POLYNOMIAL AND RATIONAL CONVERGENCE RATES FOR LAPLACE PROBLEMS ON PLANAR DOMAINS *

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Abstract. Laplace problems on planar domains can be solved by means of least-squares expansions associated with polynomial or rational approximations. Here it is shown that, even in the context of an analytic domain with analytic boundary data, the difference in convergence rates may be huge when the domain is nonconvex. Our proofs combine the theory of the Schwarz function for analytic continuation, potential theory for polynomial and rational approximation rates, and the theory of crowding of conformal maps.

Key words. Laplace problem, polynomial approximation, rational approximation, Schwarz function, analytic continuation, inverted ellipse, potential theory, crowding

MSC codes. 30E10, 35J05, 41A20, 65N80, 31A15, 65E05

1. Introduction. Suppose we wish to solve numerically the Laplace problem

$$\Delta u = 0,\ z \in \Omega, \qquad u = h,\ z \in \Gamma \tag{1.1}$$

in a simply-connected planar domain Ω bounded by an analytic Jordan curve Γ, as suggested in Figure 1, where h is a real analytic function.[1] (Everything can be generalized to less smooth geometries or data, to other types of boundary conditions, and to domains with holes. In Figure 1(b), Γ is not analytic.) For convenience we think of Ω as complex, identifying $z = x + iy$. An old idea, going back to Walsh and Curtiss nearly a century ago [7, 36], is to approximate u as the real part of a polynomial,

$$u(z) \approx \mathrm{Re}\,p(z), \tag{1.2}$$

so that the problem reduces to the approximation of h by $\mathrm{Re}\,p$ on Γ. This idea builds on the facts that u must be the real part of a function f that is analytic in $\overline{\Omega}$ [1],

$$u(z) = \mathrm{Re}\,f(z), \tag{1.3}$$

and that f can be approximated on $\overline{\Omega}$ by a polynomial,

$$f(z) \approx p(z). \tag{1.4}$$

In particular, Runge showed in 1885 that f can be approximated arbitrarily closely by polynomials in the supremum norm [26].

In the old days, computing good approximations p would have been problematic even if today's computers had been available, because of the difficulty of finding well-distributed boundary points for interpolation and also the lack of a well-conditioned basis in which to represent the polynomial. Today, the first difficulty is bypassed by the use of least-squares fitting in a large number of sample points on Γ, a proposal originating perhaps with Moler in 1969 [19], and the second is taken care of by the use

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[1] By an analytic Jordan curve we mean the one-to-one image of the unit circle under an analytic function with nonvanishing derivative. See [28, p. 2] or [37, p. 2].

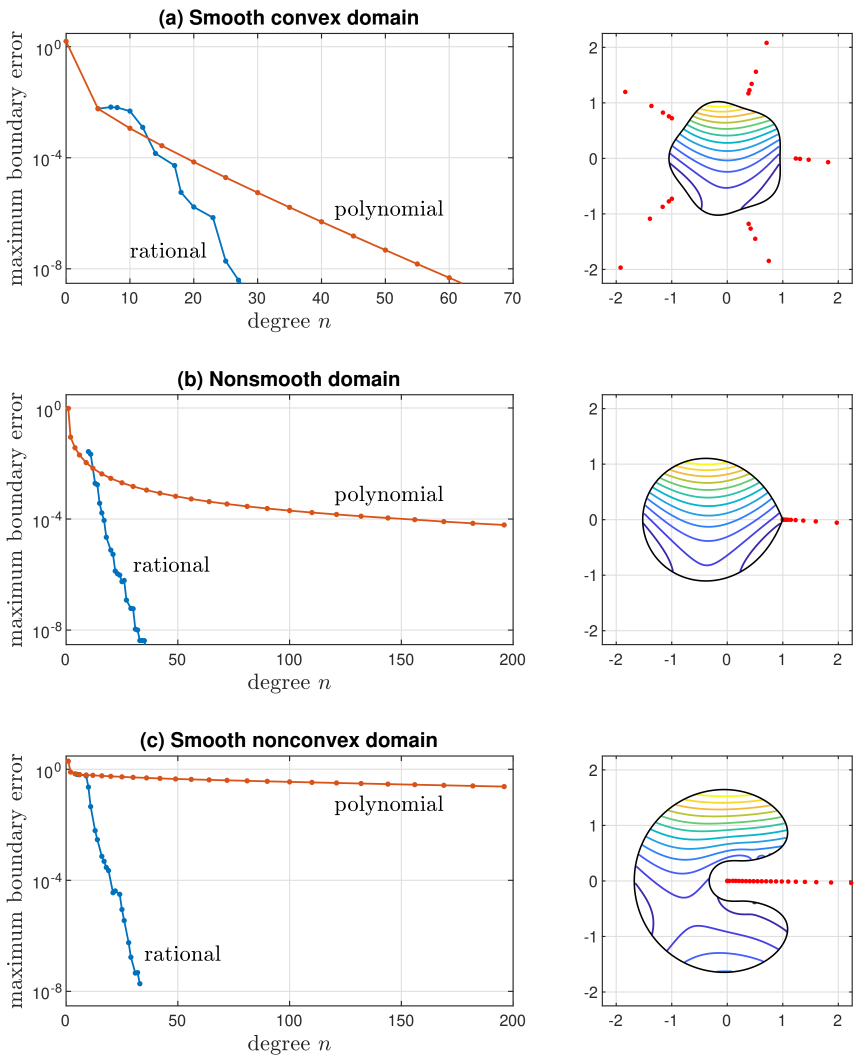

FIG. 1. *Solutions of Laplace problems by polynomials and rational functions of various degrees on three domains. The boundary function in each case is* $h(z) = (0.5 + \operatorname{Im} z)^2$. *On the smooth convex domain* (a)*, both methods converge rapidly. On the domain* (b) *with a corner singularity, rational approximations are much more efficient than polynomials, an effect going back to Newman in* 1964 [23]. *On the smooth nonconvex domain* (c) *with an inlet, rational approximations are again much more efficient, the new observation of this paper. Red dots mark the poles of rational approximations that achieve accuracy about* 10^{-8}. *These computations and comparisons are made possible by approximation algorithms developed in the last six years* [4, 6, 20].

of the Vandermonde-with-Arnoldi method of on-the-fly Stieltjes orthogonalization [4]. As a result, numerical solution of planar Laplace problems by polynomial approximation is entirely practical nowadays, so long as polynomials exist that approximate f efficiently. For many problems this is an excellent numerical method, quick and accurate and delivering a result that is exactly analytic and trivial to differentiate. Figure 1(a) is of this kind. Note that a rational function of degree n, as a quotient of polynomials of degree n, has about twice as many free parameters as a polynomial of degree n, so the convergence rates of these two curves could be regarded as about the same. For further illustrations of the power of polynomial approximations in favorable circumstances, see [32] and [4, Figure 6.1].

However, sometimes good approximating polynomials do not exist. Figure 1(b) shows a context in which this has been known since the work of Bernstein, Jackson, and de la Vallée Poussin in the 1910s, where u has a boundary singularity. Here the boundary contains a corner, and this prevents rapid convergence by any polynomial approximations.

The first purpose of this paper is to show that the same effect may arise even when Γ and h are analytic, as illustrated in Figure 1(c). This is the situation when Ω is nonconvex, containing an inlet. The figure shows that polynomial approximations may converge at a negligible rate in such cases, and we shall prove this mathematically. Although the convergence is exponential, the convergence constant is exponentially close to 1, rendering polynomial approximations useless in practice (Theorem 4).

On the other hand, instead of approximating u by a polynomial on Γ, one may approximate it by a rational function with no poles in Ω,

$$u(z) \approx \operatorname{Re} r(z), \tag{1.5}$$

which implicitly makes use of an approximation

$$f(z) \approx r(z). \tag{1.6}$$

Walsh considered this idea too [36], though it was even further out of practical range in those days, with no robust algorithms available for computing rational approximations, even if suitable computers had been at hand. Much more recently Hochman, Leviatan, and White developed a method of this kind in 2013 [16], and the AAA-least squares method, which appeared a few years later, has made these computations quick and easy [6], so that rational approximations are now a very practical method for solving Laplace problems.

The second purpose of this paper is to prove that the speedups possible with rational functions are transformative, not just in cases like Figure 1(b), where the power of rational approximations has been known for a long time, but also in cases like Figure 1(c), whose analysis is new.

Although this paper includes numerical illustrations to make the points clear, its main purpose is theoretical, and thus we do not specify computational details concerning, for example, the choice of sample points on the boundaries (which is generally not an issue so long as plenty of points are used). The power of rational approximations for solving Laplace and related problems has been illustrated abundantly by numerical experiments in other works [2, 5, 6, 10, 11, 16, 38]. The same power applies to Helmholtz problems too [11], although here there is less literature and not yet any theory to explain why rational approximations are so effective in choosing good locations for singularities of Hankel functions.

One instance of a nonconvex domain is particularly tidy, the case of an *inverted ellipse*, where everything can be worked out explicitly. For the inverted ellipse of parameter $\rho > 1$, we show that the degree of a polynomial approximation must be increased asymptotically by $(\log(10)/1.16)\exp((\pi^2/4)/(\rho-1)) \approx 2.0 \cdot (11.8)^{1/(\rho-1)}$ for each additional digit of approximation accuracy (Theorem 5). With $\rho = 1.3$, for example, hardly an extreme case, each digit of accuracy requires an increase of the polynomial degree by about 7000 (Table 1). With rational approximations, on the other hand, each new digit of accuracy requires an increase in the rational degree by just $\sim \frac{1}{2}\log(10)/(\rho-1) \approx 1.2/(\rho-1)$, so for $\rho = 1.3$, about 4 rather than 7000 (Theorem 9). See Figure 7.

There are a number of theoretical details in the upcoming pages, but the essential argument can be summarized compactly, as follows. Convergence rates of polynomial and rational approximations to Laplace solutions in 2D depend on analytic continuation outside Γ. This analytic continuation is described by the theory of the Schwarz function, which asserts that there will usually be branch point singularities not far from Γ (Theorems 1 and 2). This is true for both convex and nonconvex domains, as one can see by comparing the pole locations in Figures 1(a) and 1(c), which in both cases are near the boundary but not extremely near. The significance of nonconvexity arises at the next step of the argument, where we track the consequences of these singularities. For polynomial approximation, convergence is determined by a conformal map of the exterior of Γ to the exterior of the unit disk (Theorems 3–4 and Figures 3–4), and in the case of an inlet, the singularity will map to a point exponentially close to the disk, resulting in a convergence factor exponentially close to 1 (section 7). For rational approximation, on the other hand, poles can line up near branch cuts of the analytic continuation, leading to a much faster exponential convergence rate associated with a more forgiving doubly connected conformal mapping problem onto an annulus (Theorems 6–8 and Figures 5–6). In a word, the freedom of a rational function to place poles near Γ rather than just at ∞ eliminates the "crowding" phenomenon that causes exponential slow-down for polynomial approximations around inlets.

It is hard to find works in the literature that are close to the present paper, particularly in combining ideas of analytic continuation of Laplace solutions with polynomial and rational approximation theory. The one item I know that is exactly on target is the 1984 paper by Reichel [25], which presents a computed example with an inlet and states:

> The following discussion shows that polynomial approximants in general may converge very slowly when computing approximants on pronouncedly nonconvex regions.

Two better-known related papers are by Millar [18], who applies the Schwarz function to track singularities of analytic continuations of Helmholtz solutions across boundaries, and Barnett and Betcke [3], who apply the method of fundamental solutions [9] to Helmholtz problems, also with the aid of the Schwarz function, and show that good positioning of the singularities outside the domain is crucial to obtaining well-behaved bases. In these and other works related to analytic continuation of solutions of elliptic PDE, it is common to formulate the problem in terms of two independent complex variables, following Vekua in the 1950s [14, 34]. However, so far as I know, this is not needed in the special case of the Laplace equation.

I must end this introduction with a historical and personal remark. Most of the theory of this paper has connections with Joseph Walsh, the great mid-20th century expert on polynomial and rational approximation, who was a mathematics faculty member at Harvard during 1921–1966 until he retired and took a position at the University of Maryland. None of the algorithms that make these computations practicable were available in his day, however, not to mention computers of the necessary capabilities. Today, we can apply this Walsh-style theory in an entirely new environment. Meanwhile my own career has brought me to retirement after 26 years at Oxford, whereupon, in a strange symmetry, I have taken a position at Harvard. This is the first paper I have written at my new institution. I never met Walsh, but I had indirect connections to him through his students Ted Rivlin, Dick Varga, and Ed Saff, and with affection I think of the present paper as a continuation of Walsh's research interests into a computational era he could not have imagined.

2. Analytic continuation and the Schwarz function. The starting point of our analysis is analytic continuation across the boundary curve Γ. Let h be the boundary data function of (1.1), which we have assumed is real analytic. This implies that h can be analytically continued to a complex analytic function in a neighborhood of Γ. Similarly, let f be the complex analytic function of (1.3) whose real part is the solution u of the Laplace problem (1.1). (The imaginary part of f is determined up to a constant, which plays no role in the discussion.) Like h, f must extend to an analytic function in a neighborhood of Γ. Along with f and h, a third analytic function whose analytic continuation across Γ we shall work with is the difference,

$$d(z) = f(z) - h(z). \tag{2.1}$$

The important property of d is that it is pure imaginary on Γ, because f and h have equal real parts there. Thus h and d are analytic functions in a neighborhood of Γ that map Γ to subsets of the real and imaginary axes, respectively. If Γ were a straight line segment or an arc of a circle, we could now describe the relationship of h and d inside and outside Γ by the Schwarz reflection principle. In our more general case where Γ is an analytic arc, the generalization that comes into play is based on what is known as the Schwarz function [8, 28].

The *Schwarz function* of Γ is defined as the unique function $S(z)$ that is analytic in a neighborhood of Γ and takes the values $S(z) = \overline{z}$ for $z \in \Gamma$. Note that $S(z)$ is determined by Γ, not f or h or d. The significance of S is that its complex conjugate, $\overline{S}(z)$, maps points close to Γ on one side to their analytic reflections on the other side. Specifically, it is known that in a sufficiently small neighborhood U of Γ, S is analytic and satisfies the reflection property $\overline{S}(\overline{S}(z)) = z$. Here is how S enables analytic continuation: if h or d is analytic in the part of U interior or exterior to Γ, then it extends analytically to the other side by the reflection formula

$$h(\overline{S}(z)) = \overline{h(z)}, \quad z \in U \tag{2.2}$$

or

$$d(\overline{S}(z)) = -\overline{d(z)}, \quad z \in U, \tag{2.3}$$

respectively. In words, the values of h and d at $\overline{S}(z)$ are the reflections in the real and imaginary axes, respectively, of their values at z. For details of these developments see [8, chap. 6] or [28, Prop. 1.2].

For an idea of the shape of the Schwarz function, consider Figure 2. This shows an analytic Jordan curve Γ for which a degree 107 AAA rational approximation r to S has been computed. (This took 2/3 s on our laptop based on the Chebfun command `[r,poles] = aaa(conj(Z),Z)`, where `Z` is a vector of 2000 sample points along the boundary. The spacing of the points is unimportant for such computations involving smooth functions, so long as there are plenty of them.) For $z \in \Gamma$, r matches $S(z) = \overline{z}$ to accuracy 10^{-13}. The black dots are the poles of r, and they give an indication of the behavior of S for $z \notin \Gamma$. Roughly speaking, with rational approximations we expect poles to line up along the branch cuts of the function being approximated. More precisely, like all analytic functions, S does not intrinsically have branch cuts; these

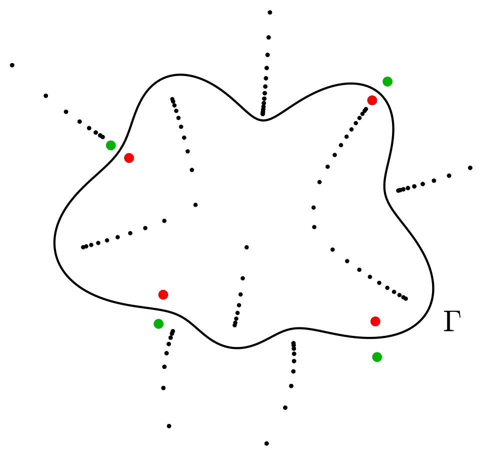

FIG. 2. *An analytic Jordan curve Γ and the poles of a rational approximation on Γ to its Schwarz function S. The clustered strings of poles indicate that S has five branch points of S near Γ in the interior and another five in the exterior. The red and green dots mark four arbitrarily chosen points on one side of Γ and their reflections on the other side. Analytic continuation of a function d analytic and imaginary on Γ is carried out by applying the reflection condition* (2.3) *at such points.*

are only introduced by humans when we wish to make the function single-valued, or by rational approximations (in the approximate fashion illustrated in the figure) as a by-product of their near-optimality. On the other hand S has branch *points*, and these are approximated by the cluster points of the poles of r. In this case it appears that there are five branch points of S near Γ on the inside and five more on the outside.

Going out from Γ beyond and around the branch points, it would make sense to speak of a multi-valued analytic Schwarz function, and this approach is taken in some of the theoretical literature [28] and will also turn up in our Theorem 8, below. However, multi-valued functions are mostly not relevant to the present work, because polynomial and rational approximations are single-valued. Therefore, throughout this paper, our attention is restricted to neighborhoods U that are narrow enough or otherwise confined in such a way that S is analytic and single-valued.

Let U be a neighborhood of Γ of this kind, with S defined in U satisfying $\overline{S}(U) = U$ and $\overline{S}(\overline{S}(z)) = z$, so that U is reflected into itself by $\overline{S}$. We can now establish a foundational theorem for this paper (compare [18, sec. 4]).

Theorem 1. Analytic continuation of f across Γ. *Let Ω, Γ, f, h, and $d = f - h$ be defined as discussed above, and let U be a neighborhood of Γ in which the Schwarz function S of Γ is analytic and satisfies $\overline{S}(U) = U$ and $\overline{S}(\overline{S}(z)) = z$. If h is analytic in the part of U outside Γ, then the same is true of f.*

Proof. Let $U_{\rm in}$ and $U_{\rm out}$ be the portions of U interior and exterior to Γ, respectively. If h is analytic in $U_{\rm out}$, then by (2.2) it is analytic in $U_{\rm in}$. By (2.1) it follows that d is analytic in $U_{\rm in}$, since f is analytic throughout Ω. By (2.3), it follows that d is analytic in $U_{\rm out}$. By (2.1) again, it follows that f is analytic in $U_{\rm out}$. ∎

To prove that polynomial approximation stagnates as illustrated in Figure 1, we will need a converse of this theorem. We would like to show that when the Schwarz function S has singularities outside Ω, they will usually block the analytic continuation of the function f associated with the Laplace problem (1.1). More precisely, I suspect it is true that if S cannot be analytically continued to a point z_c outside of Ω, then the same will apply to f if h is a nonconstant function analytic outside Ω. (The situation is different if h is a constant, since then f will be a constant too and thus analytically continuable to all of $\mathbb{C}$, regardless of the shape of Γ.)

It is worth mentioning that there are plenty of functions h that are real on Γ and analytic in $\mathbb{C}\backslash\Omega$. If Γ is the unit circle, then x and y are both functions in this class, since they can be written $x = (z + z^{-1})/2$ and $y = (z - z^{-1})/2i$. The same therefore applies on the unit circle to real polynomials in x and y, and more generally, to functions defined by Laurent series convergent for $0 < |z| < \infty$ whose coefficients have the symmetry $a_{-k} = \overline{a_k}$. For a general Jordan curve Γ, we may obtain an equivalent class of functions $h(z)$ as transplants of these Laurent series under a conformal map Φ of the exterior of Γ to the exterior of the unit disk. (We will make use of Φ in the next section in the context of polynomial approximation.) Note that it follows from here that, given any continuous real function h_0 on Γ and any $\varepsilon > 0$, we can find a function h that is real on Γ and analytic in $\mathbb{C}\backslash\Omega$ with $|h - h_0| < \varepsilon$ on Γ.

I don't know how to establish the singularity of f in the generality conjectured above. Instead, following [18, sec. 4], here is a statement restricted to the familiar situation in which S has a branch point as in Figure 2 and in our later example of the inverted ellipse.

Theorem 2. Limit to analytic continuation of f across Γ. *Let Ω, Γ, f, h, and d be defined as discussed above, and let $z_c \in \mathbb{C}\backslash\overline{\Omega}$ be such that S can be analytically continued up to z_c but has a branch point there. Then for some choices of h analytic in $\mathbb{C}\backslash\Omega$, f cannot be analytically continued to a neighborhood of z_c.*

Proof. We argue by contradiction. Given Ω and Γ, let z_c be a branch point of the Schwarz function S of Γ as indicated. Then there is an open region U_{out} exterior to Γ, as in the proof of Theorem 1, such that if we remove a branch cut C that runs from z_c to the outer boundary of U_{out}, we get an open region $\tilde{U}_{\text{out}}$ exterior to Γ on which S is analytic and single-valued.

Suppose f can be analytically continued to a neighborhood of z_c, which we may take to be this region U_{out} (by shrinking U_{out} if necessary). By (2.1) and the assumed analyticity of h exterior to Γ, the same is true of d. Thus f and d are analytic in U_{out}, hence also in the smaller domain $\tilde{U}_{\text{out}}$. Since S is analytic in $\tilde{U}_{\text{out}}$, we can use (2.1)–(2.3) to analytically continue f and d from $\tilde{U}_{\text{out}}$ across Γ into an open set $\tilde{U}_{\text{in}}$ interior to Γ.

Now consider a point b on the branch cut C such that $\overline{S}$ takes different values at b on the two sides of the cut, which we can think of as b_1 and b_2. (There must be such a point b, or z_c would not be a branch point of S.) That is, $\overline{S}(b_1) = a_1 \in \tilde{U}_{\text{in}}$ and $\overline{S}(b_2) = a_2 \in \tilde{U}_{\text{in}}$ with $a_1 \neq a_2$. Unlike $\overline{S}$, d and h are analytic and single-valued at b, so we must have $d(a_1) = d(a_2)$ and $h(a_1) = h(a_2)$. By (1.3) and (2.1), this implies $f(a_1) = f(a_2)$ and $u(a_1) = u(a_2)$.

And now the contradiction becomes apparent. Of course the solution u_0 of a Dirichlet problem with boundary data h_0 may satisfy $u_0(a_1) = u_0(a_2)$ at two points $a_1 \neq a_2$. But this cannot be true for all choices of h_0. Pick a boundary function h_0 for which $u_0(a_1) \neq u_0(a_2)$. As observed at the end of the second paragraph above this theorem, h_0 can be approached arbitrarily closely by boundary functions h that are analytic throughout $\mathbb{C}\backslash\overline{\Omega}$. By continuity of Laplace solutions with respect to boundary data (a consequence for example of the Poisson integral formula), we must then have $u(a_1) \neq u(a_2)$ when h is sufficiently close to h_0, which contradicts the conclusion of the last paragraph. ∎

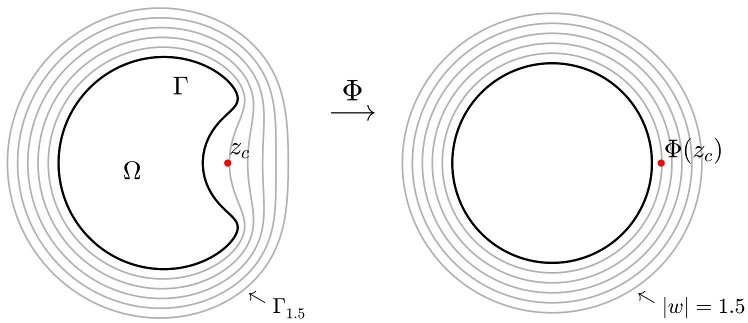

FIG. 3. *Illustration of the theory of polynomial approximation of an analytic function f on a domain Ω bounded by an analytic Jordan curve Γ. The grey curves are level lines $\{\Gamma_r\}$ of the Green's function of Γ, $r = 1.1, 1.2, \ldots, 1.5$, which can be interpreted as preimages of circles outside the unit disk in a conformal map Φ of the exterior of Γ to the exterior of the unit disk. The analyticity radius R of f is the largest $r > 1$ for which f is analytic in the region interior to Γ_r. The red dot on the left marks a point on Γ_R to which f cannot be analytically continued, and the absolute value of its image on the right is the number R, i.e., $R = |\Phi(z_c)|$. The convergence rate of best polynomial approximations is given by* (3.2).

3. Polynomial approximation. Let E_n be the minimax error of degree n polynomial approximation to f on Ω,

$$E_n = \inf_{p \in P_n} \|f - p\|, \tag{3.1}$$

where $\|\cdot\|$ is the supremum norm on $\overline{\Omega}$ and P_n denotes the space of polynomials of degree n. We now consider what Theorem 2 implies about the convergence rate of E_n to 0 as $n \to \infty$.

Assume for simplicity that f cannot be analytically continued to all of $\mathbb{C}$. According to standard theory going back to Walsh [35, 37] and presented beautifully by Levin and Saff [17], the convergence rate is then exponential, at a rate essentially R^{-n} for some $R > 1$. The constant R, which we shall call the *analyticity radius* of f on Ω, has an interpretation in terms of a conformal map Φ of the exterior of Ω to the exterior of the unit disk in the w-plane with $\Phi(\infty) = \infty$. For any $r \geq 1$, as illustrated in Figure 3, let Γ_r be the preimage of $|w| = r$ under Φ. Then R is the largest r for which f is analytic in the region enclosed by Γ_r.[2]

Here is Walsh's result.

Theorem 3. Polynomial approximation of an analytic function. *Let the domain Ω, the function f analytic in Ω, the polynomial minimax error E_n, and the analyticity radius R of f be defined as above. The minimax approximation errors satisfy*

$$\limsup_{n \to \infty} E_n^{1/n} = \frac{1}{R}. \tag{3.2}$$

We can paraphrase Theorem 3 like this:

[2]If z_c is a point on Γ_R at which f is not analytic, then we can also write $R = \exp(g(z_c))$, where g is the Green's function of $\overline{\Omega}$, that is, the harmonic function defined in $\mathbb{C} \backslash \overline{\Omega}$ with $g(z) = 0$ on Γ and with $g(z) \sim \log |z|$ as $z \to \infty$ [17, Thm 3]. The relationship between g and Φ is $g(z) = \log |\Phi(z)|$.

The rate of polynomial approximation of f on Ω
is determined by its closest singularity outside Ω.

"Closest" means in the sense of the level curves Γ_r, and the rate in question is that of (3.2). This is the "Walsh half" of the analysis of our Laplace problem. The "Schwarz half" was presented in Theorem 2 of the last section:

This closest singularity is normally the closest point z_c of nonanalyticity
of the Schwarz function S (unless h has a singularity even closer).

Combining Theorems 2 and 3 gives us our basic result on slow convergence of polynomial approximations of solutions of Laplace problems. As with Theorem 2, though this statement only mentions "some" choices of h, in practice it will be almost all of them. The theorem references the analyticity radius R of S, a notion defined above for a function analytic throughout Ω. For a function like S analytic just in a neighborhood of Γ, R is the largest $r > 1$ for which S is analytic in the region bounded between Γ and Γ_r.

Theorem 4. Polynomial approximation of a Laplace solution. *Let the Laplace problem* (1.1) *with boundary data h have solution $u(z) = \mathrm{Re} f(z)$ as in* (1.3)*, let $R > 1$ be the analyticity radius of the Schwarz function S on Ω, and let E_n be the minimax error* (3.1) *in degree n polynomial approximation. Then for some choices of h analytic throughout $\mathbb{C}\backslash\overline{\Omega}$, and assuming S has a branch point on the curve Γ_R, these errors satisfy*

$$\limsup_{n\to\infty} E_n^{1/n} = \frac{1}{R}. \tag{3.3}$$

4. Polynomials on the inverted ellipse. Theorem 4 has an elegant application to the special case in which Ω is the region bounded by an inverted ρ-ellipse, as sketched in Figure 4. For any $\rho > 1$, we define the *ρ-ellipse* E_ρ to be the image in the ζ-plane of the circle $|w| = \rho$ in the w-plane under the Joukowsky map $J(w) = (w + w^{-1})/2$. Geometrically, E_ρ is the ellipse with foci ± 1 whose semiminor and semimajor axis lengths sum to ρ. The *inverted ρ-ellipse* is the reciprocal $I_\rho = 1/E_\rho$, and our variables are related by $\zeta = z^{-1}$.

The Schwarz function for the inverted ellipse is known analytically. For example on p. 25 of [8] we find for the ellipse E_ρ

$$S(z) = \tfrac{1}{2}(\rho^2 + \rho^{-2})z - \tfrac{1}{2}(\rho^2 - \rho^{-2})\sqrt{z^2 - 1}, \tag{4.1}$$

which implies that for I_ρ we have

$$S(z) = \left(\tfrac{1}{2}(\rho^2 + \rho^{-2})z^{-1} - \tfrac{1}{2}(\rho^2 - \rho^{-2})\sqrt{z^{-2} - 1}\,\right)^{-1}. \tag{4.2}$$

For our purposes what matters is that the only singularities of S are a pair of branch points outside Γ at $z = \pm 1$ and a pair of simple poles inside Γ at $\pm i(\rho^2 - \rho^{-2})/2$. If branch cuts are drawn along $[1, \infty)$ and $(-\infty, -1]$, then S becomes meromorphic in the remaining slit domain $\mathbb{C}\backslash\{(-\infty, -1] \cup [1, \infty)\}$, analytic in the portion outside Ω.

To determine the analyticity radius for I_ρ, we accordingly need to know the image of $z = 1$ under a conformal map of the exterior of I_ρ to the exterior of the unit disk. The conformal map of the interior of I_ρ to the unit disk is elementary: it reduces to the map of the exterior of E_ρ, which is essentially the Joukowsky map. The conformal

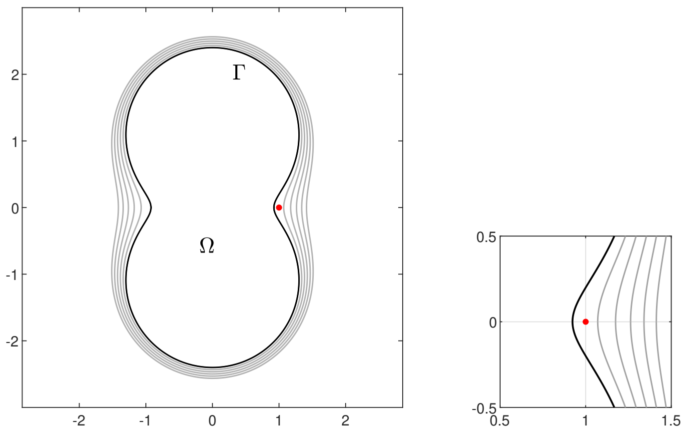

FIG. 4. *The inverted ellipse I_ρ with parameter $\rho = 1.5$, with level curves Γ_r plotted for $r = 1.02, 1.04, \dots, 1.10$. The dot at $z = 1$ corresponds to the analyticity radius $r = R \approx 1.009$ at which the Schwarz function for Ω has a branch point. This value implies that although the boundary of the region is analytic, it will still take an increase of the polynomial degree by about 250 asymptotically for each additional digit of accuracy. On the right, a close-up.*

map of the exterior of I_ρ, on the other hand, reduces to the map of the interior of E_ρ, which is not elementary. The required formula involving a Jacobian elliptic function was derived by Schwarz in 1869 and is presented in a number of sources including [22], [31], and Figure 3.2 of [13]. According to equation (21) of [31], the focus $\zeta = 1$ of the ellipse maps to the point

$$\varphi^{-1}(1) = \frac{\rho^{-1^2} + \rho^{-3^2} + \rho^{-5^2} + \cdots}{\frac{1}{2} + \rho^{-2^2} + \rho^{-4^2} + \rho^{-6^2} + \cdots}, \tag{4.3}$$

where φ denotes the conformal map from the disk to the ellipse. The analyticity radius we need follows by taking the reciprocal.

Theorem 5. Polynomial approximation on the inverted ellipse. *If Ω is the domain bounded by the inverted ρ-ellipse I_ρ for some $\rho > 1$, the associated analyticity radius R of Theorem 4 is*

$$R = \frac{\frac{1}{2} + \rho^{-2^2} + \rho^{-4^2} + \rho^{-6^2} + \cdots}{\rho^{-1^2} + \rho^{-3^2} + \rho^{-5^2} + \cdots}. \tag{4.4}$$

Asymptotically as $\rho \to 1$, R satisfies

$$R - 1 \sim 4e^{-\pi^2/(4\log\rho)} \sim Ae^{-\pi^2/(4(\rho-1))} \tag{4.5}$$

with $A = 4\exp(-\pi^2/8) \approx 1.16485$.

Proof. The derivation of (4.4) was given in the discussion above, and I am grateful to Jon Chapman of Oxford and Alex Barnett of the Flatiron Institute for proofs of (4.5). The following particularly elegant argument comes from Barnett. If we double both the numerator and the denominator of (4.4) by extending the sums to $k = -\infty$, the quotient is unchanged in value and takes the simple form

$$R = \sum_{k=-\infty}^{\infty} \rho^{-4k^2} \Bigg/ \sum_{k=-\infty}^{\infty} \rho^{-4(k+\frac{1}{2})^2} ,$$

or equivalently, on setting $a = 4 \log \rho$,

$$R = \frac{A}{B} = \sum_{k=-\infty}^{\infty} \exp(-ak^2) \bigg/ \sum_{k=-\infty}^{\infty} \exp(-a(k+\tfrac{1}{2})^2) \; . \tag{4.6}$$

This is the ratio of two infinite trapezoidal quadrature approximations to the Gaussian, which explains why it converges exponentially to 1 as $a \to 0$. By the Poisson summation formula applied to the Fourier transform pair $f(x) = \exp(-ax^2)$ and $\hat{f}(k) = c \exp(-\pi^2 k^2/a)$, where c is a constant whose value does not matter, we have for a new constant c'

$$A = c' \left(1 + 2e^{-\pi^2/a} + 2e^{-(2\pi)^2/a} + \cdots \right),$$

and, since the translation by $1/2$ leads to alternating signs in the Poisson summation formula,

$$B = c' \left(1 - 2e^{-\pi^2/a} + 2e^{-(2\pi)^2/a} - \cdots \right).$$

These formulas imply $A/B - 1 \sim 4e^{-\pi^2/a}$ as $a \to 0$. ∎

The exponential dependence of R on $\log \rho$ or $\rho - 1$ in Theorem 5 is striking. Table 1 lists R for various values of ρ decreasing toward 1. The final column shows $\log(10)/\log(R)$, the increase of degree required asymptotically for each improvement of accuracy by one digit.

TABLE 1

Analyticity radii and corresponding convergence rates for polynomial approximations to solutions of Laplace problems on the inverted ellipse I_ρ for various ρ. All numbers are rounded to 2 significant figures.

ρ	R	degree increase per digit
2	1.12	20
1.9	1.089	27
1.8	1.062	38
1.7	1.038	60
1.6	1.021	110
1.5	1.0091	250
1.4	1.0026	880
1.3	1.00033	7000
1.2	1.0000053	430,000
1.1	1.000000000023	100,000,000,000

5. Rational approximation. Polynomials are rational functions all of whose poles are constrained to lie at $z = \infty$. When rational functions without this constraint are allowed, the approximation power increases enormously on nonconvex domains. This happens because poles can now line up along branch cuts of f, or, equivalently in most cases, along branch cuts of the Schwarz function S. We saw this effect in Figure 1. The phenomenon of poles approximating branch cuts is well known in rational approximation theory, associated in particular with theorems of Stahl for Padé approximation [30], and is illustrated, for example, in the papers [2, 5, 6, 10, 11, 16, 20, 21, 33, 38].

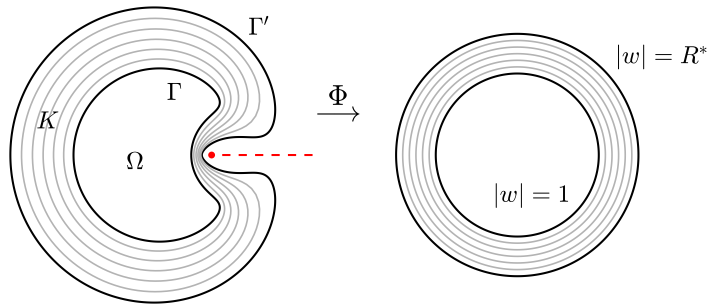

FIG. 5. *A rational companion to Figure* 3*, illustrating the theory of rational approximation of an analytic function f on a domain Ω bounded by an analytic Jordan curve Γ. We suppose f has a singularity outside Γ, marked by the red dot, but that f can be analytically continued to a condenser region K bounded between Γ and a larger Jordan curve Γ' that avoids this singularity. If Φ is a conformal map of the condenser onto a circular annulus $1 < |w| < R^*$, then rational approximations to f can converge at the exponential rates* (5.2) *or* (5.4) *determined by R^*. They achieve this by placing poles approximately along branch cuts, as suggested by the dashed line. Compare Figure* 1(c)*. The conformal map for this image was computed by the AAA-least squares method* [6].

An explanation of this behavior and of the approximation power of rational functions comes again from Walsh [37]. In Walsh's theory, key roles are played by the n poles $\{\pi_k\}$ of a rational function r of degree n, which can be thought of as point charges for a two-dimensional potential $\log|z-\pi_k|$, and any n interpolation points $\{z_k\}$ where $r(z_k) = f(z_k)$, which can be thought of as point charges of opposite sign associated with potentials $-\log|z-z_k|$. By considering these potentials in the context of the Hermite integral formula representing $f(z) - r(z)$ at other points $z \in \mathbb{C}$, one can derive error bounds associated with potential theory. A few decades after Walsh, Gonchar and others took the theory further, showing how in the limit $n \to \infty$, minimax approximation rates are governed by the equilibrium potentials for a continuum distribution.

We will not go into the details of this theory, which are summarized in [17, pp. 91–93] and more fully in sections 6–8 of [33]. We will, however, state its main conclusion, which is illustrated in Figure 5. Let f be analytic on Ω and extend analytically (i.e., as a single-valued analytic function) to some larger domain $\Omega' \subseteq \mathbb{C}$ enclosing $\overline{\Omega}$ in its interior. For simplicity we suppose that Ω' is also a Jordan domain, with boundary Γ'. Then Γ and Γ' are the inner and outer boundaries of an annular region K, called a *condenser*. Such a region can be conformally mapped onto a circular annular region in $\mathbb{C}$ whose inner boundary is the unit circle and whose outer boundary is the circle about the origin of some radius $R^* > 1$, the *modulus* of the condenser. The number R^* is uniquely determined, and the conformal map itself is also uniquely determined up to a rotation. As in (3.1), but now for rational approximations, we define

$$E_{nn} = \inf_{r \in R_n} \|f - r\|, \tag{5.1}$$

where $\|\cdot\|$ is the supremum norm on $\overline{\Omega}$ and R_n denotes the space of rational functions of degree n (i.e., with at most n poles counted with multiplicity, including any poles at ∞). Here is Walsh's result:

Theorem 6. Rational approximation of an analytic function. *Let the domain Ω, the function f, the rational minimax error E_{nn}, the condenser K, and the* *condenser modulus R^* be defined as above. The minimax errors satisfy*

$$\limsup_{n\to\infty} E_{nn}^{1/n} \le \frac{1}{R^*}. \tag{5.2}$$

Recall that Theorem 3 for polynomials involved a number R that was fully determined by f and Ω, defined as the largest $r > 1$ for which f could be extended analytically to within the Green function contour Γ_r. Here with rational functions, by contrast, the number R^* as we have defined it depends on the choice of the condenser K to which f is analytically continued. In cases where f has branch points, a single-valued continuation will need to be restricted to a domain that avoids these, and some choices will lead to larger values of R^* than others. An optimal choice can be made, leading to a maximal value of R^*, but we will not go into that here, simply accepting that Theorem 6 holds for any choice of K and its associated R^*. As a consequence the next two theorems involve inequalities, not equalities.

The crucial point is that R^* can be much bigger than R, allowing rational approximations to converge much faster than polynomials. In particular we will see in sections 6 and 7 that this happens with domains Ω involving inlets, where R is exponentially close to 1 whereas R^* may be only algebraically close.

Theorem 7. Rational approximation of a Laplace solution. *Let the Laplace problem* (1.1) *with boundary data h have solution $u(z) = \mathrm{Re} f(z)$ as in* (1.3). *Let the Schwarz function S of Γ be analytically continuable to a condenser K about Ω, as defined above, with modulus R^*. Assume that $\overline{S}$ reflects K into a subset of Ω such that $U = K \cup \Gamma \cup \overline{S}(K)$, as in Theorem* 1*, is a neighborhood of Γ in which S is analytic and satisfies $\overline{S}(U) = U$ and $\overline{S}(\overline{S}(z)) = z$. If h is analytic in K, then the rational minimax errors E_{nn} of* (5.1) *satisfy*

$$\limsup_{n\to\infty} E_{nn}^{1/n} \le \frac{1}{R^*}. \tag{5.3}$$

Proof. By Theorem 1, f is analytic in K, and (5.3) follows from Theorem 6. ∎

The $1/R^*$ convergence factor of Theorems 6 and 7 is pessimistic in many cases: the actual convergence factor is often $1/(R^*)^2$. Equivalently, we may say that $1/R^*$ is often an upper bound not just for $\limsup_{n\to\infty} E_{nn}^{1/n}$ but also for $\limsup_{n\to\infty} E_{nn}^{1/2n}$. Intuitively speaking, this factor of 2 is associated with the fact that rational functions have twice as many parameters as polynomials, a property not exploited in the theory leading to Theorems 6 and 7. These issues are summarized in section 7.5 of [33], whose substance is derived from the important paper [24] by Rakhmanov. According to what Rakhmanov calls the *Gonchar–Stahl ρ^2-theorem,* the factor of 2 speedup always applies in the case of a function that can be analytically continued to a multivalued analytic function along all curves in $\mathbb{C}$ that avoid a certain fixed set $\Sigma \subseteq \mathbb{C}$ of capacity zero. In particular, it applies when f is analytic apart from a finite or countable collection of poles, essential singularities, or branch points. We summarize the improvement to Theorems 6 and 7 as follows. Readers who would like full details are encouraged to look at the opening pages of [24], which are very clearly presented.

Theorem 8. Factor of 2 speedup for functions with algebraic branch points. *The convergence rates of Theorems* 6 *and* 7 *can be doubled to*

$$\limsup_{n\to\infty} E_{nn}^{1/n} \le \frac{1}{(R^*)^2}. \tag{5.4}$$

if f can be analytically continued to a multivalued analytic function along all curves in $\mathbb{C}$ that avoid a fixed set $\Sigma \subseteq \mathbb{C}$ of capacity zero.

Proof. The estimate (5.4), which is the upper-bound half of the Gonchar–Stahl theorem, originates in [29]. ∎

6. Rational functions on the inverted ellipse. We now apply Theorems 7–8 to the inverted ellipse I_ρ. This proves surprisingly easy. We saw in section 4 that the conformal map of the exterior of I_ρ to the exterior of the unit disk is not elementary. For rational approximation, however, since the Schwarz function is analytic in the slit domain $\mathbb{C}\backslash\{\overline{\Omega} \cup (-\infty, -1] \cup [1, \infty)\}$, the conformal map we need is of the exterior of $\overline{\Omega}$ in this slit domain onto a circular annulus. Taking reciprocals, what is at issue is the conformal map onto a circular annulus of the doubly-connected domain bounded by the ρ-ellipse E_ρ and slit along $[-1, 1]$. This is essentially just the Joukowsky map, as summarized in Figure 6.

The gives us the following theorem.

Theorem 9. Rational approximation on the inverted ellipse. *If Ω is the domain bounded by the inverted ρ-ellipse I_ρ for some $\rho > 1$, then if h can be analytically continued to all of $\mathbb{C}\backslash\Omega$, the associated modulus R^* of Theorem 8 can be taken to be any value $R^* < \rho$. It follows that for such h, minimax rational approximation errors for solutions of the Laplace problem satisfy*

$$\limsup_{n\to\infty} E_{nn}^{1/n} \le \frac{1}{\rho^2}. \tag{6.1}$$

Figure 7 shows a pair of numerical experiments confirming Theorems 5 and 9. The figure has two panels, corresponding to domains Ω bounded by inverted ellipses I_ρ with $\rho = 1.8$ and $\rho = 1.3$. Each panel compares the convergence of a polynomial approximation method (Vandermonde with Arnoldi least squares [4]) against a rational one (AAA-least squares [6]). As in Figure 1, the boundary data function is $h(x + iy) = (y + 0.5)^2$, with the shift by 0.5 introduced to break symmetry, and the boundary is discretized by 1500 points, so that these computations involve matrices of reasonably modest size with 1500 rows and a few hundred columns. One could get away with far fewer than 1500 points if they were clustered more densely near the inlets, but as mentioned in section 2, for simplicity we take plenty of points and an approximately uniform distribution.

For $\rho = 1.8$, where Ω is barely indented at all, Figure 7 shows that the polynomial method converges about 5 times more slowly than the rational method, requiring an increase of n by about 38 for each digit of accuracy as listed in Table 1. (As commented in the introduction, a rational function of degree n has about twice as many parameters as a polynomial of degree n, so it would be fairer to say that the number of parameters goes up by about 76 per digit, making the polynomials just 2.5 times slower than the rational functions.) The case $\rho = 1.3$ shows the extreme behavior predicted by Theorem 5. The rational method has slowed down by a factor of 2, whereas the polynomial method has slowed down by a factor of 200, now needing an increase of degree by 7000 for each additional digit of accuracy. Note that convergence of the polynomial method is still observed, but showing the upward-curving form associated with subexponential convergence for a non-smooth boundary rather than a straight line for exponential behavior. Evidently it would be impossible in practice to get even two digits of accuracy by this method, even though the boundary is analytic

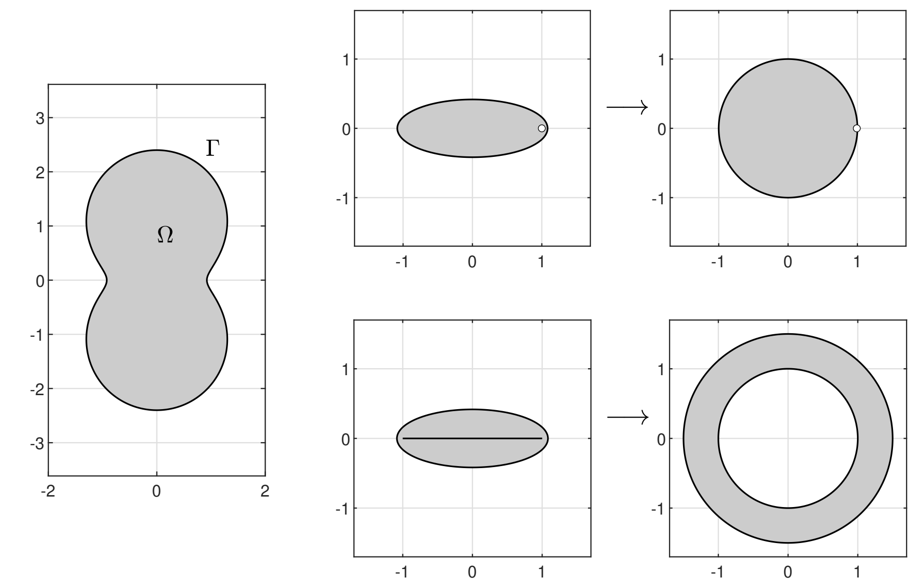

FIG. 6. *A summary in images of polynomial and rational convergence rates for the inverted ellipse I_ρ as discussed in sections 4 and 6. On the left, the domain Ω bounded by I_ρ with $\rho = 1.5$. On the right, the conformal maps of two domains bounded by the corresponding ellipse E_ρ, first without and then with a slit along the midline $[-1, 1]$. According to the arguments leading to Theorems 5 and 9 and summarized in Figures 3 and 5, these maps determine the exponential convergence rates for polynomial and rational approximation methods, respectively, for solving Laplace problems on Ω. The polynomial rate is very slow because the image of $z = 1$ in the upper map is exponentially close to 1 (about 0.9909 for this choice of ρ, marked by a white dot). The rational rate is much faster because the outer boundary of the annulus in the lower map is only algebraically close to the inner one (distance 0.5).*

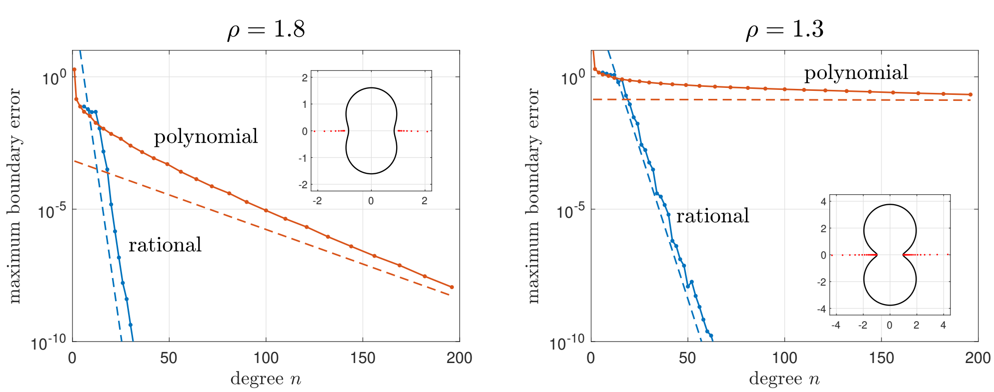

FIG. 7. *Numerical confirmation of Theorems 5 and 9 for solving a Laplace problem by polynomial and rational approximation on the domain bounded by the inverted ρ-ellipse I_ρ (Vandermonde with Arnoldi least squares [4] and AAA-least squares [6], respectively). With $\rho = 1.8$, the indentation is so mild that exponential convergence is observed for both methods, though at a lower rate for polynomials than rational functions. With $\rho = 1.3$ the exponential convergence rate for polynomial approximation is essentially zero and we see subexponential convergence, as if Ω had a corner. The rational method still converges rapidly, and the poles marked as red dots in the inset show that this is achieved by means of poles delineating approximate branch cuts $(-\infty, -1]$ and $[1, \infty)$. Dashed lines show the theoretically predicted slopes (4.5) and (6.1); their heights are arbitrary.*

and the Schwarz function extends analytically a nonnegligible distance 0.0346 outside it. This distance is just algebraically small, but its consequences are exponential.

The computations for Figure 7, about 120 numerical Laplace solutions all together, required 9 seconds on our laptop.

7. Approximation on a general domain with an inlet. The exponential effect that makes polynomial approximation on nonconvex domains problematic goes by the name of "crowding" in the literature of numerical conformal mapping. (In elasticity theory it is called Saint-Venant's principle.) The observation here is that a conformal map involving a long and narrow peninsula or finger, which hardly needs to be very long and narrow, will involve exponential distortions. Peninsulas become inlets in our context because approximation on a domain Ω depends on the conformal map of its complement $\mathbb{C}\backslash\overline{\Omega}$.

The most systematic analysis of crowding that I know of appears in [12]. The following definition is used there for an analytic Jordan domain Ω:

> We say that Ω *contains a finger of length* $L > 0$ if there is a rectangular channel of width 1 defined by a pair of parallel line segments of length L, disjoint from Ω, such that Ω extends all the way through the channel with parts of Ω lying outside both ends.

On the basis of this definition it is shown that the associated harmonic measure (Theorem 2 of [12]), conformal mapping derivative (Theorem 3), radius of univalence (Theorem 4), and approximating polynomial degrees (Theorem 5) all scale in ways controlled by the factor $\exp(\pi L)$. Thus, for example, peninsulas of length-to-width ratios 1, 2, and 3 induce distortions of magnitudes on the order of 23, 540, and 12,000.

In the context of the present paper, it follows from these theorems that the degrees of polynomial approximations to solutions of Laplace problems on any domain Ω will have to grow by a factor at least as large as order $\exp(\pi L)$ for each additional digit of accuracy, if Ω contains an inlet of length-to-width ratio L. We do not spell out precise theorems.

It is interesting to check how closely this general result matches Theorem 5 for the case of the inverted ellipse I_ρ. Setting $\varepsilon = \rho - 1$, we find that for I_ρ the inlet parameter L scales as[3]

$$L \sim \frac{1}{8\varepsilon}, \quad \varepsilon \to 0. \tag{7.1}$$

This implies by the $\exp(\pi L)$ results above that the analyticity radius R of sections 3 and 4 satisfies

$$R - 1 \le O(\exp(-\pi/8\varepsilon)), \tag{7.2}$$

which is looser by the factor 2π than the actual result of (4.5),

$$R - 1 \sim O(\exp(-\pi^2/4\varepsilon)). \tag{7.3}$$

We have seen that polynomial approximations of Laplace solutions can be expected almost always to be ineffective on domains with inlets. Conversely, will rational approximations almost always be effective? Certainly not always, for the boundary

[3]Equation (7.1) is derived by considering the rectangle $1 \le \operatorname{Re} z \le 2$, $-4\varepsilon \le \operatorname{Im} z \le 4\varepsilon$, with aspect ratio $L = 1/8\varepsilon$. For $\varepsilon < 0.225$, the right inward-pointing finger of I_ρ is contained within the top and bottom sides of this rectangle while extending outside both the left and right ends.

may be nonsmooth in the sense of having singularities of the Schwarz function very close by. But as a smooth example it is interesting to consider the inverted ellipse once more. From (7.1), the finger length is about $1/8\varepsilon$, whereas from Theorem 9, the convergence constant is $\rho^2 \sim 1 + 2\varepsilon$. Thus in this case at least, the convergence rate for rational approximations slows down only linearly with the length of the finger.

8. Discussion. We have shown that polynomial approximations to solutions of 2D Laplace problems are essentially useless on non-convex domains, whereas rational approximations are often very effective. In principle these results might have been obtained decades ago, but in practice, everything has changed since the AAA and AAA-least squares algorithms have made rational functions an easy tool for numerical computation.

The Laplace equation $\Delta u = 0$ is the limit of the Helmholtz equation $\Delta u + k^2 u = 0$ as $k \to 0$, and poles of rational functions can be regarded as the limits of point singularities of certain Hankel functions [11]. It is known to experts in the solution of the Helmholtz equation by the method of fundamental solutions that when a region has an inlet, it is crucial to include some Hankel singular points therein [3]. Thus the present paper could be regarded as investigating a limiting case of an effect that is known in the Helmholtz context, though we are not aware that theorems are to be found in that literature analogous to what we have developed here.

It is natural to wonder about the extension of these methods and results to three-dimensional problems. Here one would have to go beyond univariate polynomials and rational functions, and as yet, no algorithms have been proposed comparable to what is available in the 2D case. This is an exciting area for future research, and the mathematics of analytic continuation and Schwarz functions in higher dimensions [28] may again ultimately prove relevant.

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