The physical meaning of nonsymmetric eigenvalue problems

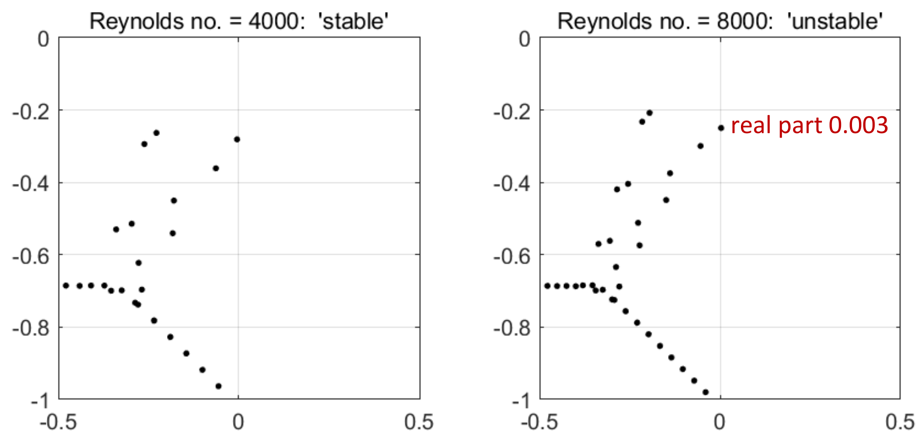
Nick Trefethen, June 2021

In mathematics, we idealize things. This is an essential and powerful part of what we do. All of us who have any contact with applications know that we must be on our guard to make sure the idealizations don't lose contact with the scientific problem they are aiming to shed light on.

This is true all across mathematics, but I believe that nonsymmetric eigenvalue problems are a particularly extreme case, so extreme that in a sizeable fraction of applications, nonsymmetric eigenvalues don't have the significance people think they have.

To explain what I mean I want to show a picture of the nonsymmetric eigenvalue problem that has possibly been studied the most in the sciences: the Orr-Sommerfeld equation. The field is stability of fluid flows. The scientific question is, when and how do high speed fluid flows go turbulent?

To answer this question, since Orr and Sommerfeld in 1908, fluid mechanics people have investigated eigenvalues. It's a nonlinear problem, and these are the eigenvalues of a linearization around a smooth, nonturbulent solution. The idea is that if all the eigenvalues are in the left half-plane, the flow is stable, and if there's an eigenvalue in the right half-plane, it is unstable and you'll end up with turbulence.



So let's look at the picture. The Reynolds number Re is the nondimensionalized speed of the flow, and on the left, at $Re = 4000$, the flow is eigenvalue stable. On the right, at $Re = 8000$, it is eigenvalue unstable. An incredible amount is known about these eigenvalues, and Steve Orszag got famous in part for calculating that the critical Reynolds number at which the rightmost eigenvalue moves into the right half-plane is 5772.22. So the traditional view is that something suddenly changes when Re hits 5772. And of course there are theorems that prove that in certain senses this is true.

And yet for a century, laboratory experiments have almost never fitted this picture. Actual flows don't show a clear Reynolds number at which there is transition to turbulence. In fact in most experiments the flow would normally be turbulent at both $Re = 4000$ and 8000. And now I want to draw your attention to the number shown in red. What exactly is implied by this eigenvalue? It is that flow perturbations can grow at the incredibly slow rate $\exp(0.003t)$. That means that by $t = 300$, they can be amplified by a factor of e. This translates to a channel 300 times as long is at is wide, which is pretty much at the limit of the kind of channel you could build in the lab. And of course a factor of merely e is not going to drive turbulence; you'll need more amplification than that. So if you think about this picture quantitatively, you find that the "instability" it presents is one that should be unobservable in experiments. And sure enough, it is not observed in experiments.

Yet fluid flows go turbulent. The reason is related to other parts of the spectrum, well in the stable left half-plane, associated with strong nonnormality, not shown in these pictures.

In short: when an eigenvalue goes "unstable," nothing suddenly changes. All that changes is the potential fate of certain trajectories *if* the system remains unchanged and unperturbed for a sufficiently long time. There are systems that behave like that, usually featuring symmetric or nearly symmetric matrices. But in plenty of other cases, certainly in high Reynolds number fluid mechanics, eigenvalue analysis has brought a great deal of confusion. In areas like ecology and food webs, with all their complexities and time-dependencies, the idea of inferring anything precise from whether or not there are eigenvalues in the right half-plane is really very nebulous.