Is Gauss quadrature better than Clenshaw-Curtis?

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For $f \in C[-1,1]$, define

$$I = \int_{-1}^{1} f(x)\,dx, \qquad I_n = \sum_{k=0}^{n} w_k f(x_k)$$

where $\{x_k\}$ are nodes in $[-1,1]$ and $\{w_k\}$ are weights such that $I = I_n$ if f is a polynomial of degree $\le n$.

Newton-Cotes: $x_k = -1 + 2k/n$ **diverges as $n \to \infty$ (Runge phenomenon)**

Clenshaw-Curtis: $x_k = \cos(k\pi/n)$ **converges as $n \to \infty$**

Gauss: x_k = k th root of Legendre poly P_{n+1} **converges as $n \to \infty$**

C-C is easily implemented via FFT ($O(n \log n)$ flops).
Gauss involves an eigenvalue problem ($O(n^2)$ flops).

(HANDOUT)

We think of Gauss as “twice as good” as C-C:

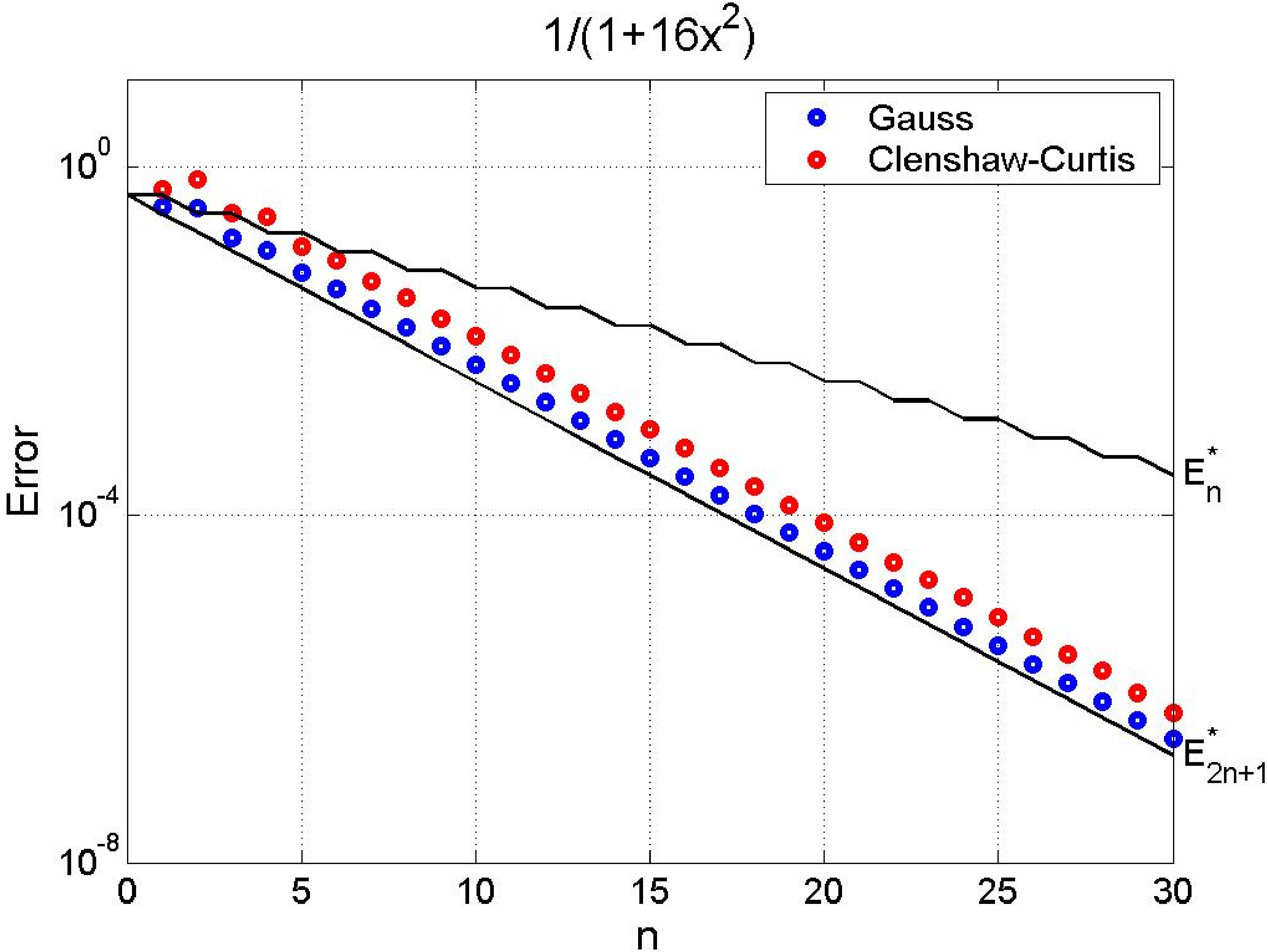
THEOREM

C-C: $| I - I_n | \leq 4\, E_n^*$

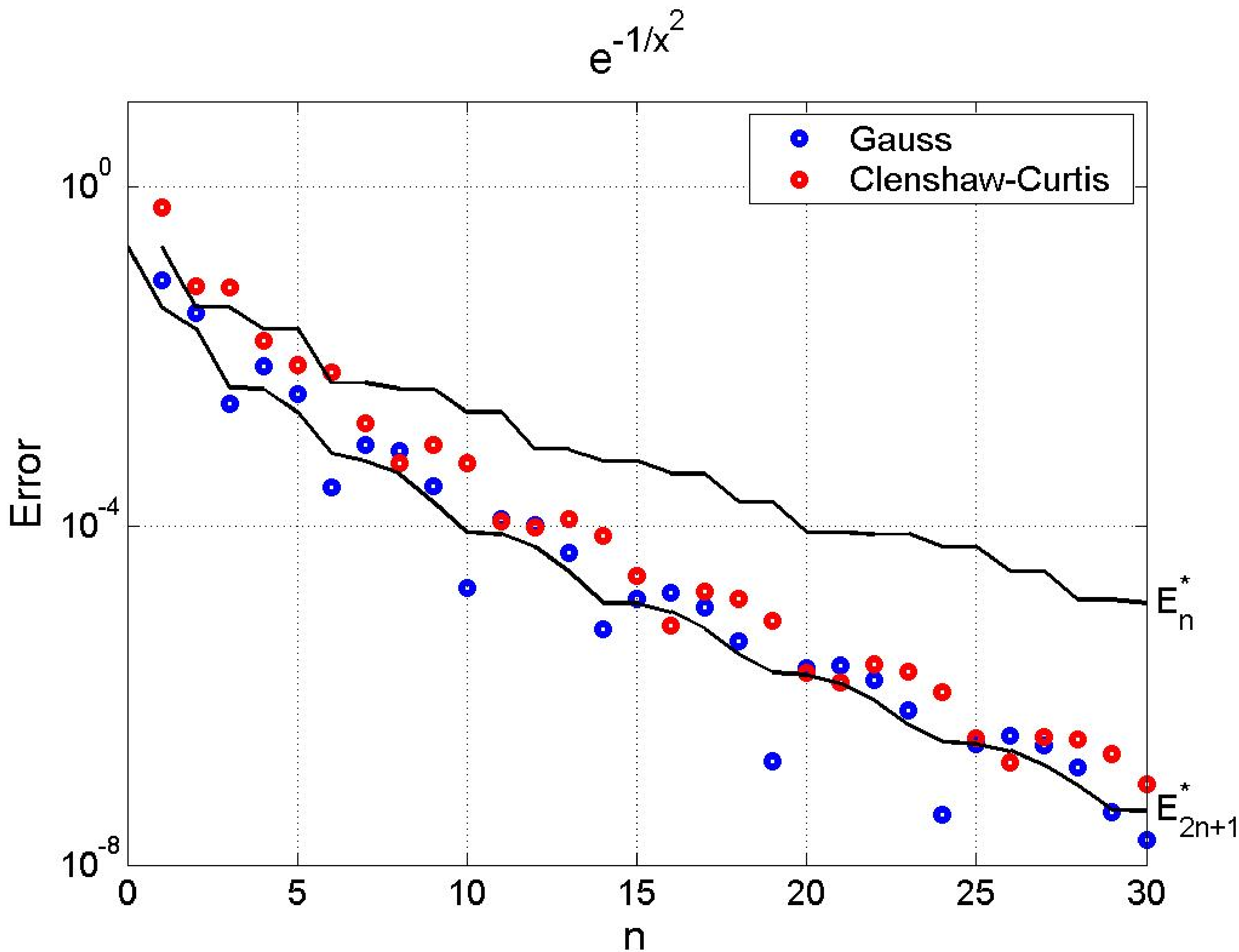
Gauss: $| I - I_n | \leq 4\, E_{2n+1}^*$

best approximation errors for polynomials of degrees *n* , 2*n*+1

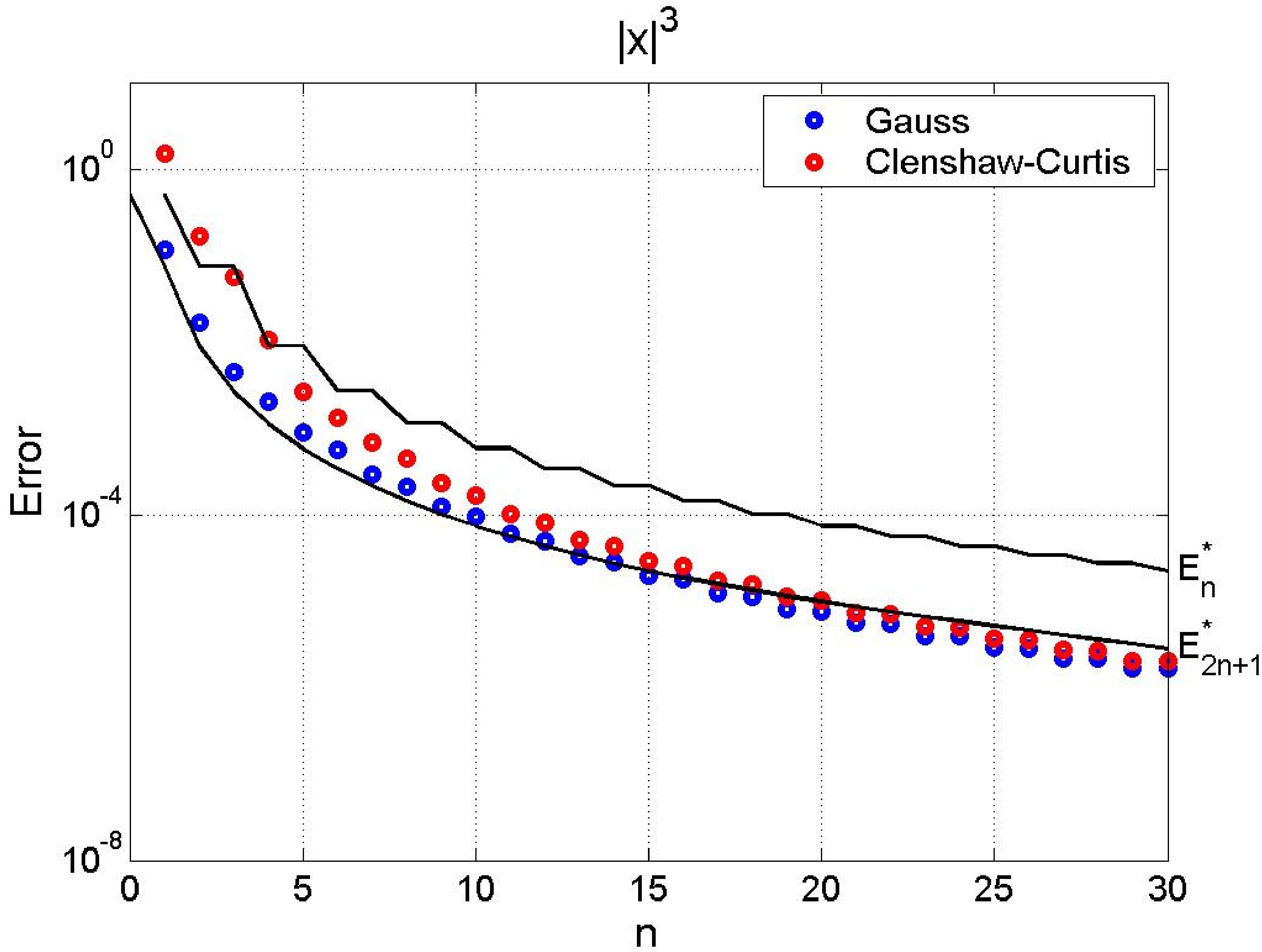
Yet in experiments, this factor of 2 often doesn’t appear.



1/(1+16x^2)
Gauss
Clenshaw-Curtis
Error
n
10^0
10^-4
10^-8
0
5
10
15
20
25
30
E*_n
E*_2n+1



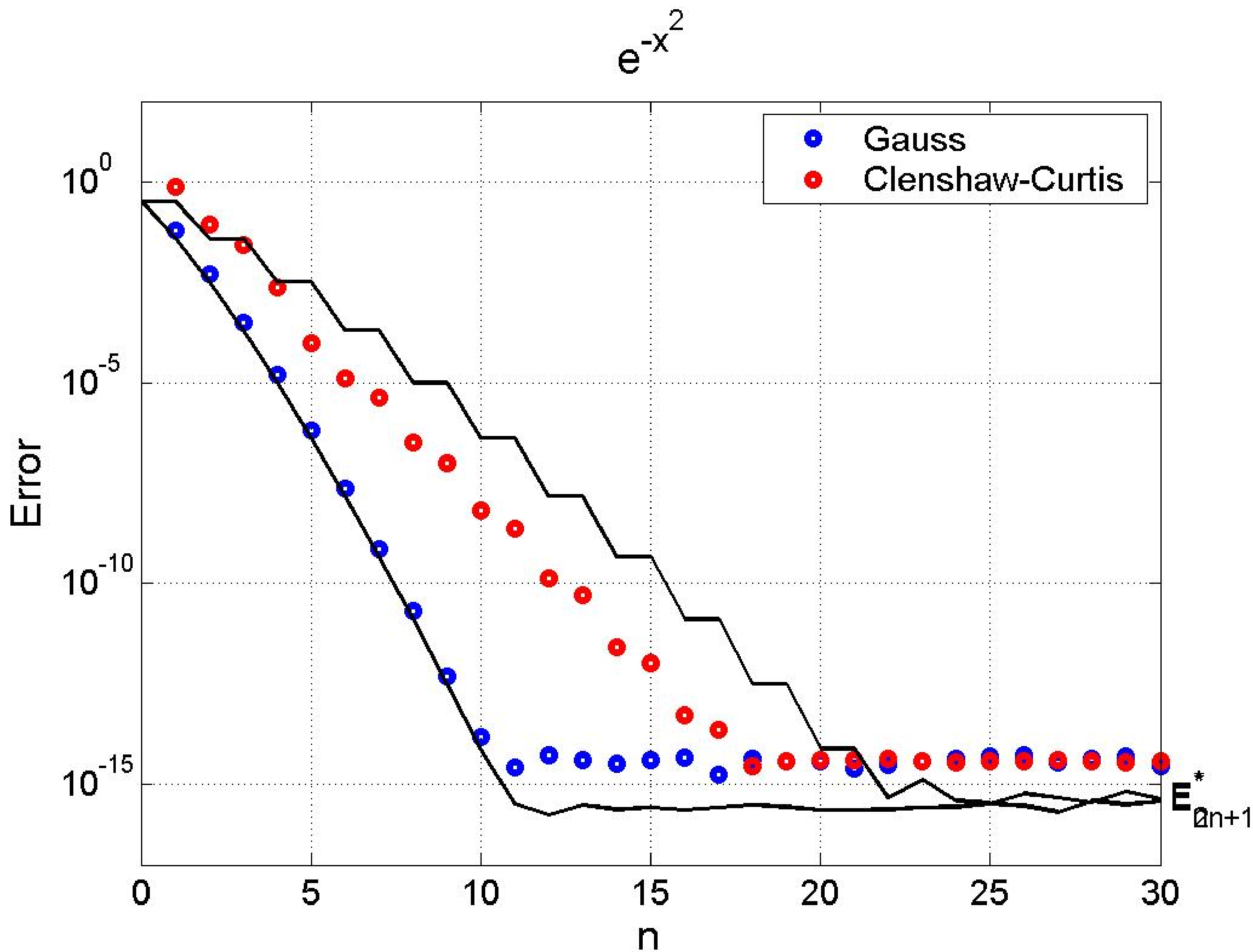
e^{-1/x^2}
Gauss
Clenshaw-Curtis
Error
10^0
10^{-4}
10^{-8}
0
5
10
15
20
25
30
n
E_n^*
E_{2n+1}^*



$|x|^3$
Gauss
Clenshaw-Curtis
Error
10^0
10^{-4}
10^{-8}
0
5
10
15
20
25
30
n
E_n^*
E_{2n+1}^*

In fact, Gauss beats C-C only for functions analytic in a big neighborhood of [–1,1].

And even then rarely by a full factor of 2.



e^{-x^2}
Gauss
Clenshaw-Curtis
Error
n
10^{0}
10^{-5}
10^{-10}
10^{-15}
0
5
10
15
20
25
30
E^*_{2n+1}

The Gauss ≈ C-C phenomenon was noted by O'Hara and Smith (*Computer J.* 1968), but no theorems were proved.

Here's a theorem. ("Variation" involves a certain Chebyshev-weighted total variation, and $C = 64/15\pi$.)

THEOREM. Let $f^{(k)}$ have variation $V < \infty$. Then for $n \geq k/2$, the Gauss quadrature error satisfies

$$| I - I_n | \leq C\, k^{-1} (2n+1-k)^{-k} \,. \qquad (*)$$

THEOREM. For suff. large n, the C-C error satisfies $(*)$ too!

Proofs: based on Chebyshev coefficients and aliasing.

But really I came here to show you some pictures.

Suppose f is analytic on [–1,1]. Let Γ be a contour in the region of analyticity of f enclosing [–1,1].

The following identity was used e.g. by Takahasi and Mori ≈1970 but more or less goes back to Gauss. (See Gautschi's wonderful 1981 survey of G. quad. formulas.)

THEOREM. For any interpolatory quadrature formula with nodes $\{x_k\}$ and weights $\{w_k\}$,

$$I - I_n = (2\pi i)^{-1} \int_\Gamma f(z) \left[\log((z+1)/(z-1)) - r_n(z) \right]$$

where $r_n(z)$ is the type $(n, n+1)$ rational function with poles $\{x_k\}$ and corresponding residues $\{w_k\}$.

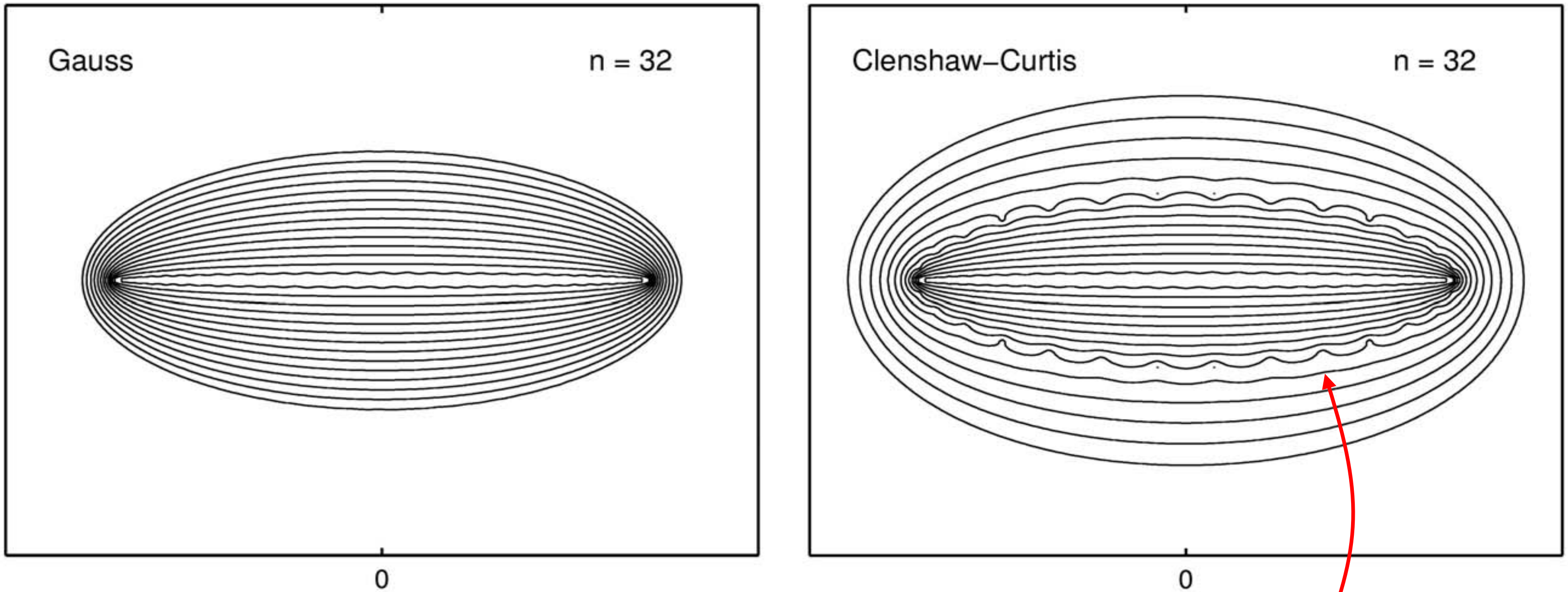
Proof: Cauchy integral formula.

So convergence of a quadrature formula depends on accuracy of rational approximations: $\log((z+1)/(z-1)) \approx r_n(z)$.

Contour lines $|\log((z+1)/(z-1)) - r_n(z)| = 10^0, 10^{-1}, 10^{-2}, \ldots$

(from inside out)

$n = 32$



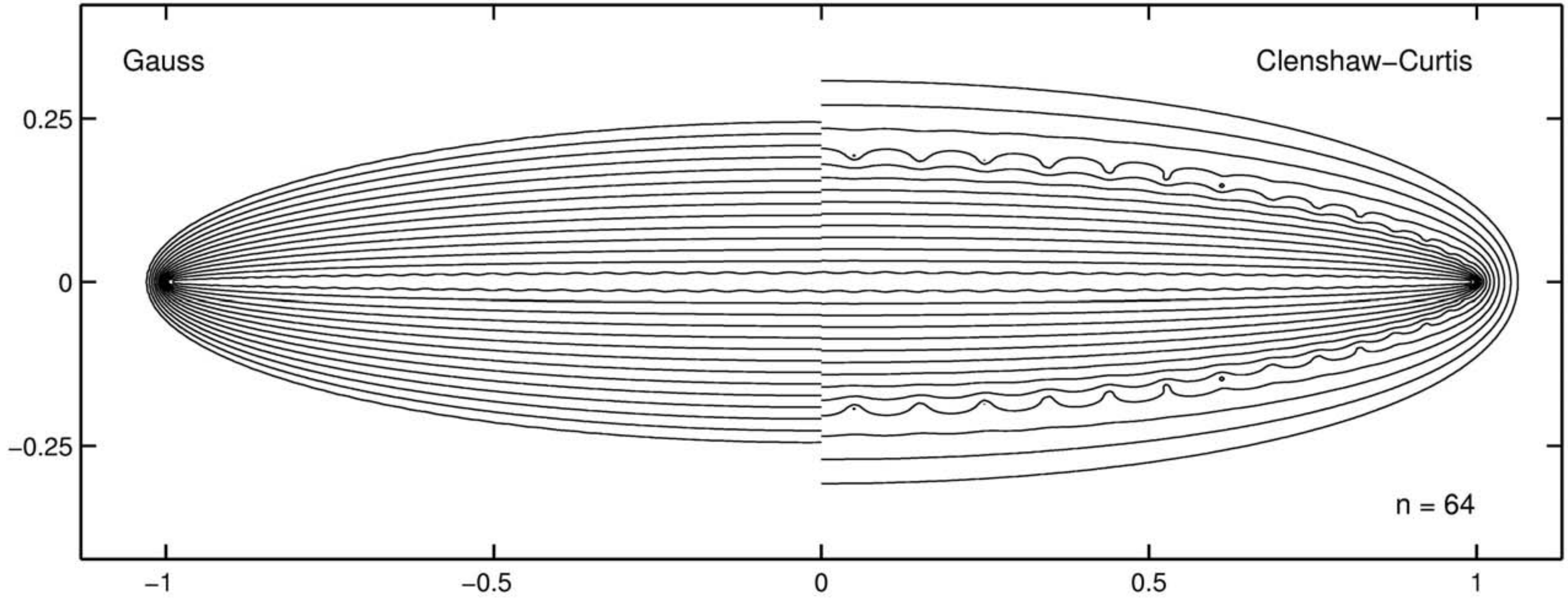
For Gauss quadrature, there are $2n+3$ interpolation points, all at ∞

Thus r_n **is a Padé approximant.**
(This is how Gauss himself derived Gauss quad.!)

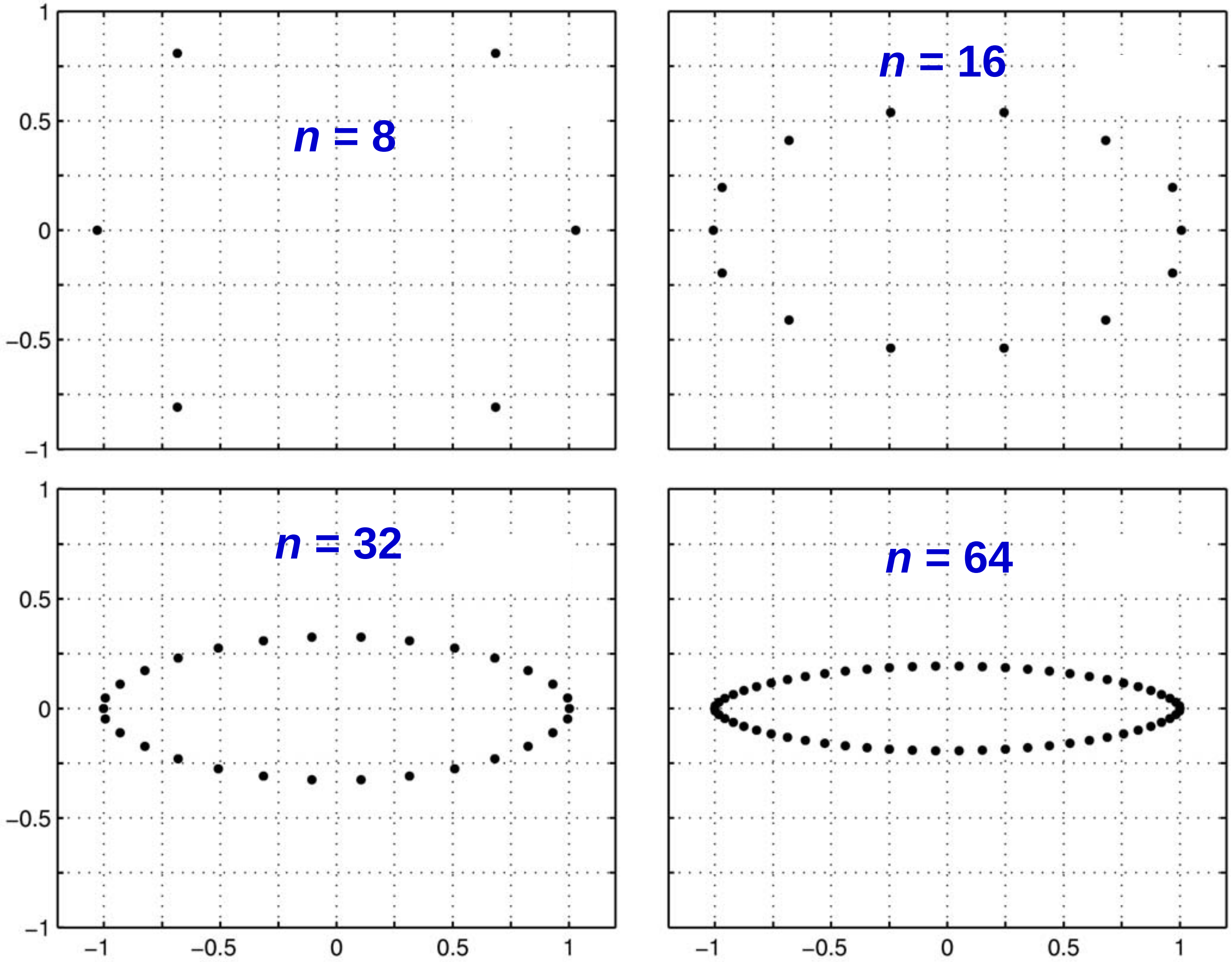
Scallops reveal interpolation points — $n-2$ of them (as well as $n+3$ at ∞)

Contour lines $|\log((z+1)/(z-1)) - r_n(z)| = 10^0, 10^{-1}, 10^{-2}, \ldots$

$n = 64$

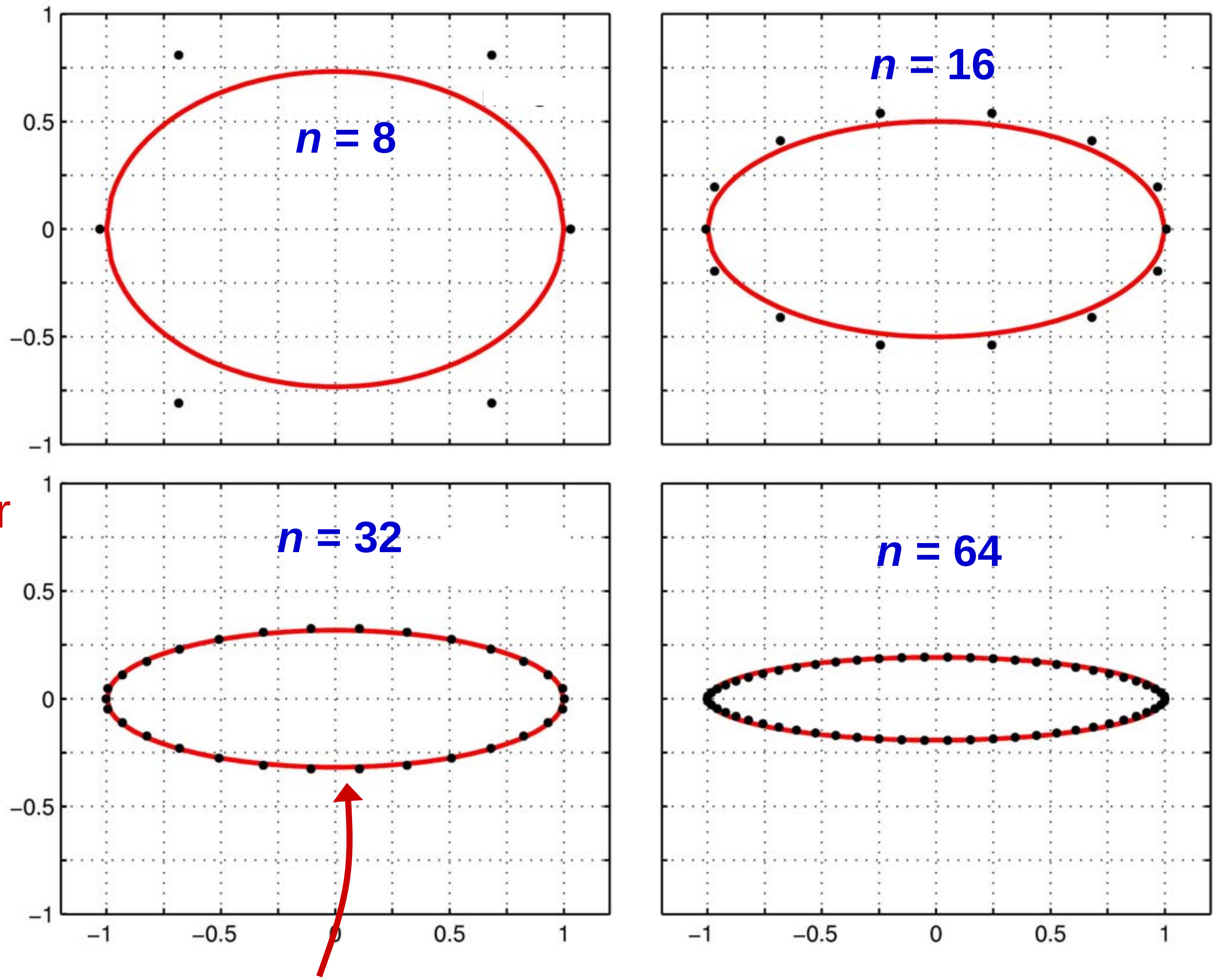


Interpolation pts — zeros of $\log((z+1)/(z-1)) - r_n(z)$



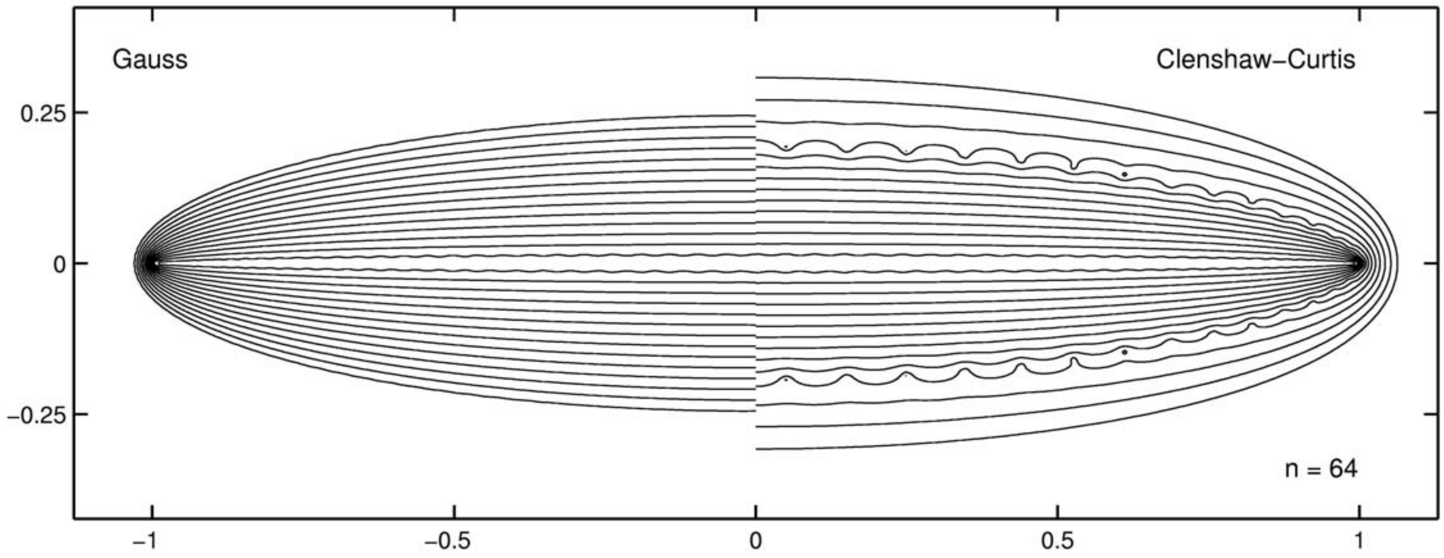
Weideman has shown that these ovals are close to ellipses of semiaxis lengths 1 and $3 \log n / n$.

Interpolation pts — zeros of $\log((z+1)/(z-1)) - r_n(z)$



I suspect the essence of the matter is potential theory — "balayage"

Weideman has shown that these ovals are close to ellipses of semiaxis lengths 1 and 3 log *n* / *n* .

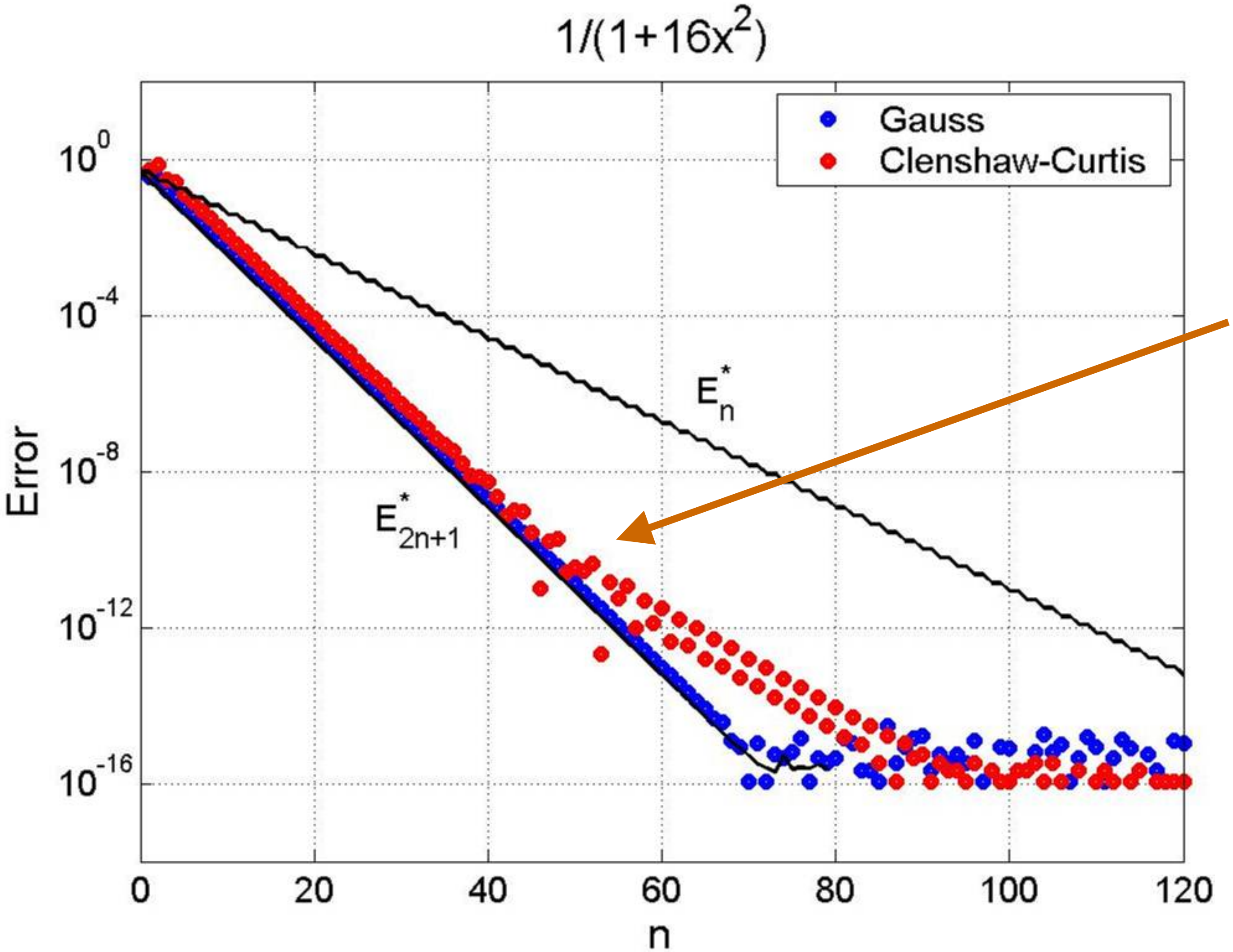


These observations suggest a prediction:

C-C is as good as Gauss when the region of analyticity of f is smaller than the magic oval.

This is just what we observe.
We finish with an experiment to illustrate.

Same experiment as before, carried to higher n . As n increases, the oval shrinks and cuts across the pole of f .



Thus Weideman's analysis explains why this kink appears where it does. Paper to appear.