Unbounded Growth of Band-Limited Functions

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Let f be a real or complex function of $\mathbb{R}$. The idea of f being *band-limited* is that it is composed of components e^{ikx} spanning a finite range of wave numbers k. To be concrete, let us take the range to be $k \in [-1, 1]$. The basic example of a function with wave numbers in this range is the *sinc function,*

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x}, \tag{1}$$

with $\operatorname{sinc}(0) = 1$ and $\operatorname{sinc}(n\pi) = 0$ for the other integers n, as illustrated in Figure 1.

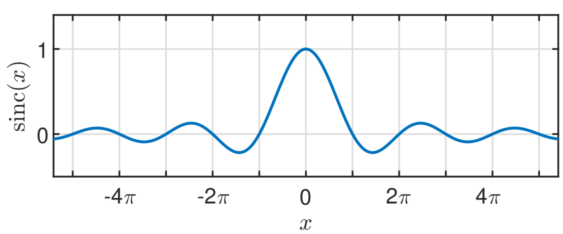

Figure 1: The sinc function is the basic example of a band-limited function, and the starting point of sampling theory.

A fundamental result for making such ideas precise is the *Paley-Wiener theorem* for band-limited functions in L^2 [Sim15, Thm. 11.1.2], [HJ94]. If $F \in L^2(\mathbb{R})$ has compact support in $[-1, 1]$, then its inverse Fourier transform

$$f(x) = \int_{-\infty}^{\infty} F(k)\, e^{ikx} dk \tag{2}$$

belongs to $L^2(\mathbb{R})$ and extends to an entire function of $z = x + iy$ satisfying

$$|f(z)| \le Ce^{|y|}, \quad z \in \mathbb{C} \tag{3}$$

for some C. (Throughout this note, C is a generic positive constant, changing from one appearance to the next.) Conversely, if $f \in L^2(\mathbb{R})$ extends to an entire function of $z = x + iy$ that satisfies (3) for some C, then its Fourier transform

$$F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-ikx} dx \tag{4}$$

belongs to $L^2(\mathbb{R})$ and has compact support in $[-1, 1]$. In the case of $\operatorname{sinc}(x)$, the Fourier transform is π times the characteristic function of $[-1, 1]$, and since $\sin(x) = (e^{ix} - e^{-ix})/2i$, it is obvious that $\operatorname{sinc}(x)$ satisfies (3).

Band-limited functions became a subject of concerted attention especially through the work of Henry Landau, Henry Pollak, and David Slepian at Bell Labs in the early 1960s. Like their colleagues Claude Shannon and Richard Hamming, these men were concerned with *sampling theory,* the study of relationships beteween a continuous signal and its samples taken at regular intervals. The *Nyquist sampling theorem* asserts that an L^2 function f that is band-limited to $[-1, 1]$ can be recovered from its samples $\{f(\pi n)\}$, and this observation highlights the importance of the sinc function: the recovery formula is

$$f(x) = \sum_{n=-\infty}^{\infty} f(\pi n) \operatorname{sinc}(x - \pi n). \tag{5}$$

This brings us to a puzzle I encountered two or three years ago, which led to a conjecture. The application area is *numerical analytic continuation* of

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an analytic function f beyond the real or complex domain where it is known. Analytic continuation is a standard notion that can be effected in theory by the Weierstrass chain-of-disks method, but what about practical algorithms applicable to functions just known numerically? The best general method seems to be to make use of rational approximations to f, and from this work emerged a curious empirical observation.

Here is the *one-wavelength principle* [Tre23]. It seems that in the usual 16-digit computer arithmetic, all kinds of oscillatory functions can be numerically analytically continued about one wavelength beyond their domain of definition (assuming this domain is big enough) before accuracy is lost. This is a rough observation, not tied to any precise notions of accuracy or wavelength. The appearance of the number *one* in the principle is a coincidence related to computations typically employing approximations to 13 digits of accuracy: with 26 or 39 digits one can track about 2 or 3 wavelengths, respectively. Figure 2 illustrates the effect. In this example, the function $f(x)$ being extrapolated is a portion of the trajectory of one of the three components of a solution to the Lorenz equations. More examples are given in [Tre23].

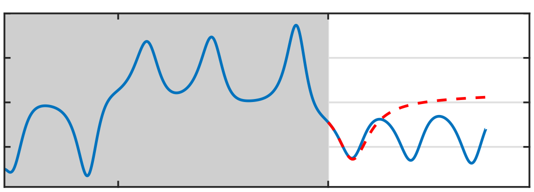

Figure 2: The blue curve is an analytic function $f(x)$. Numerical analytic continuation [Tre23] to the right of the gray interval gives the dashed red curve, which matches $f(x)$ for about one wavelength.

The one-wavelength principle is a rough observation, and I tried to develop a possible explanation of it by investigating what seemed a simple model of a problem of this kind. Let $\mathcal{B}$ be the class of functions $f \in L^2(\mathbb{R})$ that are band-limited to $[-1,1]$ and satisfy $|f(x)| \le 1$ for $x \le 0$. Define

$$M(x) = \sup_{f\in\mathcal{B}} |f(x)|, \quad x > 0. \tag{6}$$

How fast does M grow as a function of x? The conjecture was that it grows exponentially as $x \to \infty$. For suppose this were true with an approximate growth rate

$$M(x) \approx e^{Cx}, \quad C \approx \frac{\log(10^{13})}{2\pi} \approx 4.8, \tag{7}$$

which corresponds to a factor of 10^{13} over an interval of length 2π. Then this would give some kind of explanation of the observed behavior.

A 1986 paper by Landau [Lan86] (1931–2020) is the sole publication I have found that considers $M(x)$, and it is encouragingly consistent with the conjecture. Landau states the following theorem in the language of sampling theory.

> *Theorem 2:* When sample measurements are accurate only to within $\varepsilon > 0$ in amplitude or in total energy, good extrapolation is possible for only a bounded distance (having an order of magnitude $-\log\varepsilon$) beyond the interval of observation, regardless of the amount of data used.

Landau does not give quantitative bounds, but his proof of the theorem can be unwound to show that $M(x)$ grows at least exponentially with a constant C about 1/8 of that proposed in (7). He established this exponential growth by considering translates of the band-limited functions $\mathrm{sinc}(x), (\mathrm{sinc}(x/2))^2, (\mathrm{sinc}(x/3))^3, \ldots$.

Since 1986, Landau's paper has had impact in discussions of *superresolution, superoscillation,* and *prolate spheroidal wave functions,* which had been introduced by Landau, Pollak, and Slepian themselves in their earlier work; a few of the many references in this area are [Ber94, BZAT19, Lin12, ORX13, RR20]. Landau's theorem is cited for example as establishing that extrapolations of band-limited functions can be "exponentially unstable" [Lin12] or "highly unstable" [RR20]. All these areas of discourse are concerned with matters of how rapidly a band-limited function can change from one region to another, putting them very much in the realm of the e^{Cx} conjecture.

To try to find a proof and a derivation of C, I discussed the problem with various colleagues. One of these was John Urschel at MIT, who shared it with Alex Cohen, a graduate student at MIT working with Larry Guth. Cohen, to my surprise, saw that the conjecture is false. In fact, there is no bound on how fast band-limited functions can grow. Here is Cohen's result.

Theorem. *With the definition* (6)*,* $M(x) = \infty$ *for all* $x > 0$*.*

In a word, if you can bound a band-limited function on one side, that gives no constraints on its magnitude. (It's different if the function is bounded on both sides.) Thus the one-wavelength principle, if it is true in some sense, must find another mathematical explanation.

Cohen's proof. Consider the function

$$g(x) = \cos(a\sqrt{-x}\,) = \cosh(a\sqrt{x}\,), \tag{8}$$

where $a > 0$ is a parameter. This function is entire (the evenness of cosh takes care of the square root) and it satisfies $|g(x)| \le 1$ for $x \le 0$. For any $x > 0$, it grows without bound as $a \to \infty$. What's missing is that g is not in $L^2(\mathbb{R})$, so it is not in the class $\mathcal{B}$. However, this can be fixed by multiplying it by a rapidly-decaying band-limited function ψ, so that in fact our counterexample becomes the a-dependent family of functions

$$f(x) = g(x)\psi(x). \tag{9}$$

Specifically, we choose ψ to be a nonzero entire function satisfying

$$|\psi(z)| \le e^{|y|/2 - |x|^\sigma}, \quad z = x + iy \in \mathbb{C} \tag{10}$$

for some $\sigma \in (1/2, 1)$. Since $g(x) = O(\exp(a|x|^{1/2}))$ and $\sigma > 1/2$, such a choice guarantees that $f \in L^2(\mathbb{R})$ for all a. The reason for requiring $\sigma < 1$ is that this allows such a bandlimited function to exist. There can be no band-limited function with decay $\psi(x) = O(\exp(-C|x|))$, because its Fourier transform would have to be analytic (by another Paley-Wiener theorem), which precludes compact support. But it is known that band-limited functions exist satisfying (10) for any $\sigma < 1$ [BM62], [Bjö66, Thm. 1.4.1]. To show that $M(x) = \infty$ for a given $x > 0$ and thus prove the theorem, we just have to pick such a ψ that is nonzero at this value of x to ensure that $|f(x)| \to \infty$ as $a \to \infty$.

To finish the argument, it remains to confirm that f is band-limited as required, satisfying the condition (3). Following (8), (9), and (10), we calculate

$$|f(x+iy)| \le \exp(a|x+iy|^{1/2} - |x|^\sigma + |y|/2).$$

Now since $|x+iy| \le 2\max\{|x|,|y|\}$ and therefore $|x+iy|^{1/2} \le 2(|x|^{1/2} + |y|^{1/2})$, we have

$$|f(x+iy)| \le \exp(2a|x|^{1/2} + 2a|y|^{1/2} - |x|^\sigma + |y|/2).$$

The proof is completed by noting that for some constant C,

$$\exp(2a|x|^{1/2} + 2a|y|^{1/2} - |x|^\sigma) \le C\exp(|y|/2).$$

The value of C depends on a, but uniformity with respect to a is not needed. ∎

The function $\cosh(a\sqrt{x})$ has remarkable properties. For $x \le 0$ it is just an oscillatory cosine of an argument varying with x, but the oscillation gets faster as a increases, as shown in Figure 3. One would hardly guess from the plots that this whole a-dependent class of functions is uniformly band-limited, but this is the case—and precisely so in the L^2 sense once g is multiplied by ψ. What makes this possible is the exponentially great scale of $g(x)$ for $x > 0$, growing rapidly as $a \to \infty$. Thus $x \le 0$ lies at the edge of the main signal, and as is well known to experts in superoscillation and prolate spheroidal wave functions, almost anything is possible in the edges.

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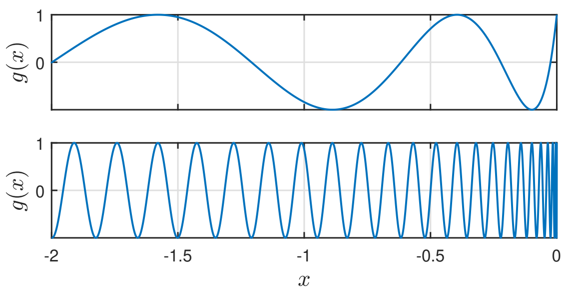

Figure 3: The function $g(x)$ of (8) on the negative real axis for $a = 10$ (above) and 100 (below). Despite the increasingly rapid oscillations as $a \to \infty$, all such functions are uniformly band-limited after multiplication by the fixed envelope $\psi(x)$, an example of superoscillation. For $x > 0$, they take huge values.

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