W, M) \longrightarrow \mathrm{Emb}\ (K, M). \tag{2.2}$$

When $W = M$ the space of embeddings $\mathrm{Emb}\ (W, M)$ may be replaced by the space of (compactly supported) diffeomorphisms $\mathrm{Diff}(W)$, see [L].

Proof. Using (2.2) with $K = D$, we can show that the following are maps of (horizontal) fibre bundles

$$\begin{tikzcd}
\mathrm{Diff}_1(W) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Diff}(W) \arrow[r] \arrow[d, "="] & \mathrm{Emb}(D, W) \arrow[d, "\simeq"] \\
\mathrm{Diff}^1(W; w_1 \cup \partial_0; T_{w_1}W) \arrow[r] \arrow[d] & \mathrm{Diff}(W) \arrow[r] \arrow[d, "="] & \mathrm{Fr} \arrow[d] \\
\mathrm{Diff}^1(W) \arrow[r] & \mathrm{Diff}(W) \arrow[r] & W
\end{tikzcd}$$

where the right horizontal maps are the restriction maps to D, $Iso(T_0D, T_{w_1}W)$ and w_1. Furthermore, evaluation at $0 \in D$ and projection from the frame bundle to W give rise to compatible fiber bundles

$$\begin{tikzcd}
\mathrm{Emb}(D, 0; W, w_1) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Emb}(D, W) \arrow[r] \arrow[d, "\simeq"] & W \arrow[d, "="] \\
\mathrm{GL}_d(\mathbb{R}) \arrow[r] & \mathrm{Fr} \arrow[r] & W.
\end{tikzcd}$$

We will now argue that the left arrow is a homotopy equivalence, hence so is the middle one, and therefore also the two arrows in the previous diagram which are labelled as homotopy equivalences.

The space of collars for the image of D in W is contractible for any embedding. Hence,

$$\mathrm{Emb}(N(D), 0; W, w_1) \simeq \mathrm{Emb}(D, 0; W, w_1) \tag{2.3}$$

where $N(D)$ is an open disk containing D. The fiber over the identity of the forgetful map

$$\mathrm{Emb}(N(D), 0; W, w_1) \longrightarrow \mathrm{Fr}\,|_{w_1} \simeq \mathrm{GL}_d(\mathbb{R}) \tag{2.4}$$

is the space of tubular neighborhoods of w_1 in W and hence is contractible. (The intuition is that by combing from the origin w_0 any diffeomorphism fixing its tangent space can be homotoped to the identity in some neighbourhood of w_0.) $\square$

We will be interested in a variation of this lemma. Fix $G \subset O(d)$ and denote by $\mathrm{Diff}_G(W_1)$ the group of diffeomorphisms of $W_1 = W \setminus \mathrm{int}(D)$ that restrict on (a neighbourhood of) the boundary sphere to an element in G. There are canonical injective homomorphisms

$$\mathrm{Diff}_1(W) \longrightarrow \mathrm{Diff}_G(W_1) \xrightarrow{\iota} \mathrm{Diff}^1(W)$$

where the left map is the inclusion, which is defined for all G, and the right map is defined by extending the diffeomorphisms canonically from the boundary sphere S^{d-1} to the disk D^d. This is where we use the condition that $G \subset O(d)$. Note that the midpoint w_1 of D^d is fixed and the induced map on its tangent space is the same element of $O(d)$ as that defining the map on S^{d-1}. Let $\mathrm{Diff}^1_G(W)$ denote the subgroup of $\mathrm{Diff}^1(W)$ consisting of those diffeomorphisms whose induced map on the tangent space $T_{w_1}W$ is an element in G.

Lemma 2.5. *The map* $\mathrm{Diff}_G(W_1) \xrightarrow{\iota} \mathrm{Diff}^1_G(W)$ *is a homotopy equivalence. In particular, the map* $\mathrm{Diff}_{O(d)}(W_1) \xrightarrow{\iota} \mathrm{Diff}^1(W)$ *is a homotopy equivalence.*

Proof. Consider the following commutative diagram of fibre bundles

$$\begin{tikzcd}
\mathrm{Diff}_1(W) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Diff}_G(W_1) \arrow[r] \arrow[d, "\iota"] & G \arrow[d, "="] \\
\mathrm{Diff}(W; w_1 \cup \partial_0; T_{w_1}W) \arrow[r] & \mathrm{Diff}^1_G(W) \arrow[r] & G,
\end{tikzcd}$$

where the vertical arrows are the canonical inclusion maps and the right horizontal arrows are given by restriction to the boundary and the tangent space at w_1 respectively. Indeed, the top row is the part of the bundle

$$\mathrm{Diff}(W_1) \longrightarrow \mathrm{Diff}(\partial W_1 \setminus \partial W)$$

lying over the subgroup G. Similarly, the bottom row is the part of the bundle

$$\mathrm{Diff}^1(W) \longrightarrow \mathrm{GL}_d\,\mathbb{R}$$

lying over the subgroup G. As the map on the base spaces is the identity and the map on the fibre spaces is a homotopy equivalence by Lemma 2.4, it follows that also the map of total spaces is a homotopy equivalence. The second statement in the lemma follows from the homotopy equivalence $O(d) \simeq \mathrm{GL}_d(\mathbb{R})$. $\square$

We finish this section with some remarks that put our choices of groups above into context. Taking $W = M = S^d$ and $K = D^d$ in (2.2) we get a fibre bundle

$$\mathrm{Diff}(D^d;\ \mathrm{rel}\ \partial) \longrightarrow \mathrm{Diff}(S^d) \longrightarrow \mathrm{Emb}\ (D^d, S^d) \tag{2.5}$$

for all $d > 0$. The orthogonal group $O(d+1)$ injects via $\mathrm{Diff}(S^d)$ into $\mathrm{Emb}\ (D^d, S^d)$. Following this by the evaluation map at the origin $0 \in D^d$ gives rise to a map of fibre bundles

$$\begin{tikzcd}
O(d) \arrow[r] \arrow[d, "\simeq"] & O(d+1) \arrow[r] \arrow[d, "\simeq"] & S^d \arrow[d, "="] \\
\mathrm{Emb}\ (D^d, 0; S^d, w_1) \arrow[r] & \mathrm{Emb}\ (D^d, S^d) \arrow[r] & S^d
\end{tikzcd}$$

where all vertical maps are homotopy equivalences by (2.3) and (2.4). We thus have a homotopy equivalence of spaces

$$\mathrm{Diff}(S^d) \simeq O(d+1) \times \mathrm{Diff}(D^d;\ \mathrm{rel}\ \partial). \tag{2.6}$$

Note that here $\mathrm{Diff}(S^d)$ includes the orientation reversing diffeomorphisms. Restricting to the orientation preserving diffeomorphisms this is Proposition 4 of [C1], page 127.

Remark 2.6. By theorems of Smale [Sm1] and Hatcher [H] the group $\mathrm{Diff}(D^d, \mathrm{rel}\ \partial)$ is contractible for $d = 2, 3$. For $d \geq 5$, the mapping class group $\pi_0(\mathrm{Diff}(D^d, \mathrm{rel}\ \partial))$ is known to be isomorphic to the group Θ_{d+1} of exotic spheres in dimension $d+1$ by theorems of Smale [Sm2] and Cerf [C2]. The exotic spheres are constructed by a clutching construction that glues two disks D^{d+1} together using elements in $\mathrm{Diff}(D^d; \mathrm{rel}\ \partial) \subset \mathrm{Diff}(S^d)$.

The question arises in our context which diffeomorphism of the k boundary spheres is the restriction of some symmetric diffeomorphism of $W \sharp_k Q$. First note, that if $\alpha \in \mathrm{Diff}_0(S^{d-1})$ is an element in the identity component, then it is such a restriction as any path to the identity defines a diffeomorphisms $\hat{\alpha}$ of $S^{d-1} \times [-1, 1]$ with $\hat{\alpha}|_{S^{d-1}\times\{0\}} = \alpha$ and $\hat{\alpha}|_\partial = 1$ which can be extended to the whole of $W \sharp_k Q$ by the identity. By the Lemma 2.3 then also any element in the wreath product $\Sigma_k \wr \mathrm{Diff}_0(W \sharp_k Q)$ is in the image of the restriction map. When W is not orientable, this holds also if the -1 component defined by the non-zero element in $\pi_0(GL_d(\mathbb{R}))$ is included. Our methods here however force us to restrict ourselves to the case when $G \subset O(d)$ is a closed subgroup.

2.3 Spectral sequence arguments and stable splittings

We will use the following well-known spectral sequence arguments repeatedly.

Lemma 2.7. *Let $f : E^\bullet_{p,q} \to \widetilde{E}^\bullet_{p,q}$ be a map of homological first quadrant spectral sequences. Assume that*

$$f : E^2_{p,q} \xrightarrow{\simeq} \widetilde{E}^2_{p,q} \qquad \text{for } 0 \leq p < \infty \text{ and } 0 \leq q \leq l.$$

Then f induces an isomorphism on the abutments in degrees $* \leq l$.

Proof. The fact that f is an isomorphism on $E^2_{p,q}$ for all p allows us to control higher differentials with targets of bidegree (p, q) with $p + q \leq l$. Also note that that r-th differential has bi-degree $(-p, q - 1)$. So the differentials from terms with bidegree (p, q) and $p + q \leq l$ have targets with total degree no greater than l. The details are left to the reader. □

We can improve the situation when we have further constraints. Consider the following abstract version of the situation described in Theorem 2.1. For $k \geq 1$, let G be a topological group and C_k be G-spaces with G-equivariant maps $C_k \to C_{k+1}$ for $k \geq 0$ that are stably G-equivariantly split injective. Consider the map induced on Borel constructions

$$f_k : EG \times_G C_k \longrightarrow EG \times_G C_{k+1}.$$

We have the following helpful tool.

Lemma 2.8. *If $C_k \to C_{k+1}$ is stably G-equivariantly split injective, then so is f_k. In particular f_k is split injective in homology.*

Proof. The assumptions on the C_k imply that it is G-equivariantly stably homotopy equivalent to $V_k = \bigvee_{j=0}^k D_j$ with $D_j = C_j/C_{j-1}$. The maps $C_k \to C_{k+1}$ correspond G-equivariantly to the split injective maps $V_k \to V_{k+1}$. These also induce split injective maps on the Borel constructions as

$$EG \times_G V_k \simeq \bigvee_{j=0}^{k} EG \ltimes_G D_j$$

where $X \ltimes Y$ denotes the half-smash $X \times Y / X \times *$. □

3 Proofs of the main theorems

We will first define explicit stabilisation maps and then prove the theorems in sequence using the results of the previous section.

3.1 Definition of stabilisation maps

We will construct maps of the diffeomorphisms groups that will induce the maps claimed in the three theorems of the introduction. They correspond on the underlying manifolds to connected sum but more precisely will be constructed by extending diffeomorphisms from W to W^+ as encountered in section 2.1.

As before, let ∂_0 be an embedded $(d-1)$-dimensional disk in the boundary ∂W and Q be another d-dimensional manifold. Let D be a parametrised closed disk in Q and D_0 be a closed disk with centre z_0 contained in $\partial_0 \times [0,1] \subset \partial W \times [0,1]$. Remove the two disks and identify their boundary to form a new manifold Q'. To form the connected sum we take the union of W and Q' and identify ∂W in W with $\partial W \times \{0\}$. Note that the resulting manifold $W \sharp Q$ is indeed diffeomorphic to the usual connected sum of W and Q. It comes equipped with the embedded disk $\partial_0^+ = \partial_0 \times \{1\}$ in its boundary, and we may thus repeat the process to form the connected sum $W \sharp_k Q$ as illustrated in Figure 3.1.

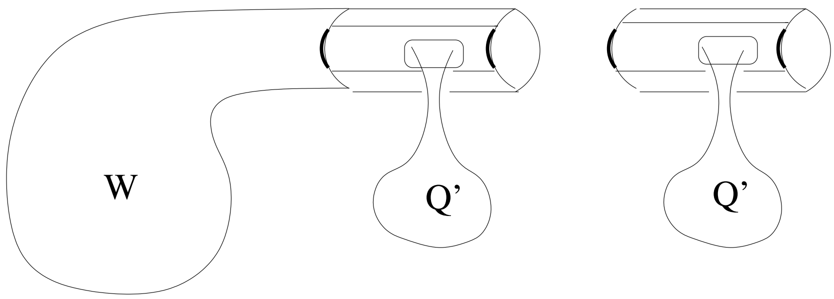

Figure 3.1: The connected sum of W with copies of Q.

The k-fold connected sum $W \sharp_k Q$ contains k copies of $Q_1 = Q \backslash \mathrm{int}(D)$. The group $\Sigma_G \operatorname{Diff}(W \sharp_k Q)$ of the introduction consists of the elements in the group $\operatorname{Diff}(W \sharp_k Q; \coprod_k Q_1; \partial_0)$ of diffeomorphisms that fix ∂_0 point-wise, map $\coprod_k Q_1$ to itself and restrict on each boundary component to an element in $G \subset O(d)$. Let $G \subset \mathrm{SO}(d)$ when W is orientable, or $G \subset O(d)$ when not. When G is the trivial group e, this group is simply the wreath product

$$\operatorname{Diff}_k(W) \wr \operatorname{Diff}_1(Q) = \operatorname{Diff}_k(W) \ltimes (\operatorname{Diff}_1(Q))^k.$$

The diffeomorphisms of $W \sharp_k Q$ that fix ∂_0 point-wise can be extended to diffeomorphisms of $W \sharp_{k+1} Q$ using the identity on the additional copy of Q_1. We have thus well-defined homomorphisms $\sigma : \operatorname{Diff}(W \sharp_k Q) \to \operatorname{Diff}(W \sharp_{k+1} Q)$ that restrict to homomorphisms

$$\sigma : \Sigma_G \operatorname{Diff}(W \sharp_k Q) \longrightarrow \Sigma_G \operatorname{Diff}(W \sharp_{k+1} Q).$$

Note that when $Q_1 = D_0 \setminus z_0$ is a disk with a puncture, $W \sharp_k Q \simeq W^k$, and when $Q_1 = D_0 \setminus \mathrm{int}(D)$ is a disk without its interior, $W \sharp_k Q \simeq W_k$. By abuse of notation we will suppress this diffeomorphism and write

$$\sigma : \operatorname{Diff}^k(W) \longrightarrow \operatorname{Diff}^{k+1}(W) \quad \text{and} \quad \sigma : \operatorname{Diff}_k(W) \longrightarrow \operatorname{Diff}_{k+1}(W)$$

for the resulting stabilisation maps. The iterated stabilisation maps $\sigma^k : \mathrm{Diff}(W) \to \mathrm{Diff}^k(W)$ and $\sigma^k : \mathrm{Diff}(W) \to \mathrm{Diff}_k(W)$ have homotopy left inverses. To define these, identify $\mathrm{Diff}^k(W)$ and $\mathrm{Diff}_k(W)$ with diffeomorphisms that fix a set of k points respectively of k parametrised disks. The left inverse is given by the forgetful maps:

$$\mathrm{Diff}^k(W) \longrightarrow \mathrm{Diff}(W) \quad \text{and} \quad \mathrm{Diff}_k(W) \longrightarrow \mathrm{Diff}(W). \tag{3.1}$$

We note here that by Remark 2.6, Theorem 1.1 and Theorem 1.2 are *not* special instances of Theorem 1.3. In particular, $\Sigma_e \mathrm{Diff}(W \sharp_k D^d)$ is homotopy equivalent to $\mathrm{Diff}_k(W)$ when $d = 2$ or 3 but not in general. Nevertheless, all three are special cases of Theorem 3.4 below.

We will prove the three theorems of the introduction in sequential order in the next sections.

3.2 Proof of Theorem 1.1

We will prove the following slightly stronger statement in which the split injection of homology groups is promoted to a stable (in the sense of stable homotopy theory) split injection of spaces.

Theorem 3.1. *The stabilisation map $\sigma : B\,\mathrm{Diff}^k(W) \to B\,\mathrm{Diff}^{k+1}(W)$ is stably split injective and a homology isomorphism in degrees $* \leq k/2$.*

Proof. Fix k points $x_1, \dots, x_k$ in the interior of W and consider the evaluation map

$$E_k : \mathrm{Diff}(W) \longrightarrow C_k(W)$$

which maps a diffeomorphism ψ to $\psi(x_1), \dots, \psi(x_k)$. Note that this map is surjective as by Lemma 2.4 (i) the diffeomorphism group of W is k-transitive. It is a fibre bundle with fibre $\mathrm{Diff}^k(W)$ by [Pal]; see (2.2). Extending this fibration twice to the right we get a new fibration

$$C_k(W) \longrightarrow B\,\mathrm{Diff}^k(W) \longrightarrow B\,\mathrm{Diff}(W). \tag{3.2}$$

This fibration can be constructed geometrically as follows. Fix an embedding $\partial_0 \to \{0\} \times \mathbb{R}^\infty$ and consider the space of smooth embeddings

$$E\,\mathrm{Diff}(W;) := \mathrm{Emb}^{\partial_0}(W; (\infty, 0] \times \mathbb{R}^\infty)$$

extending it and which furthermore are of the form $1 \times h|_{\partial W}$ near $(-\epsilon, 0] \times \mathbb{R}^\infty$. This is a contractible space by the Whitney embedding theorem and the diffeomorphism group $\mathrm{Diff}(W)$ acts freely on it. The orbit space, the space of embedded manifolds W' diffeomorphic to W, is a model for $B\,\mathrm{Diff}(W)$. Similarly, the orbit space for the restricted action of $\mathrm{Diff}^k(W)$, the space of embedded manifolds W' with k marked points, gives a model for $B\,\mathrm{Diff}^k(W)$. With these models the right hand map in the above fibration (3.2) is the map that forgets the marked points on the embedded manifold and the fibre over W' is its k-fold configuration space.

The fibration is compatible with stabilisation in the sense that we have maps of fibrations

$$\begin{tikzcd}
C_k(W) \arrow[r] \arrow[d, "\sigma"] & B\,\mathrm{Diff}^k(W) \arrow[r] \arrow[d, "\sigma"] & B\,\mathrm{Diff}(W) \arrow[d, "\sigma"] \\
C_{k+1}(W) \arrow[r] & B\,\mathrm{Diff}^{k+1}(W) \arrow[r] & B\,\mathrm{Diff}(W)
\end{tikzcd}$$

with a homotopy equivalence of the base spaces. This gives rise to a map of spectral sequences

$$\begin{array}{ccc} E^2_{**} = H_*(B\operatorname{Diff}(W); H_*(C_k(W))) & \Longrightarrow & H_*(B\operatorname{Diff}^k(W)) \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ E^2_{**} = H_*(B\operatorname{Diff}(W); H_*(C_{k+1}(W))) & \Longrightarrow & H_*(B\operatorname{Diff}^{k+1}(W)) \end{array}$$

By Theorem 2.2, the map of E^2_{pq}-terms is an isomorphism for $q \leq k/2$ and all p. Hence, by Lemma 2.7, it is also an isomorphism of the total spaces in degrees $* \leq k/2$ and the homological statement Theorem 1.1 follows.

Our geometric model for the fibration (3.2) identifies

$$B\operatorname{Diff}^k(W) \simeq E\operatorname{Diff}(W) \times_{\operatorname{Diff}(W)} C_k(W).$$

Thus Theorem 2.1 and Lemma 2.8 imply furthermore that the stabilisation map σ is stably split injective. $\square$

3.3 Proof of Theorem 1.2

We will prove the following slightly stronger statement.

Theorem 3.2. *The stabilisation map $\sigma : B\operatorname{Diff}_k(W) \to B\operatorname{Diff}_{k+1}(W)$ is stably split injective and a homology isomorphism in degrees $* \leq k/2$.*

Proof. As in the proof of Theorem 3.1, we fix k points $\boldsymbol{x} = (x_1, \ldots, x_k)$ in the interior of W. We modify the evaluation map to also record the map of tangent spaces at these points. Thus a diffeomorphism ψ is mapped to $(\psi(x_1), \ldots, \psi(x_k); D_{x_1}\psi, \ldots, D_{x_k}\psi)$ with each $D_{x_i}\psi \in \operatorname{Iso}(T_{x_i}W, T_{\psi(x_i)}W)$. After fixing an isomorphism $T_{x_i}W \simeq \mathbb{R}^d$ for each i this defines a map

$$E^T_k : \operatorname{Diff}(W) \longrightarrow C_k(W; \mathrm{Fr}),$$

to the twisted configuration space where $\mathrm{Fr} : \operatorname{Iso}(\mathbb{R}^d, -) \to W$ is the frame bundle of the tangent bundle of W with fibre $\operatorname{Iso}(\mathbb{R}^d; T_wW)$ over w. It is understood here that in the case when W is oriented, the linear isomorphisms preserve the orientation and the fibre is isomorphic to the group of orientation preserving invertible matrices $\mathrm{GL}^+_d(\mathbb{R})$. Otherwise it is isomorphic to the whole linear group $\mathrm{GL}_d(\mathbb{R})$. By Lemma 2.3.(ii) the evaluation map E^T_k is surjective and its fibre over $\boldsymbol{x}$ and the identity linear transformations is

$$\operatorname{Diff}(W; \{x_1, \ldots, x_k, T_{x_1}W, \ldots, T_{x_k}W\}; \partial_0),$$

the group of diffeomorphisms that fix ∂_0 point-wise, fix the points $\{x_1, \ldots, x_k\}$ as a set and their tangent spaces point-wise up to permutation. By an obvious extension of Lemma 2.4, this group is homotopic to $\operatorname{Diff}_k(W)$.

The remainder of the proof follows now the pattern of the proof of Theorem 1.1. We consider the associated fibre bundle

$$C_k(W; \mathrm{Fr}) \longrightarrow B\operatorname{Diff}_k(W) \longrightarrow B\operatorname{Diff}(W). \tag{3.3}$$

The stabilisation maps commute with all maps in the fibre sequence and hence induce a map of associated Serre spectral sequences with E^2-term

$$E^2_{**} = H_*(B\operatorname{Diff}(W); H_*(C_k(W; \mathrm{Fr}))).$$

By Theorem 2.2 with twisted labels in the frame bundle Fr, the stabilisation map induces an isomorphism on homology groups $H_*(C_k(W;\mathrm{Fr}))$ in degrees $* \leq k/2$, thus on the E^2-term of the Serre spectral sequence and the abutment by Lemma 2.7, which proves the homological statement Theorem 1.2.

A geometric model for the total space is given by the space of embedded manifolds in infinite Euclidean space with k marked points and framings for their tangent spaces, that is

$$B\operatorname{Diff}_k(W) \simeq \operatorname{Emb}^{\partial_0}(W;(\infty,0]\times\mathbb{R}^\infty)) \times_{\operatorname{Diff}(W)} C_k(W;\mathrm{Fr}).$$

Recall that it is understood that if W is oriented then the framings are compatible with the orientation. We note here that the action of the diffeomorphism group on the label space is non-trivial. The full strength of Theorem 2.1 and Lemma 2.8 imply the stable splitting. $\square$

3.4 Proof of Theorem 1.3

To prove Theorem 1.3 we will consider two fibration sequences. Recall from the previous section that the evaluation map gives rise to a fibration (up to homotopy)

$$\operatorname{Diff}_k(W) \longrightarrow \operatorname{Diff}(W) \xrightarrow{E_k^T} C_k(W;\mathrm{Fr}).$$

The second fibration is

$$P\operatorname{Diff}_k(W) \longrightarrow P_G\operatorname{Diff}(W_k) \longrightarrow G^k$$

where the right map sends a diffeomorphism to its restriction to the boundary spheres; $P\operatorname{Diff}_k(W)$ and $P_G\operatorname{Diff}(W_k)$ denote the subgroups of $\operatorname{Diff}_k(W)$ and $\Sigma_G\operatorname{Diff}(W_k)$ that do not permute the boundary spheres. Extending the fibrations twice to the right and dividing out by the (free) symmetric group action on base and total space, we can combine them into the following diagram where each column and row is a fibration (up to homotopy):

$$\begin{tikzcd}
G^k \arrow[r] \arrow[d, "="] & C_k(W;\mathrm{Fr}) \arrow[r] \arrow[d] & C_k(W;\mathrm{Fr}/G) \arrow[d] \\
G^k \arrow[r] \arrow[d] & B\operatorname{Diff}_k(W) \arrow[r] \arrow[d] & B\Sigma_G\operatorname{Diff}(W_k) \arrow[d] \\
* \arrow[r] & B\operatorname{Diff}(W) \arrow[r, "="] & B\operatorname{Diff}(W).
\end{tikzcd}$$

Here the bundle Fr/G over W is the fibrewise quotient space of the frame bundle Fr by the natural action of G. Note that the right column is associated to the evaluation fibration

$$E_k^G : \operatorname{Diff}(W) \longrightarrow C_k(W;\mathrm{Fr}/G)$$

which is E_k^T from (3.3) composed with the quotient map. As in the proof of Theorem 3.2, we can now apply Theorem 2.2 and Lemma 2.7 as well as Theorem 2.1 and Lemma 2.8 to the fibration of the right column to prove the following proposition which generalises both Theorems 3.1 and 3.2.

Theorem 3.3. *For $G \subset O(d)$ a closed subgroup, the map $\sigma : B\Sigma_G\operatorname{Diff}(W_k) \to B\Sigma_G\operatorname{Diff}(W_{k+1})$ is a stably split injection and induces an isomorphism in homology of degrees $* \leq k/2$.* $\square$

We will now 'decorate' the diffeomorphism group of W_k at the embedded parametrised disks with a label group H which comes equipped with a continuous surjective homomorphism $\rho : H \to G$. Define $\Sigma_G^H \operatorname{Diff}_k(W)$ to be the pullback under the induced map on wreath products and the natural restriction map $\Sigma_G \operatorname{Diff}_k(W) \to \Sigma_k \wr G$:

$$\begin{tikzcd}
\Sigma_G^H \operatorname{Diff}_k(W) \arrow[r] \arrow[d] & \Sigma_G \operatorname{Diff}_k(W) \arrow[d] \\
\Sigma_k \wr H \arrow[r, "\rho"] & \Sigma_k \wr G.
\end{tikzcd}$$

Write H_1 for the kernel of ρ. Then there is a fibration sequence

$$BH_1 \longrightarrow O(d) \times_G BH_1 \longrightarrow O(d)/G$$

where we take $BH_1 = EH/H_1$ so that it has a free G action. Combining this with the diagram above we have the following (up to homotopy) commutative diagram with fibrations in each column and row

$$\begin{tikzcd}
BH_1^k \arrow[r] \arrow[d, "="] & C_k(W; \operatorname{Fr} \times_G BH_1) \arrow[r] \arrow[d] & C_k(W; \operatorname{Fr}/G) \arrow[d] \\
BH_1^k \arrow[r] \arrow[d] & B\Sigma_G^H \operatorname{Diff}_k(W) \arrow[r] \arrow[d] & B\Sigma_G \operatorname{Diff}(W_k) \arrow[d] \\
* \arrow[r] & B \operatorname{Diff}(W) \arrow[r, "="] & B \operatorname{Diff}(W).
\end{tikzcd} \tag{3.4}$$

Thus we have a description of $B\Sigma_G^H \operatorname{Diff}_k(W)$ as a fibre bundle over $B \operatorname{Diff}(W)$ with fibre the configuration space $C_k(W; \operatorname{Fr} \times_G BH_1)$. As in the proof of Theorem 3.2, an application of Theorem 2.2 and Lemma 2.7 and invoking the full strength of Theorem 2.1 and Lemma 2.8 gives the following most general statement.

Theorem 3.4. *The stabilisation map $\sigma : B\Sigma_G^H \operatorname{Diff}_k(W) \to B\Sigma_G^H \operatorname{Diff}_{k+1}(W)$ is a stably split injection and an isomorphism in homology of degrees $* \le k/2$.* □

It is now easy to see that our previous results are special cases of this. Let $H = \operatorname{Diff}_G(Q_1)$ be the group of diffeomorphisms of Q_1 that restrict to an element of G on the spherical boundary component created by deleting a disk from Q. Then

$$\Sigma_G^H \operatorname{Diff}_k(W) = \Sigma_G \operatorname{Diff}(W \sharp_k Q).$$

Thus Theorem 3.4 specialises to Theorem 1.3. When $H = \{e\} = G$ is the trivial group, then

$$\Sigma_G^H \operatorname{Diff}_k(W) = \operatorname{Diff}_k(W)$$

and Theorem 3.4 gives Theorems 1.2 and 3.2. To recover Theorems 1.1 and 3.1 put $H = G = O(n)$ respectively $H = G = SO(n)$ in the orientable case with ρ the identity homomorphisms. Then, by Lemma 2.5, we have a homotopy equivalence

$$\Sigma_G^H \operatorname{Diff}_k(W) \simeq \operatorname{Diff}^k(W).$$

4 Fundamental groups of configuration spaces

In order to prove the mapping class group analogues of our main results we will need the analogues of Theorem 2.1 and 2.2. The role of the configuration spaces will be taken by their fundamental groups which we will now compute first.

Lemma 4.1. *Assume* $\dim(W) \geq 3$ *and let* $\pi : E \to W$ *be a fibre bundle on* W *with path-connected fibre* F*. Then*

$$\pi_1(C_k(W;\pi)) = \Sigma_k \wr \pi_1(E).$$

For $\pi = id : W \to W$ this is the braid group of W and the result is well-known. For completeness we include here a proof of the more general case.

Proof. Consider the fibre bundle for the ordered configuration spaces

$$E \setminus E|_{\boldsymbol{w}} \longrightarrow \widetilde{C}_{k+1}(W;\pi) \xrightarrow{eval_{1,\dots,k}} \widetilde{C}_k(W;\pi) \tag{4.1}$$

where the map $eval_{1,\dots,k}$ remembers the first k points of the ordered configurations and forgets the last one. The fibre over the configuration $\boldsymbol{x} = \{x_1,\dots,x_k\}$ with $\pi(\boldsymbol{x}) = \boldsymbol{w}$ in $\widetilde{C}_k(W)$ is a point in E which is not in any of the fibres $E|_{\boldsymbol{w}}$ over $\boldsymbol{w}$. The stabilisation map for ordered configuration spaces $\widetilde{\sigma} : \widetilde{C}_k(W;\pi) \to \widetilde{C}_{k+1}(W;\pi)$ defines a section (up to isotopy) of the above fibre bundle. Hence there is a split exact sequence of fundamental groups

$$0 \longrightarrow \pi_1(E \setminus E|_{\boldsymbol{w}}) \longrightarrow \pi_1(\widetilde{C}_{k+1}(W;\pi)) \xrightarrow{eval_{1,\dots,k}} \pi_1(\widetilde{C}_k(W;\pi)) \longrightarrow 0. \tag{4.2}$$

The natural inclusion $incl : E \setminus E|_{\boldsymbol{w}} \hookrightarrow E$ factors as the composition of the inclusion of the fibre in (4.1) and the map $eval_{k+1} : \widetilde{C}_{k+1}(W;\pi) \to E$ that maps an ordered configuration $\boldsymbol{x} = \{x_1,\dots,x_k,x_{k+1}\}$ to its last point x_{k+1}.

Claim. *The inclusion* $incl : E \setminus E|_{\boldsymbol{w}} \hookrightarrow E$ *induces an isomorphism on fundamental groups.*

Proof of claim. Note that $incl$ defines a map of fibre bundles covering the inclusion $incl_W : W \setminus \boldsymbol{w} \hookrightarrow W$. We thus have a map of exact sequences of homotopy groups

$$\begin{tikzcd}
\pi_2(W \setminus \boldsymbol{w}) \arrow[r] \arrow[d, "incl_W"] & \pi_1(F) \arrow[r] \arrow[d, "="] & \pi_1(E \setminus E|_{\boldsymbol{w}}) \arrow[r] \arrow[d] & \pi_1(W \setminus \boldsymbol{w}) \arrow[r] \arrow[d, "incl_W"] & \pi_0(F) \arrow[d, "="] \\
\pi_2(W) \arrow[r] & \pi_1(F) \arrow[r] & \pi_1(E) \arrow[r] & \pi_1(W) \arrow[r] & \pi_0(F).
\end{tikzcd}$$

As $\dim(W) \geq 3$, $incl_W$ induces an isomorphism on fundamental groups and a surjection on second homotopy groups. An application of the Five Lemma finishes the proof of the claim. □

Using the claim we see that the group extension (4.2) is doubly split and thus trivial:

$$\pi_1(\widetilde{C}_{k+1}(W;\pi)) \simeq \pi_1(E) \times \pi_1(\widetilde{C}_k(W;\pi)).$$

Hence, by induction, starting with $\widetilde{C}_1(W;\pi) = \pi_1(E)$, we get $\pi_1(\widetilde{C}_k(W;\pi)) \simeq \pi_1(E)^k$.

To deduce the result we want for unordered configurations spaces consider the fibre bundle

$$\Sigma_k \longrightarrow \widetilde{C}_k(W;\pi) \longrightarrow C_k(W;\pi)$$

and the associated short exact sequence of groups

$$\pi_1(\widetilde{C}_k(W;\pi)) \longrightarrow \pi_1(C_k(W;\pi)) \longrightarrow \Sigma_k. \tag{4.3}$$

Let $D \subset W$ be an embedded disk of codimension zero. The restricted bundle $\pi|_D$ is trivialisable and hence in particular has a section. We can thus define inclusions $C_k(D) \hookrightarrow C_k(D; \pi|_D) \hookrightarrow C_k(W; \pi)$. As $\dim(W) \geq 3$, $\pi_1(C_k(D)) = \Sigma_k$ and the inclusion gives a splitting of the short exact sequence (4.3). By definition Σ_k acts by permuting the k points in E and we can deduce that the fundamental group of $C_k(W; \pi)$ is the wreath product. $\square$

Consider a general wreath product $\Sigma_k \wr G$ for a group G. Its classifying space has a configuration space model given by

$$B(\Sigma_k \wr G) \simeq E\Sigma_k \times_{\Sigma_k} (BG)^k \simeq C_k(\mathbb{R}^\infty; BG).$$

The automorphism group $\mathrm{Aut}(G)$ of G acts on BG by the naturallity of the bar construction. Applications of Theorem 2.1 and Theorem 2.2 give the following general result:

Lemma 4.2. *The stabilisation map $B(\Sigma_k \wr G) \to B(\Sigma_{k+1} \wr G)$ is (i) Aut(G)-equivariantly split injective and (ii) induces an isomorphisms in homology in degrees $* \leq k/2$.* $\square$

We note now that the mapping class group $\Gamma(W; \pi) := \pi_0(\mathrm{Diff}(W; \pi))$ acts on the fundamental group of W and more generally on the fundamental groups of the configuration spaces $C_k(W; \pi)$; here we choose the base-point to be on ∂_0 so that it is fixed by the elements in $\mathrm{Diff}(W; \pi)$. Applying Lemma 4.2 with $G = \pi_1(E)$ and $\Gamma(W; \pi) \subset \mathrm{Aut}(G)$ and invoking Lemma 4.1 gives the following desired result.

Corollary 4.3. *Assume $\dim(W) \geq 3$ and let $\pi : E \to W$ be a fibre bundle on W with path-connected fibres. The stabilisation map $\sigma : \pi_1(C_k(W; \pi)) \to \pi_1(C_{k+1}(W; \pi))$ induces (i) a $\Gamma(W; \pi)$-equivariant stably split injective map on classifying spaces; and (ii) an isomorphism on homology in degrees $* \leq k/2$.* $\square$

5 Homology stability for mapping class groups

We prove here the analogues of the homology stability of diffeomorphism groups for the associated mapping class groups. Theorem 5.1 was also proved by Hatcher and Wahl [HW] with different methods. When W is a surface, which we assume has non-empty boundary, the mapping class groups are homotopic to the diffeomorphism groups, see [ES]. So the analogues for the mapping class groups follow immediately from the results for the diffeomorphism groups proved in section 3. (See also [BT] for results closely related to Theorem 5.1 and 5.2.) We therefore have to prove the statements below only in the case when $\dim(W) \geq 3$ and Corollary 4.3 applies.

Let $\Gamma^k(W) := \pi_0(\mathrm{Diff}^k(W))$ be the mapping class group of W with k punctures.

Theorem 5.1. *The map $\sigma : B\Gamma^k(W) \to B\Gamma^{k+1}(W)$ is a stable split injection and an isomorphism in homology for degrees $* \leq k/2$.*

Define $\Gamma_k(W) := \pi_0(\mathrm{Diff}_k(W))$.

Theorem 5.2. *The map $\sigma : B\Gamma_k(W) \to B\Gamma_{k+1}(W)$ is a stable split injection and an isomorphism in homology for degrees $* \leq k/2$.*

Define $\Sigma_G\Gamma(W\sharp_k Q) := \pi_0(\Sigma_G \mathrm{Diff}(W\sharp_k Q))$ where $G \subset O(d) \subset \mathrm{Diff}(S^{d-1})$ is a closed subgroup. More generally, for any group H and homomorphisms $\rho : H \to G$ define

$$\Sigma_G^H \Gamma_k(W) := \pi_0(\Sigma_G^H \mathrm{Diff}_k(W)).$$

As explained at the end of section 3, the following generalises Theorems 5.1 and 5.2.

Theorem 5.3. *The map $\sigma : B\Sigma_G^H\Gamma_k(W) \to B\Sigma_G^H\Gamma_{k+1}(W)$ is a stable split injection and an isomorphism in homology for degrees $* \leq k/2$.*

Proof. From the fibration of the middle column of diagram (3.4)

$$C_k(W; \mathrm{Fr} \times_G BH_1) \longrightarrow B\Sigma_G^H \operatorname{Diff}_k(W) \longrightarrow B\operatorname{Diff}(W)$$

we get a long exact sequence of homotopy groups

$$\pi_1(\Sigma_G^H \operatorname{Diff}_k(W)) \longrightarrow \pi_1(\operatorname{Diff}(W)) \longrightarrow \pi_1(C_k(W; \mathrm{Fr} \times_G BH_1)) \longrightarrow \Sigma_G^H\Gamma_k(W) \longrightarrow \Gamma(W) \longrightarrow 0.$$

Using the maps in (3.1) we see that the first map is split surjective, and hence we have a short exact sequence of discrete groups

$$0 \longrightarrow \pi_1(C_k(W; \mathrm{Fr} \times_G BH_1)) \longrightarrow \Sigma_G^H\Gamma_k(W) \longrightarrow \Gamma(W) \longrightarrow 0.$$

For different k these short exact sequences are compatible with the stabilisation maps. Consider the Leray-Serre spectral sequence

$$E^2_{p,q} = H_p(\Gamma(W), H_q(\pi_1(C_k(W; \mathrm{Fr} \times_G BH_1)))) \Longrightarrow H_{p+q}(\Sigma_G^H\Gamma_k(W)).$$

The result now follows from Corollary 4.3 and an application of Lemmas 2.7 and 2.8. □

6 Variations and extensions

Homology stability of configuration spaces has been studied extensively in recent years and the different versions and strengthenings of Theorem 2.2 lead immediately to variants of our main theorems. We mention a few of them here.

6.1 Changing coefficients

The stability range of $k/2$ in our main theorems is a direct consequence of the stability range for the homology of the configuration spaces in Theorem 2.2. The latter stability range can be improved when we change the coefficients of the homology theory. For example, in [CP] it is shown that for $\dim W \geq 3$ the stability range can be improved to $* \leq k-1$ if we take homology with coefficients in $\mathbb{Z}[1/2]$. Hence, Theorems 1.1, 1.2, 1.3 and their mapping class group analogues Theorems 5.1, 5.2, 5.3 hold for such manifolds with this improved stability range. See [CP] and papers cited there for a discussion of field coefficients more generally.

6.2 Boundary conditions and subgroups

In order to define stabilisation maps, we considered only diffeomorphisms that fix point-wise a disk ∂_0 in the boundary of ∂W. We may introduce any other compatible boundary conditions on the diffeomorphisms and analogues of our main theorems will hold.

We may also consider any subgroups of $\operatorname{Diff}(W)$ as long as the evaluation map E_k restricts to a surjective map onto $C_k(W)$ and induces homotopy fibre sequences analogous to those in (3.2) and (3.3). Similarly, Theorem 3.4 allows us already to replace the group $\operatorname{Diff}_G(Q_1)$ by any of its subgroups as long as the restriction to the boundary defines a fibration over G.

6.3 Replacing the symmetric by the alternating group

The three types of diffeomorphism groups that we consider all come equipped with a natural surjection to the symmetric group Σ_k by considering the permutation of the k punctures, deleted disks and copies of $Q \setminus \operatorname{int}(D)$ respectively. This defines in particular short exact sequences of groups:

$$P_G \operatorname{Diff}(W\sharp_k Q) \longrightarrow \Sigma_G \operatorname{Diff}(W\sharp_k Q) \longrightarrow \Sigma_k.$$

We refer to these therefore as symmetric diffeomorphism groups as in the title. Similarly, we can define alternating diffeomorphism groups as subgroups of the symmetric diffeomorphism groups fitting into the extension sequence

$$P_G \operatorname{Diff}(W\sharp_k Q) \longrightarrow A_G \operatorname{Diff}(W\sharp_k Q) \longrightarrow A_k.$$

Here $A_k \subset \Sigma_k$ denotes the alternating group. In [P] Palmer considered oriented configuration spaces with labels in a space X. These are defined as the orbit spaces

$$C_k^+(W;X) := \widetilde{C}_k(W;X)/A_k$$

of ordered configuration spaces under the action of the alternating group A_k. Palmer proves homology stability for these spaces: the stabilisation maps induce in homology isomorphisms for $* \leq (k-5)/3$ and surjections for $* \leq (k-2)/3$. This is enough for the analogues of Theorems 1.1, 1.2 and 1.3 with these stability conditions to be proved for the alternating diffeomorphism and mapping class groups *as long as* W is parallelisable and the frame bundle Fr is trivial.

6.4 Group completion

Take $W = D$ to be the d-dimensional disk. The union

$$\mathbb{M} := \coprod_{k\geq 0} B\Sigma_G \operatorname{Diff}(D\sharp_k Q)$$

forms a monoid induced by connected sum. Taking models of $B\Sigma_G \operatorname{Diff}(D\sharp_k Q)$ of embedded manifolds in Euclidean space, one can construct an action of the standard little d-disk operad on $\mathbb{M}$. Thus its group completion $\Omega B\mathbb{M}$ is a d-fold loop space. As $d \geq 2$ the monoid $\mathbb{M}$ is therefore homotopy commutative. Define

$$\Sigma_G \operatorname{Diff}(D\sharp_\infty Q) := \operatorname{hocolim}_k \Sigma_G \operatorname{Diff}(D\sharp_k Q).$$

By the strengthened group completion theorem, see [RW2] [MP] and references therein, the fundamental group of the classifying space of the monoid $\mathbb{M}$ has perfect commutator subgroup and

$$\Omega B\mathbb{M} \simeq \mathbb{Z} \times B\Sigma_G \operatorname{Diff}(D\sharp_\infty Q)^+; \tag{6.1}$$

here X^+ denotes the Quillen plus construction of X with respect to the maximal perfect subgroup of the fundamental group $\pi_1(X)$. As this space is a homotopy commutative H-space, its homology is a bi-commutative Hopf algebra. The structure of these has been studied in [MM]. In particular, if its homology with rational coefficients is of finite type then it is a graded polynomial algebra, that is a polynomial algebra on even dimensional generators tensor an exterior algebra on odd dimensional generators. By the homology stability theorem, Theorem 1.3, we deduce that

$$H_*(B\Sigma_G \operatorname{Diff}(D\sharp_k Q);\mathbb{Q}) = \mathbb{Q}[\, q_i\,; i \in I]$$

is a polynomial algebra in degrees less or equal to $k/2$ on generators q_i indexed by some set I. Similarly, we may consider the monoid built from mapping class groups,

$$\mathbb{M}_Q^\Gamma := \coprod_{k\geq 0} B\Sigma_G\Gamma(D\sharp_k Q).$$

Define $\Sigma_G\Gamma(D\sharp_\infty Q) := \lim_k \Sigma_G\Gamma(D\sharp_k Q)$. Then by the same reasoning as above, its commutator subgroup is perfect and

$$\Omega B\mathbb{M}^\Gamma \simeq \mathbb{Z} \times B\Sigma_G\Gamma(D\sharp_\infty Q)^+ \tag{6.2}$$

is a d-fold loop space. If its rational homology is of finite type then it is a graded polynomial algebra.

References

[B] C.-F. Bödigheimer, *Stable splittings of mapping spaces*, Algebraic topology (Seattle, Wash., 1985), Lecture Notes in Math., vol. 1286, Springer, Berlin, 1987, pp. 174–187.

[BT] Carl-Friedrich Bödigheimer and Ulrike Tillmann, *Stripping and splitting decorated mapping class groups.* Cohomological methods in homotopy theory (Bellaterra, 1998), 47–57, Progr. Math., **196**, Birkhuser, Basel, 2001.

[CP] Federico Cantero and Martin Palmer, *On homological stability for configuration spaces on closed background manifolds*, arXiv:1406.4916

[C1] Jean Cerf, *Sur les difféomorphismes de la sphere de dimension trois,* $\Gamma_4 = 0$, Springer Lect. Notes Math. **53** (1968).

[C2] Jean Cerf, *La stratification naturelle des espaces de fonctions différentiables réelles et le théorme de la pseudo-isotopie*, (French) Inst. Hautes tudes Sci. Publ. Math. No. **39** (1970), 5–173.

[ES] Clifford F. Earle and A. Schatz, *Teichmüller theory for surfaces with boundary*, J. Differential Geometry **4** (1970) 169–185.

[GMTW] Soren Galatius, Ib Madsen, Ulrike Tillmann and Michael Weiss, *The homotopy type of the cobordism category*, Acta Math. **202** (2009), no. 2, 195-239.

[GRW1] Soren Galatius and Oscar Randal-Williams, *Stable moduli spaces of high dimensional manifolds*, Acta Math. **212** (2014), no. 2, 257–377.

[GRW2] Soren Galatius and Oscar Randal-Williams, *Homological stability for moduli spaces of high dimensional manifolds*, arXiv:1203.6830

[H] Allen Hatcher, *A proof of the Smale conjecture,* $\mathrm{Diff}(S^3) \simeq O(4)$, Ann. of Math. (2) **117** (1983), no. 3, 553–607.

[HW] Allen Hatcher and Nathalie Wahl, *Stabilization for mapping class groups of 3-manifolds*, Duke Math. J. **155** (2010), no. 2, 205-269.

[I] Kiyoshi Igusa, *The stability theorem for smooth pseudoisotopies*, K-Theory **2** (1988), no. 1-2, vi+355pp

[KM] Alexander Kupers and Jeremy Miller, *E_n-cell attachments and a local-to-global principle for homological stability*, arXiv:1405.7087

[L] Elon L. Lima, *O*n the local triviality of the restriction map for embeddings, Comment. Math. Helv. **38** (1964), 163–164.

[MW] Ib Madsen and Michael Weiss, *The stable moduli space of Riemann surfaces: Mumford's conjecture*, Ann. of Math. (2) **165** (2007), no. 3, 843–941.

[MT] Richard Manthorpe and Ulrike Tillmann, *Tubular configurations: equivariant scanning and splitting*, Jour. London Math. Soc. **90** (2014), 940–962.

[McD] Dusa McDuff, *Configuration spaces of positive and negative particles*, Topology **14** (1975), 91–107.

[MP] Jeremy Miller and Martin Palmer, *A twisted homology fibration criterion and the twisted group-completion theorem*, arXiv:1409.4389.

[MM] John W. Milnor and John C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) bf 81 (1965), 211-264.

[Pal] Richard Palais, *Local triviality of the restriction map for embeddings*, Comment. Math. Helv. **3**4 (1960) 305-312.

[P] Martin Palmer, *Homological stability for oriented configuration spaces*, Trans. Amer. Math. Soc. **365** (2013), no. 7, 3675-3711.

[Pe] Nathan Perlmutter, *Homological stability for the moduli spaces of products of spheres*, to be published in Trans. Amer. Math. Soc.

[RW1] Oscar Randal-Williams, *Homological stability for unordered configuration spaces*, Q. J. Math. **64** (2013), no. 1, 303-326.

[RW2] Oscar Randal-Williams, *'Group-completion', local coefficient systems and perfection*, Q. J. Math. **64** (2013), no. 3, 795–803.

[Se] Graeme Segal, *The topology of spaces of rational functions*, Acta Math. **143** (1979), no. 1-2, 39–72.

[Sm1] Stephen Smale, *Diffeomorphisms of the 2-sphere*, Proc. Amer. Math. Soc. **10** (1959), 621–626.

[Sm2] Stephen Smale, *Generalized Poincaré's conjecture in dimensions greater than four*, Ann. of Math. (2) **74** (1961) 391–406.

[W] Friedhelm Waldhausen, *Algebraic K-theory of spaces* in Algebraic and geometric topology (New Brunswick, N.J., 1983), 318-419, Lecture Notes in Math., 1126, Springer, Berlin, 1985.

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