ODE IVPs/BVPs
AND A NEW TEXTBOOK

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Purpose: Chebfun is a tool for numerical computing with functions.

Idea: overload Matlab’s vectors/matrices to functions/operators.

Algorithms: based on piecewise Chebyshev polynomial interpolants.

www.chebfun.org
This is Chebfun’s syntax for solving ODE BVPs $Lu = f$. $L$ can be linear or nonlinear and includes BCs.

Inputs: $f$ is a chebfun, $L$ is a chebop.
Output: $u$ is a chebfun.

Algorithm: adaptive Chebyshev spectral collocation embedded in a Newton iteration. Formulation via \textit{block operators} $\rightarrow$ rectangular matrices (Driscoll + Hale)

What about IVPs? Here Chebfun’s algorithm is completely different: marching with \texttt{ode113}, then conversion of the result to a chebfun.

Until 2015, you had to call \texttt{chebfun.ode113}. In 2014, Birkisson folded this into the syntax $u = L\backslash f$. This was nontrivial because higher-order problems must be converted silently to first order for \texttt{ode113}.

So now, Chebfun solves all ODE problems with $u = L\backslash f$. 

\[ u = L\backslash f \]
Airy equation  (scalar linear BVP)

\[ y'' - xy = 0, \ x \in [-10,10], \ y(-10) = y(10) = 1. \]

\[
L = \text{chebop}(-10,10);
L\.\text{op} = @(x,y) \text{diff}(y,2) - x*y;
L\.lbc = 1; L\.rbc = 1;
y = L\backslash 0; \text{plot}(y)
\]

\[
\begin{align*}
y(0) & \quad \text{ans} = 8.82249323 \\
\max(y) & \quad \text{ans} = 13.31113744
\end{align*}
\]
Airy equation (scalar linear BVP)

\[ y'' - xy = 0, \quad x \in [-10, 10], \quad y(-10) = y(10) = 1. \]

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L.lbc = 1; L.rbc = 1;
y = L\backslash 0; \text{plot}(y)
\]

plotcoeffs(y,'.'), ylim([1e-18, 1e2])
van der Pol equation  (scalar nonlinear IVP)

\[ 0.3y'' - (1 - y^2)y' + y = 0, \quad t \in [0,20], \quad y(0) = 1, \quad y'(0) = 0. \]

\[
N = \text{chebop}(0,20);
N.op = @(t,y) 0.3*diff(y,2) - (1-y^2)*diff(y) + y;
N.lbc = @(y) [1; 0];
y = N\0; plot(y)
\]

```
[0.6 secs.]
[ val, pos ] = max(y,'local');
diff(pos)
ans = 3.16508501
    4.07245435
    4.07253777
    4.07253777
    4.07253777
```

[6/17]
van der Pol equation  (scalar nonlinear IVP)

\[ 0.3y'' - (1 - y^2)y' + y = 0, \quad t \in [0,20], \ y(0) = 1, \ y'(0) = 0. \]

\[
N = \text{chebop}(0,20);
N.op = @(t,y) 0.3*diff(y,2) - (1-y^2)*diff(y) + y;
N.lbc = @(y) [1; 0];
y = N\0; plot(y)

plotcoeffs(y,'.'), ylim([1e-18,1e2])

Chebyshev coefficients
Lorenz equations  (system of nonlinear IVPs)

\[ u' = 10(v-u), \quad v' = u(28-w) - v, \quad w' = uv - (8/3)w, \]
\[ t \in [0,30], \quad u(0) = v(0) = -15, \quad w(0) = 20. \]

```matlab
N = chebop(0,30);
N.op = @(t,u,v,w) [diff(u)-10*(v-u); ...
    diff(v)-u*(28-w)+v; diff(w)-u*v+(8/3)*w];
N.lbc = @(u,v,w) [u+15; v+15; w-20];
[u,v,w] = N\0; plot(u,w)
```

```
N.lbc = @(u,v,w) [u+15.000001;v+15;w-20];
[u2,v2,w2] = N\0;
tt = 0:.05:30;
semilogy(tt,abs(u2(tt)-u(tt)),'.');
```

[1.8 secs.]
3-body problem (system of nonlinear IVPs)

\[
N = \text{chebop}(0,100); \quad u0 = 0; \quad v0 = 3; \quad w0 = 4i;
\]
\[
N.\text{op} = @(t,u,v,w) \begin{bmatrix}
    \text{diff}(u,2) + (u-v)/\text{abs}(u-v)^3 + (u-w)/\text{abs}(u-w)^3; \\
    \text{diff}(v,2) + (v-u)/\text{abs}(v-u)^3 + (v-w)/\text{abs}(v-w)^3; \\
    \text{diff}(w,2) + (w-u)/\text{abs}(w-u)^3 + (w-v)/\text{abs}(w-v)^3
\end{bmatrix};
\]
\[
N.\text{lbc} = @(u,v,w) \begin{bmatrix}
    u-u0; \\
    v-v0; \\
    w-w0; \\
    \text{diff}(u); \\
    \text{diff}(v); \\
    \text{diff}(w)
\end{bmatrix};
\]
\[
y = N\backslash\left[0; v0; w0; \text{diff}(u); \text{diff}(v); \text{diff}(w)\right];
\]
\[
\text{plot}(y)
\]

[roots(real(v)+2)]
\[
\text{ans} = 90.1543
\]

[9.0 secs.]
Harmonic oscillator (scalar linear eigenvalue problem)

\[-y'' + x^2y = \lambda y, \quad x \in [-6,6], \quad y(-6) = y(6) = 0.\]

```matlab
L = chebop(-6,6);
V = chebfun('x.^2',[-6,6]);
plot(V,'k'), hold on
L.op = @(x,y) -diff(y,2) + V*y;
L.lbc = 0; L.rbc = 0;
[Y,D] = eigs(L,12);
for k = 1:12, plot(Y(:,k)+D(k,k)), end
```

d = diag(D); d(1:5)

ans = 1.0000000
      3.0000000
      5.0000000
      7.0000000
     9.0000000

[1.0 secs.]
Interior layer equation  (scalar linear BVP)

\[0.001y'' + xy' + xy = 0, \ x \in [-2, 2], \ y(-2) = -4, \ y(2) = 2.\]

```matlab
L = chebop(-2,2);
L.op = @(x,y) 0.001*diff(y,2) + x*diff(y) + x*y;
L.lbc = -4; L.rbc = 2;
y = L\0; plot(y)
```

```
ans = 186.918317
```

[0.9 secs.]
Resonance  (scalar linear IVP)

\[ y'' + y = 0, \quad t \in [0,200], \quad y(0) = y'(0) = 0. \]

t = chebfun('t',[0 200]);
L = chebop(0,200);
L.op = @(t,y) diff(y,2) + y;
L.lbc = [0; 0];

\[ y = L\cos(0.9*t) \]
\[ \text{plot}(y) \]
\[ \text{norm}(y) \]
\[ \text{ans} = 72.7106 \]

[2.9 secs.]

\[ y = L\cos(0.95*t) \]
\[ \text{plot}(y) \]
\[ \text{norm}(y) \]
\[ \text{ans} = 148.8003 \]
Damping switched on/off  (scalar linear IVP, discontinuous)

\[ y'' + d(t)y' + y = 0, \quad t \in [0,200], \quad y(0)=1, \ y'(0) = 0. \]

\[
t = \text{chebfun}('t',[0 \ 200]);
d = 0.1*(\text{abs}(t-90)<10);
L = \text{chebop}(0,200);
L.op = @(t,y) \text{diff}(y,2) + d*\text{diff}(y) + y;
L.lbc = @(y) [1; 0];
y = L\0; \text{plot}(y)
\]

\[
\text{max}(y\{0,50\}) \quad \text{ans} = 1.0000000
\]
\[
\text{max}(y\{150,200\}) \quad \text{ans} = 0.3582228
\]

[3.2 secs.]
Even easier: Birkisson’s graphical user interface chebgui.
There are lots of ODE books.
Our aim is to write a book quite unlike the others: focusing on ODE behavior, illustrating everything effortlessly.

Enabled by numerics, but not a numerical book.

Mathematically mature, but not technically advanced.

Nonlinear problems and BVPs throughout.

A book that certain instructors will choose to teach a course from, and they will all want to look at. A book that the top 10% of students in any ODE course will be excited to discover.

Cheap from SIAM, and freely available online.
Exploring ODEs

Lloyd N. Trefethen, Ásgeir Birkisson, and Tobin A. Driscoll

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Still to be written:
- Linearization and classification of fixed points
- Bifurcation
- Appendix B: 100 more examples

Please email me pointers to good examples and any other suggestions.