Notes of a Numerical Analyst

Double Exponential Bump Functions

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I want to tell you about my new favourite function,

$$\tau(x) = \tanh\left(\frac{\pi}{2}\sinh(3.2x)\right), \qquad (1)$$

and its derivative rescaled to height 1,

$$d(x) = \cosh(3.2x)\,\mathrm{sech}^2\left(\frac{\pi}{2}\sinh(3.2x)\right). \qquad (2)$$

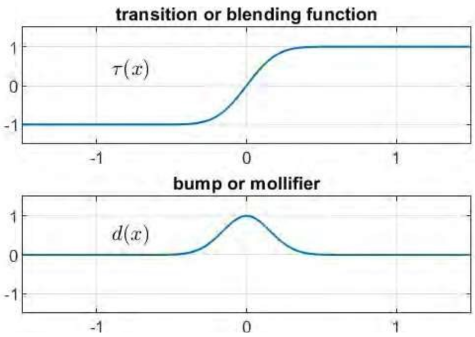

Figure 1. Double exponential blending and bump functions.

These are analytic functions, the first making a transition from -1 to 1 and the second defining a bump or (if multiplied by 0.8π) mollifier. They approach their limiting values "double exponentially"—not just exponentially like $\tanh(x)$ or exponentially-squared like $\exp(-x^2)$.

How could anyone's favourite function have a decimal in it like 3.2? Well, this constant has been chosen so that $\tau(x)$ rounds to exactly ± 1 for $|x| \geq 1$ in 16-digit IEEE floating-point arithmetic. So the transition region of τ has compact support in $[-1, 1]$ in the standard arithmetic of computational science and engineering.

The uncertainty principle states that a nonzero function and its Fourier transform cannot both have compact support. This implies a trade-off between locality and smoothness, and the well-known balanced compromise is a Gaussian $\exp(-x^2)$, where both f and $\hat{f}$ are entire functions with exponential-squared decay. The functions (1) and (2) make a different choice, prioritizing locality over smoothness. The locality is spectacular (we have $|d(x)| < 10^{-400}$ for $|x| \geq 2$) yet the smoothness is still excellent, since the functions are analytic. (The constant $\pi/2$ arises since smaller values give poorer localization and larger ones give a narrower strip of analyticity.) Figure 2 illustrates how $\tau(x)$ can be used to blend one analytic function into another. I think of the transition region as a "fat branch cut".

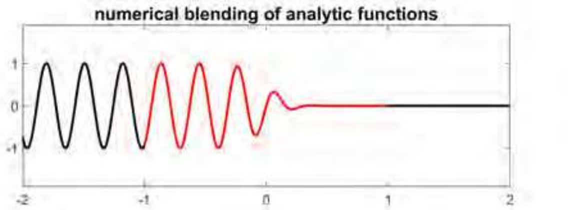

Figure 2. Numerical blending of $\sin(20x)$ for $x \leq -1$ and 0 for $x \geq 1$. The blend function is analytic and matches the pieces to machine precision.

Of course, for proving a theorem, one may need true compact support, and the standard choices are C^∞ functions composed of pieces, such as $\exp(-1/(1-x^2))$ for $|x| < 1$ and 0 for $|x| \geq 1$. When it comes to applying such ideas computationally, however, it is not obvious that C^∞ functions are the best starting point. The $\tanh((\pi/2)\sinh(x))$ combination was introduced by Takahasi and Mori in 1974 for numerical quadrature of functions with singularities [1], but I am not aware that it has been proposed for other applications.

FURTHER READING

[1] M. Mori, Discovery of the double exponential transformation and its developments, Publ. Res. Inst. Math. Sci., 41 (2005), 897–935.

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