QUADRATURE WITH EQUISPACED NODES

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Abstract. Quadrature formulas for integrating a continuous function f based on n equispaced samples in $[-1,1]$ range from the composite trapezoidal rule, which converges as $n \to \infty$ for all continuous f but only very slowly even when f is smooth, to the Newton–Cotes formulas, which converge exponentially when f is entire but are exponentially unstable and do not converge at all in general. We prove an equispaced quadrature analogue of the "impossibility theorem" for equispaced approximation, establishing that the best convergence rate for analytic integrands achievable by any stable equispaced quadrature formula is root-exponential: $O(C^{-\sqrt{n}})$ for some $C > 1$. One class of such formulas is in the literature: the least-squares "EquiLS" method of Wilson (1970) and Huybrechs (2009), which is based on least-squares fitting of the function samples by a polynomial of degree $O(\sqrt{n})$. Like all polynomial quadrature or approximation methods, EquiLS is nonuniform in its properties, having extra resolution for $x \approx \pm 1$ at the cost of less resolution for $x \approx 0$. We introduce a new "EquiRat" method based instead on least-squares fitting by rational functions with poles equally spaced along the sides of a strip of width $O(1/\sqrt{n})$. We show that EquiRat has uniform resolution along $[-1,1]$ and conjecture that it is root-exponentially convergent for analytic integrands. We show numerically that it is weakly unstable, with weights growing at a rate about $\exp(0.3\sqrt{n})$ as $n \to \infty$, too slowly to cause trouble normally in practice. A MATLAB code for stable computation of EquiRat weights based on Cauchy with Arnoldi orthogonalization is given. Convergence and other properties of EquiLS and EquiRat are compared in numerical experiments, and numerical comparisons are also made with the Gregory and Floater–Hormann methods of equispaced quadrature. Throughout the discussion, we make use of the duality implicit in all quadrature formulas between rational approximation of an integrand $f(x)$ and rational approximation of $2\pi i$ times the associated Cauchy transform $\mathcal{C}(s)$.

Key words. quadrature, rational approximation, equally spaced data, impossibility theorem, Cauchy transform, EquiLS, EquiRat, AAA approximation, Cauchy with Arnoldi

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1. Introduction.

Suppose an integral

$$I(f) = \int_{-1}^{1} f(x)\,dx \tag{1.1}$$

with $f \in C([-1,1])$ is to be approximated by a quadrature formula

$$I_n(f) = \sum_{k=1}^{n} w_k f(x_k) \tag{1.2}$$

defined on a set of n equispaced nodes $X_n = \{x_k\}$,

$$x_k = -1 + (k-1)h, \quad h = \frac{2}{n-1}, \quad 1 \le k \le n \tag{1.3}$$

for some $n \ge 2$. What is a good choice of the weights $\{w_k\}$? This would appear to be as basic a question of quadrature as could be asked, yet there is no established answer. (For an extensive discussion, see the excellent book by Brass and Petras [3].) The aim of this paper is to review the reasons for that, prove some relevant theorems,

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compare various possibilities, and introduce a new family of what we call *EquiRat* quadrature formulas.

One famous choice, the (composite) *trapezoidal rule*, takes $w_k = h$ for $2 \le k \le n-1$ and $w_1 = w_n = h/2$. The limitation here is that the convergence rate for smooth integrands is only $O(n^{-2})$ as $n \to \infty$. At the other extreme are the *Newton–Cotes* formulas [3], where $\{w_k\}$ are chosen so that $I_n(f)$ is the integral of the unique degree $n-1$ polynomial interpolant to the data $\{f(x_k)\}$. These formulas have order of accuracy n (actually $n+1$ when n is odd), so one might expect convergence at a rate $O(C^{-n})$ for some $C > 1$ for analytic integrands, but in fact they have exponentially growing positive and negative weights and don't converge at all in general, even for analytic integrands in the absence of rounding errors. The failure of Newton–Cotes formulas as $n \to \infty$ must have been obvious to experts after Runge published his paper in 1901 on the exponential divergence of equispaced polynomial interpolants, the *Runge phenomenon* [25], but a theorem confirming this was not published until Pólya in 1933 [24].[1] For a quadrature formula (1.2), the ∞-norm of the functional I_n is equal to the 1-norm of the weight vector,

$$\|I_n\| = \|w\|_1, \tag{1.4}$$

and Pólya showed that a consistent formula will converge for all continuous integrands as $n \to \infty$ if and only if these norms are uniformly bounded. (Consistency means that the formula converges as $n \to \infty$ for all polynomials, or for some other dense set of functions.) Nowadays we can interpret this result as a first instance of what Strang calls the *Fundamental Theorem of Numerical Analysis* [26, sec. 6.5]: a consistent numerical discretization scheme is convergent if and only if it is stable. (The more famous later instances involved discretizations of ODEs and PDEs [5, 19].)

As Huybrechs puts it in [15] (speaking specifically of composite Newton–Cotes formulas),

> The low-order rules converge slowly and the high-order rules are numerically unstable.

The sweet spot on the spectrum between the trapezoidal rule and the Newton–Cotes formulas is *root-exponential convergence*, at a rate $O(C^{-\sqrt{n}})$ for some $C > 1$ for analytic integrands. In Theorem 2.1 we establish the optimality of this convergence rate, showing that any scheme that converges faster for analytic integrands must be unstable and accordingly nonconvergent for continuous integrands. This is the analogue for quadrature of the "impossibility theorem" of [23] for function approximation, which asserts that a stable method for approximation from equispaced data can converge no faster than root-exponentially.

We begin the paper with a statement and proof of Theorem 2.1 in section 2. Section 3 summarizes half a dozen quadrature schemes for equispaced points, of which one, least-squares quadrature, has the property of stable root-exponential convergence. Section 4 then presents our proposed new scheme that at least comes close to having this property, *EquiRat* quadrature. One way to derive this is from the fundamental connection between quadrature and rational approximation of a Cauchy transform that was introduced implicitly by Gauss in 1814 [11], exploited explicitly

[1] In 1925 Ouspensky established the exponential divergence of the weights [22], commenting (in French) that "One sees that the [weights] tend to infinity, making it apparent that the Cotes formula loses all practical value when the number of samples grows large." Related work on interpolation is also due to Hahn in 1918 [12].

by Takahasi and Mori in 1971 [27], and stated in a general form recently by ourselves [14]. Section 5 presents a MATLAB code for stable computation of EquiRat quadrature weights by the Cauchy with Arnoldi method [8]. Various properties of EquiRat quadrature are then examined and compared with least-squares quadrature in section 6: rational approximation interpolation points and accuracy contours, quadrature weights, and convergence for sample integrands. The reason for our focus on the comparison of EquiRat with least-squares quadrature is that the two methods are close in spirit and effectiveness, the fundamental difference between them being that least-squares quadrature aims at integration of polynomials of maximal stable degree—implying much more power for $x \approx \pm 1$ than for $x \approx 0$—whereas EquiRat aims at uniform approximation all along $[-1, 1]$. The comparisons are extended to other quadrature methods in section 7. Section 8 discusses the accuracy of EquiRat and EquiLS for analytic integrands and section 9 their stability. Finally, section 10 begins with a systematic summary of the contributions of the paper and then comments on practical implications.

2. Impossibility theorem: root-exponential convergence. The impossibility theorem for function approximation, Theorem 3.1 of [23], can be summarized as follows. Consider an arbitrary scheme, linear or nonlinear, for approximating functions from n equispaced sample values on $[-1, 1]$. Let E be a compact subset of $\mathbb{C}$ containing $[-1, 1]$ in its interior, and let $\mathcal{A}(E)$ be the Banach space of functions continuous on E and analytic in the interior with norm $\|f\|_E = \sup_{x\in E} |f(x)|$. If the approximations converge exponentially for all $f \in \mathcal{A}(E)$, then the scheme must be exponentially unstable. Theorem 3.4 of [23] generalizes this result to subexponential convergence and subexponential instability. In particular, it asserts the following root-exponential optimality condition. If the approximations converge at a rate $C^{-n^{1/2+a}}$ for some $C > 1$ and $a \in (0, 1/2]$, then the scheme must be unstable, with condition numbers growing at the rate at least $D^{n^{2a}}$ for some $D > 1$.

The classic application of the impossibility theorem is to the approximation scheme considered by Runge, namely, equispaced polynomial interpolation. In this case the convergence is indeed exponential provided the set E is big enough to encompass a certain region around $[-1, 1]$ sometimes known as the "Runge football." (Conjecture 8.2 will describe an alternative equispaced approximation scheme where the football is not needed: analyticity on $[-1, 1]$ is enough.) The interpolation scheme is exponentially unstable, however, with small perturbations being amplified by factors of the order of approximately 2^n. As mentioned earlier, Pólya extended this result to the quadrature formulas derived from global polynomial interpolation, namely the Newton–Cotes formulas. He showed that the Newton–Cotes formulas diverge as $n \to \infty$ in general not only for continuous integrands but even for analytic ones.

Just as Runge's specialized result leads to a theorem for quadrature, so does the more general impossibility theorem. The following statement of the new theorem covers just the usual case of linear quadrature formulas (1.2). However, the result can be extended as in [23] to nonlinear quadrature approximations, which might be based for example on adaptive Floater–Hormann or AAA approximation of the integrand.

THEOREM 2.1. *Consider a family $\{I_n\}$ of equispaced quadrature formulas* (1.2)–(1.3) *indexed by n. Suppose that for some compact subset E of $\mathbb{C}$ containing* $[-1, 1]$ *in its interior, the convergence is faster than root-exponential for all functions $f \in \mathcal{A}(E)$:*

$$|I_n(f) - I(f)| \le MC^{-n^{1/2+\alpha}} \|f\|_E, \quad C > 1 \tag{2.1}$$

for some $\alpha \in (0, 1/2]$ and $M > 0$. Then for all sufficiently large n, the ∞-norms of $\{I_n\}$ satisfy

$$\|I_n\| \geq D^{n^{2\alpha-\varepsilon}}, \quad D > 1 \tag{2.2}$$

for any $\varepsilon > 0$.

Proof. The following is adapted from [23], where in particular, the caption of Figure 3.1 gives a particularly lean summary of the argument.

We take $\alpha = 1/2$ in (2.1) and (2.2); the general case $\alpha \in (0, 1/2]$ is similar. In view of the wiggle room provided by the exponent ε in (2.2), we can ignore the constant M for sufficiently large n and suppose that the quadrature approximation gives

$$|I_n(f) - I(f)| \leq C^{-n}\|f\|_E, \quad C > 1 \tag{2.3}$$

for all $f \in \mathcal{A}(E)$ for sufficiently large n. In particular, we have

$$|I_n(p) - I(p)| \leq C^{-n}\|p\|_E \tag{2.4}$$

for any polynomial p, for sufficiently large n. Now if (2.3) holds for a set E, it also holds for any larger set $E' \supseteq E$, since $\mathcal{A}(E') \subseteq \mathcal{A}(E)$ and $\|f\|_{E'} \geq \|f\|_E$ for any $f \in \mathcal{A}(E')$ by the maximum modulus principle. Therefore we can assume without loss of generality that E is the Bernstein ρ-ellipse for some $\rho > 1$, i.e., the ellipse with foci ± 1 whose semimajor and semiminor axis lengths sum to ρ. Now by Bernstein's lemma,

$$\|p\|_E \leq \rho^d \|p\|_{[-1,1]}, \tag{2.5}$$

where d is the degree of p. Combining (2.4) and (2.5) gives

$$|I_n(p) - I(p)| \leq C^{-n}\rho^d \|p\|_{[-1,1]} \tag{2.6}$$

for sufficiently large n. Fixing $d \approx \gamma n$ for a sufficiently small constant γ converts this to

$$|I_n(p) - I(p)| \leq C^{-n}\|p\|_{[-1,1]} \tag{2.7}$$

with a new constant C, or by the triangle inequality,

$$|I_n(p)| \geq |I(p)| - C^{-n}\|p\|_{[-1,1]}. \tag{2.8}$$

At this point we invoke Nikolskii's inequality [7, Thm. 4.2.6],

$$\int_{-1}^{1} |p(x)|\,dx \geq \frac{1}{4d^2}\|p\|_{[-1,1]}, \tag{2.9}$$

assuming $d \geq 1$.[2] This gives us

$$|I_n(p)| \geq Cn^{-2}\|p\|_{[-1,1]} \tag{2.10}$$

[2]We are grateful to Endre Süli for suggesting the use of Nikolskii's inequality here. Previously we had included a lemma to derive an inequality like (2.9) from Markov's inequality. Süli has also shown, in work not reported here, that the ε correction can be removed from (2.2).

with a new constant C, provided that p is nonnegative on $[-1,1]$. But now according to a theorem of Coppersmith and Rivlin [4], there exist universal constants $C_1 > 1$, $C_2 > 1$, and $n_1 \ge 1$ such that

$$C_1^{d^2/n} \le \sup_{\deg p = d} \frac{\|p\|_{[-1,1]}}{\|p\|_{X_n}} \le C_2^{d^2/n} \tag{2.11}$$

for all $n \ge n_1$ and $d \le n-1$. Moreover, the same bounds hold if we restrict attention to nonnegative polynomials, since we may take p to be the square of a polynomial of half the degree and this squares the quotient. With $d \approx \gamma n$, this lower bound grows exponentially with n, so we see that there exist polynomials of these degrees that are exponentially larger on $[-1,1]$, as a function of n, than on X_n. It follows from this observation and (2.10) that for these polynomials, $I_n(p)$ must be exponentially larger than $\|p\|_{X_n}$, as claimed in (2.2), where the wiggle term ε is again employed to counter the n^{-2} factor of (2.10). ∎

As an instance of the Fundamental Theorem of Numerical Analysis [26], Theorem 2.1 has a corollary about convergence.

COROLLARY 2.2. *Suppose a family* $\{I_n\}$ *of equispaced quadrature formulas as in Theorem* 2.1 *converges faster than root-exponentially for analytic integrands in the sense of* (2.1). *Then it is nonconvergent for continuous integrands in general.*

Proof. This follows from Theorem 2.1 together with Pólya's theorem that a consistent quadrature scheme is convergent as $n \to \infty$ if and only if the 1-norms of the weights are uniformly bounded [24]. For textbook references, see Theorem 3.1.2 of [3] or p. 130 of [6]. The hypothesis (2.1) implies consistency, since every polynomial p belongs to $\mathcal{A}(E)$ and thus $I_p \to I(p)$ as $n \to \infty$. ∎

3. Existing quadrature schemes. We now review a number of existing equispaced quadrature rules (1.2)–(1.3), as well as a couple of nonlinear alternatives. For related lists in the context of function approximation rather than quadrature, see section 2 of [16] and section 2 of [23].

The two extreme cases already mentioned in the introduction, the trapezoidal rule and the Newton–Cotes formulas, have no free parameters. Most other methods have a parameter that determines the balance between the number of sample points n and the order of accuracy d of the approximation. The number d is traditionally defined as one more than the largest degree of polynomials that the formula integrates exactly, but it may be defined in other ways too, or it may not be precisely defined at all. As one might expect from Theorem 2.1, in several cases a good balance as $n \to \infty$ is obtained with d on the order of $\sqrt{n}$.

Trapezoidal rule. As mentioned in the second paragraph, this is the rule defined by $w_k = h$ for $2 \le k \le n-1$ and $w_1 = w_n = h/2$. This rule dates to ancient times, and is equivalent to interpolating the data by a piecewise-linear function and then integrating the interpolant. The order of accuracy is $d = 2$, implying convergence at the rate $O(n^{-2})$ if the integrand is smooth. The trapezoidal rule is also equivalent to interpolating the data by a 2-periodic trigonometric polynomial (interpolating the mean data value at $x = \pm 1$) and then integrating *that* interpolant, which explains why the accuracy improves to exponential for analytic periodic integrands [29].

Newton–Cotes. These are the formulas defined by the condition that I_n is the integral of the unique degree $n-1$ polynomial interpolant through the data values $f(x_1), \dots, f(x_n)$. The name recognizes work by Roger Cotes published posthumously in 1722. The order of accuracy is $d = n$ (n even) or $n+1$ (n odd), hence unbounded in the limit $n \to \infty$, meaning that polynomials of all degrees are integrated exactly once n is large enough (in the absence of rounding errors). However, convergence fails in general for analytic functions because of exponential instability, with exponentially large positive and negative weights $\{w_k\}$ as $n \to \infty$ [22, 24].

Composite Newton–Cotes. For centuries the practical use of Newton–Cotes formulas has usually involved the repeated application of formulas of a fixed order d like 4 (Simpson's rule), 6, or 8. Any fixed choice of d will render composite Newton–Cotes formulas stable as $n \to \infty$, but they will be unstable for any choice of d that is unbounded as $n \to \infty$, and in particular if d is chosen to scale in proportion to $\sqrt{n}$.

Gregory formulas. Fifteen years before Newton and Leibniz made their contributions to the invention of calculus, and 45 years before Brook Taylor or Roger Cotes, James Gregory was manipulating Taylor series to design accurate equispaced quadrature formulas—a striking fact of history highlighted in recent years by Fornberg [10]. For any polynomial order of accuracy $d \geq 3$, the dth order Gregory formula starts from the trapezoidal rule but then adjusts $d-1$ weights at each end of the interval by constants that can be derived from Pascal's triangle. Thus the formula has the appealing property of constant (=trapezoidal) weights everywhere except near the boundaries. It is known that Gregory formulas are equivalent to the exact integration of certain interpolants consisting of a sum of a polynomial and a trigonometric term [17]. Gregory himself reported coefficients for formulas up to order 10, involving fractions like 11371/30240 and 3498217/3628800. In view of the absence of computing machines in his day capable of applying such formulas in practice, this must count as an example of 17th century pure mathematics!

EquiLS. To calculate Newton–Cotes coefficients, one can formulate an $n \times n$ linear system of equations imposing the conditions that n unknown weights $\{w_k\}$ give order n accuracy; details can be found in the next section. To derive more stable formulas, a powerful idea is to require just order of accuracy $d < n$ by retaining only the first d of these equations, so that the matrix is reduced to rectangular form and the system of equations is underdetermined. A solution vector $\{w_k\}$ of minimal 2-norm can then be calculated by standard methods of numerical linear algebra, and this defines the method of *least-squares quadrature* or (our name) *EquiLS*. An equivalent formulation with an overdetermined matrix problem derives the weights by integration of a degree $d-1$ polynomial discrete least-squares approximation to the data. EquiLS was investigated by P. J. Davis's student Wilson many years ago [30, 31] and more recently by Huybrechs [15], who found numerically that for $d \approx B\sqrt{n}$ with $B \leq 3.3$, all the weights are positive, making the quadrature formula perfectly stable. We have confirmed Huybrechs' claim in numerical experiments of our own (see Figure 9.2), and it is consistent with Theorem 2.1 of the present paper and with Theorem 2.1 of [30], which shows, conversely, that all-positive weights are not possible if $B > 2\pi$. We take $B = 3.3$ in the experiments of this paper. Huybrechs further showed that if least-squares fits are computed constrained by the condition of nonnegative weights, what he calls *nonnegative least-squares quadrature*, then formulas with positive weights of slightly higher order of accuracy d can be found for approximately $B \leq 3.8$.

Regularized Gregory quadrature. Fornberg recognized in [10] that the Gregory formulas are unstable if d is not fixed, with negative weights appearing first at order 9 and growing exponentially in magnitude as the order increases. In the same spirit as in EquiLS and the EquiRat method to be introduced in the next section, he proposed applying a rectangular matrix with more columns than rows to compute weights in a regularized fashion. Although there is no systematic discussion of stability, this proves effective.

Floater–Hormann quadrature. An influential paper of Floater and Hormann in 2005 introduced a linear method of function approximation based on barycentric rational interpolants [9]. For each d there is an order d Floater–Hormann rational interpolant, and if one integrates these interpolants, one gets a family of what could be called *Floater–Hormann quadrature* formulas. These were considered by Klein and Berrut [18], who used the name DRQ (Direct linear Rational Quadrature). To derive a method with better than algebraic convergence as $n \to \infty$, one could scale d according to a fixed formula, for example with $d = O(\sqrt{n})$. Such an approach will be unstable as $n \to \infty$ but is likely to be quite effective in practice since the instability will be weak.

Adaptive Floater–Hormann. Floater–Hormann quadrature as just defined is linear, but it can also be made adaptive and consequently nonlinear following an idea of Georges Klein implemented in the `'equi'` flag of Chebfun, as discussed in [13]. Here, the degree d is chosen on the fly based on the behavior of the function approximation.

AAA quadrature. Finally, a highly nonlinear but powerful possibility is to use the AAA algorithm [20] to approximate equally spaced data as proposed in [16] and then integrate the approximant, e.g. by a high order Gauss–Legendre formula. AAA approximations might have poles in $[-1, 1]$, in which case one could define the integral via a principal value, or, probably safer in practice, one could discard the offending poles and construct the approximation by a least-squares fit in partial fractions form. Figures 6.3 and 7.6 show that AAA quadrature errors may decrease startlingly fast. At least a part of the reason for this is that AAA approximations to an integrand $f(x)$ can place poles adaptively near its singularities, whereas a fixed formula (1.2) has its poles constrained a priori to locations independent of f.

4. EquiLS and EquiRat. In this section we propose a new family of what we call *EquiRat* quadrature formulas. These form a natural pair with the EquiLS method mentioned in the last section, and we present the two in parallel. Both methods are based on least-squares calculations, which may be equivalently formulated by an overdetermined system for an approximation (fit the integrand samples, then integrate) or an underdetermined system for the weights (define weights by a smaller than maximal set of moment conditions, then compute the weight vector with minimal ℓ^2 norm). EquiLS is derived from polynomial approximation of the integrand, which we will see is associated with rational interpolation of the Cauchy transform (4.11) at $s = \infty$, whereas EquiRat is derived from rational approximation of the integrand, and corresponds to rational interpolation of the Cauchy transform at points along the sides of a strip in the complex s-plane.

We begin with the mathematics of EquiLS [15], which we present in its simplest form in terms of monomials though a more stable algorithm uses Chebyshev polynomials. For any $d \le n$, let V denote the $n \times d$ Vandermonde matrix for the nodes $x_1, \ldots, x_n$ and F the n-vector of function samples,

$$V = \begin{pmatrix} x_1^0 & \cdots & x_1^{d-1} \\ \vdots & & \vdots \\ x_n^0 & \cdots & x_n^{d-1} \end{pmatrix}, \quad F = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}. \tag{4.1}$$

The discrete least-squares fit of F by a polynomial $p(x)$ of degree $d-1$ is given by Vc, where c is the coefficient vector

$$c = V^\dagger F, \tag{4.2}$$

with $V^\dagger$ denoting the $d \times n$ pseudoinverse of V. If m denotes the d-vector of moments of $x^0, \dots, x^{d-1}$ over $[-1, 1]$,

$$m = \begin{pmatrix} \int_{-1}^{1} x^0 \, dx \\ \vdots \\ \int_{-1}^{1} x^{d-1} \, dx \end{pmatrix}, \tag{4.3}$$

then the quadrature formula $I_n = \int p(x)dx$ of (1.2) amounts to

$$I_n = c^T m = F^T (V^T)^\dagger m, \tag{4.4}$$

which means that the vector of quadrature weights is

$$w = (V^T)^\dagger m. \tag{4.5}$$

In the transpose of this calculation, which is equivalent and is the formulation used by Wilson and Huybrechs [15, 31], we consider instead the underdetermined problem

$$V^T w = m, \tag{4.6}$$

which expresses the condition that polynomials of degree $d-1$ are integrated exactly. We compute a solution w in the standard ℓ^2-minimizing sense, that is, $w = (V^T)^\dagger m$, which is the same result (4.6) as before.[3]

For EquiRat, instead of approximating f by a polynomial, we approximate it by a rational function whose poles are not at ∞. Obviously rational approximation is a generalization of polynomial approximation, but why is it a generalization of particular importance? The conceptual answer is that *every* quadrature formula can be interpreted in terms of rational approximations, as is discussed in section 11 of [14] and can also be seen in equation (4.12) below. The practical answer is that by generalizing from polynomials to rational functions, we can develop quadrature formulas with uniform behavior along $[-1, 1]$.

Motivated by this aim, we have explored various approximation strategies. The one we settled on for its simplicity, practical stability, and natural relationship to EquiLS is as follows. Let $A > 0$ be a parameter, which is set to 3 in the experiments of the next section, and consider the line segments in the complex x-plane

$$[-1, 1] \pm \frac{iA}{\sqrt{n}}, \tag{4.7}$$

which lie on either side of $[-1, 1]$ at a distance that scales with $n^{-1/2}$. Assuming for simplicity that n is divisible by 4, set $d = n/2$ and let $s_1, \dots, s_d$ denote the $n/4$ points with real parts from -1 to 1 equispaced along each of these segments—$n/2$ points all

[3]We thank Yuji Nakatsukasa for showing us this elegant argument using the pseudoinverse.

told. EquiRat consists of discrete least-squares approximation of the vector F at the quadrature nodes $\{x_k\}$ by a rational function

$$q(x) = \sum_{j=1}^{d} \frac{a_j}{s_j - x} \quad (d = n/2). \tag{4.8}$$

For the linear algebra, we take V in (4.1) to be an $n \times d$ Cauchy matrix instead of a Vandermonde matrix,

$$V = \begin{pmatrix} \frac{1}{s_1 - x_1} & \cdots & \frac{1}{s_d - x_1} \\ \vdots & & \vdots \\ \frac{1}{s_1 - x_n} & \cdots & \frac{1}{s_d - x_n} \end{pmatrix}, \tag{4.9}$$

and the moment vector m of (4.3) becomes

$$m = \begin{pmatrix} \int_{-1}^{1} \frac{dx}{s_1 - x} \\ \vdots \\ \int_{-1}^{1} \frac{dx}{s_d - x} \end{pmatrix}. \tag{4.10}$$

The weight vector that defines the quadrature formula is then given by (4.5). As before, we can transpose the formulation and compute w as the vector of minimal 2-norm that satisfies d moment conditions, namely the exact integration of $1/(s_j - x)$ for $1 \le j \le d$.

The descriptions of EquiLS and EquiRat just given are set in the x variable, involving integration of approximations of the integrand or exact reproduction of certain moments over $[-1,1]$. However, as shown in [14], there is a dual interpretation of these methods involving complex rational approximation of the Cauchy transform. This was the approach Gauss implicitly took in deriving Gauss quadrature, and it was also our original route to EquiRat. In this approach, the starting point is the approximation of $2\pi i$ times the Cauchy transform over $[-1,1]$,

$$2\pi i \mathcal{C}(s) = \int_{-1}^{1} \frac{dx}{s - x} = \log\left(\frac{s+1}{s-1}\right), \tag{4.11}$$

by a degree n rational function of the form

$$r(s) = \sum_{k=1}^{n} \frac{w_k}{s - x_k} \tag{4.12}$$

in some part of $\mathbb{C} \cup \{\infty\} \backslash [-1,1]$. The branch of the logarithm in (4.11) is chosen so that $\log((s+1)/(s-1)) \sim 2/s$ as $s \to \infty$. In the present paper, $\{x_k\}$ are the equispaced nodes (1.3), so this is a problem of rational approximation with "preassigned poles," hence linear at least in some formulations (e.g. in the ℓ^2 norm, though not in the ℓ^∞ norm). Note that since $r(\infty) = 0$, (4.12) automatically interpolates $2\pi i \mathcal{C}(s)$ at least once at $s = \infty$. The residues $\{w_k\}$ of $r(s)$ are then taken as the weights of the quadrature formula. For EquiLS, we approximate $2\pi i \mathcal{C}(s)$ by the rational function

(4.12) of minimal 2-norm weight vector that interpolates $2\pi i \mathcal{C}(s)$ $d+1$ times at ∞. For EquiRat, we approximate it by the rational function (4.12) of minimal 2-norm weight vector that interpolates $2\pi i \mathcal{C}(s)$ at the points $s_1, \dots, s_d$. The equivalence of the Cauchy transform approach with the derivations in the x variable follows from the Cauchy integral formula, as spelled out in section 11 of [14] .

A minimal MATLAB code to fully specify the definition of EquiRat can be put together as follows. We would not use this in practice except for small values of n; a more stable code is given in the next section.

```
function [w,x] = equirat(n)
x = linspace(-1,1,n);
C = @(s) log((s+1)./(s-1));
A = 3; ep = A/sqrt(n);
Z = [linspace(-1,1,round(n/4))' + 1i*ep
     linspace(-1,1,round(n/4))' - 1i*ep];
A = 1./(Z-x); F = C(Z);
w = real(lsqminnorm(A,F));
```

The matrix A grows ill-conditioned for large n, so for $n > 200$, as just mentioned, alternative calculations are needed: see the next section.

As commented earlier, the motivation for our decision to derive quadrature formulas by placing poles or interpolating the Cauchy transform along the sides of a strip about $[-1, 1]$ is that this gives uniform resolution along $[-1, 1]$. This is in contrast to Gauss and other polynomial-based formulas, which have much better resolution for $x \approx \pm 1$ than for $x \approx 0$ [13, 28]. It is tempting to associate the nonuniformity of such formulas with the clustering of their nodes near ± 1, but in fact, as we shall see in the next section, it applies even with EquiLS, despite its equispaced nodes, since it is derived from polynomials.

In any quadrature problem, there is always the question of whether to work in the x variable with weight vectors and approximations of the integrand $f(x)$ (the traditional route), or in the s variable with residue vectors and approximations of the Cauchy transform $\mathcal{C}(s)$ (following [11, 14, 27]). The Cauchy transform approach has the advantage that nonlinear approximations with quadrature nodes unknown a priori can be considered, notably with the help of the AAA algorithm [21], but this is less relevant to the present paper since the quadrature nodes are fixed. However, it also has a further advantage that is certainly relevant to the present case, and that is that it provides a means of assessing quadrature accuracy in an integrand-independent fashion via the Cauchy integral

$$I_n - I = \frac{1}{2\pi i} \int_\Gamma f(s) \left[r_n(s) - 2\pi i \mathcal{C}(s) \right] ds, \tag{4.13}$$

where Γ is any closed contour enclosing $[-1, 1]$ on and inside of which f is analytic. This is the idea introduced by Takahasi and Mori [27, eq. (2.8)], and we will see it in action in sections 6 and 7. As emphasized in [14], this kind of analysis and interpretation via rational approximation of the Cauchy transform applies to every quadrature formula (1.2), regardless of how it was derived.

5. Computation of EquiRat weights by Cauchy with Arnoldi. EquiRat quadrature weights are derived by working with a space of functions spanned by d poles which lie at a distance from the approximation interval $[-1, 1]$ that grows as $n \to \infty$, relative to the spacing between them. The obvious basis for this space, consisting of the poles themselves as in the columns of the matrix V of (4.9), therefore suffers from

```
function [w,x] = equiratCA(n)
x = linspace(-1,1,n);                        % Equirat quadrature nodes
A = 3; ep = A/sqrt(n);
d2 = round(n/4); d = 2*d2;
Z = [linspace(-1,1,d2)' + 1i*ep
     linspace(-1,1,d2)' - 1i*ep];
[xg,wg] = legpts(2*n);                       % Gauss quad nodes and weights
Q = zeros(n,d); U = zeros(2*n,d);            % rat funs on equi and Gauss grids
q = 1./(x'-Z(1)); u = 1./(xg-Z(1));
Q(:,1) = q/norm(q); U(:,1) = u/norm(q);
for k = 2:d                                  % rational Arnoldi iteration
   q = ((x'-Z(k-1))./(x'-Z(k))).*Q(:,k-1);
   u = ((xg-Z(k-1))./(xg-Z(k))).*U(:,k-1);
   for pass = 1:2                            % MGS reorthogonalization
      h = Q(:,1:k-1)'*q;
      q = q - Q(:,1:k-1)*h;
      u = u - U(:,1:k-1)*h;
   end
   Q(:,k) = q/norm(q); U(:,k) = u/norm(q);
end
b = (wg*U).';
w = real(conj(Q)*b);                         % EquiRat quadrature weights
```

FIG. 5.1. *MATLAB code for stable computation of EquiRat quadrature weights by the Cauchy with Arnoldi method.*

ill-conditioning of the kind investigated memorably by Barnett and Betcke [2]. For successful computation of EquiRat weights in ordinary-precision arithmetic, a better basis is needed, and for this we have turned to the Cauchy with Arnoldi method proposed by Faghih, et al. [8], based on discrete orthogonal rational functions. This is a rational analogue of the method used for polynomials in [15], but more crucial for computation since in the latter case, stability can be achieved simply by using the basis of Chebyshev polynomials.

The idea is to replace V and m in (4.9)–(4.10) by

$$V = \begin{pmatrix} \phi_1(x_1) & \cdots & \phi_d(x_1) \\ \vdots & & \vdots \\ \phi_1(x_n) & \cdots & \phi_d(x_n) \end{pmatrix}, \quad m = \begin{pmatrix} \int \phi_1(x) \\ \vdots \\ \int \phi_d(x) \end{pmatrix}, \tag{5.1}$$

where the basis vectors $\{\phi_i(x_k)\}$, spanning the same space as the columns of V in (4.9), satisfy the discrete orthonormality condition

$$\sum_{k=1}^{n} \phi_i(x_k)\overline{\phi_j(x_k)} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases} \tag{5.2}$$

Thanks to the orthonormality, the pseudoinverse is just the conjugate transpose, $V^\dagger = V^*$. The Cauchy with Arnoldi method is a numerical process for constructing the vectors $\{\phi_i\}$. Define $\mathbf{1} = (1,\dots,1)^T$ and $D_x = \mathrm{diag}(x_1,\dots,x_n)$. The columns of V of (4.9) form a basis for the space

$$\mathcal{K}(s_1,\dots,s_d) = \mathrm{span}\{(s_1 I - D_x)^{-1}\mathbf{1},\dots,(s_d I - D_x)^{-1}\mathbf{1}\}. \tag{5.3}$$

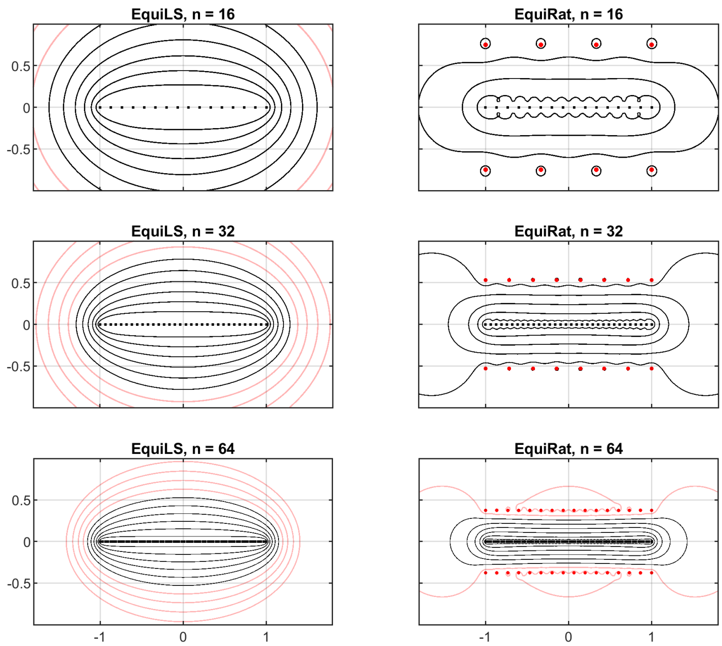

FIG. 6.1. *Takahasi–Mori contours* $\log_{10}|r_n(s)-2\pi i\,\mathcal{C}(s)| = -6,\dots,-1$ *(black) and* $-10,\dots,-7$ *(pink), from outside in, in rational approximation of the Cauchy transform for EquiLS and EquiRat with* $n = 16$*, 32, and 64. The red dots mark the EquiRat interpolation points; the interpolation points for EquiLS are at* ∞*. EquiRat has the advantage of uniform behavior along* $[-1,1]$*, whereas EquiLS has the advantage of approximation improving all the way out to* $s=\infty$ *rather than confined to a strip.*

We can regard this as a rational Krylov space, since the $(j+1)$st column of V can be computed from the jth column by applying the linear transformation $c_{j+1} = (s_jI - D_x)(s_{j+1}I - D_x)^{-1}c_j$, with the process initialized by $c_1 = (s_1I - D_x)^{-1}\mathbf{1}$. By combining this sequence of transformations with modified Gram–Schmidt orthogonalization, Cauchy with Arnoldi computes an orthonormal basis $q_1,\dots,q_d$ for $\mathcal{K}(s_1,\dots,s_d)$. The moment vector m is computed numerically by Gauss–Legendre quadrature. A MATLAB function for computing EquiRat weights is given in Figure 5.1, and we have verified its accuracy by comparison against Advanpix extended precision [1]. A less rigorous check on the fly comes from making sure that the computed weight vector w is symmetric.

6. Comparing EquiLS and EquiRat.

We now compare EquiLS and EquiRat from a few points of view. As mentioned earlier, the experiments are based on fixed parameter choices $B = 3.3$ for EquiLS (i.e., matrix dimension approximately $n \times 3.3\sqrt{n}$ in (4.1)), following [15], and $A = 3$ for EquiRat (i.e., strip half-width $3/\sqrt{n}$), based on numerical experiments such as Figure 9.1 below.

The first comparison, in Figure 6.1, looks at Takahasi–Mori error contours in rational approximation of the Cauchy transform for $n = 16, 32, 64$. In each case the levels $\log_{10}|r_n(s) - 2\pi i\,\mathcal{C}(s)| = -10,\dots,-1$ are plotted, with errors smallest on the outside and increasing toward the branch cut along $[-1,1]$. The EquiLS contours have the familiar elliptical shape associated with polynomial approximations, looking approximately like contours for Gauss quadrature of degree $1.8\sqrt{n}$ (this approximate

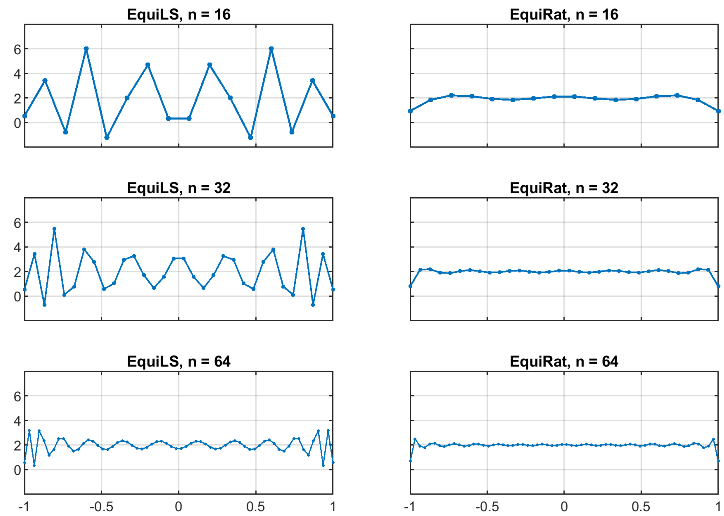

FIG. 6.2. *EquiLS and EquiRat scaled weight vectors* $\{(n-1)w_k\}$ *for the same EquiLS and EquiRat cases as in Figure* 6.1.

scaling is based on further experiments not shown). This will give rapid convergence especially for integrands that are highly analytic in the sense of having domains of analyticity extending far from $[-1,1]$. The EquiRat contours are much more uniform near $[-1,1]$, but show no further decrease outside the strip of interpolation points. These will be well adapted to functions that are not analytic, or with singularities close to the interval. It is interesting to compare these figures with those of [13], where quadrature formulas with uniform behavior are constructed by conformal mapping by Jacobi elliptic functions rather than rational approximation.

Figure 6.2 plots the weight vectors, scaled by $n-1$ so that the numbers approach the length of the interval 2, for the same three values of n. According to Pólya's theorem mentioned in the introduction, stability is equivalent to the weights being uniformly bounded in 1-norm as $n \to \infty$. A convenient proxy for this is that the weights should all be positive for sufficiently large n (this condition is sufficient but not necessary for stability), and we see that they are indeed positive in each case except for EquiLS with $n = 16$ and 32. One should probably not read too much into the differences between the weights in the two columns, as they will change with the parameter choices. If B is decreased from 3.3 to 3.0, the negative EquiLS weights become positive, and if it is decreased further to 2.5, the EquiLS weight vectors cease to be more oscillatory than those of EquiRat. Moreover, as we shall see in section 9, EquiRat is not quite stable as $n \to \infty$: negative weights appear at $n = 242$, though $\sum |w_k|$ does not exceed 10 until $n = 838$.

It is noticeable that even for large n, the weight vectors w have small oscillations in the middle of the interval. No doubt quadrature formulas could be designed with more rapid approach to the limiting value 2 for $x \approx 0$, but we have not investigated this possibility.

Figure 6.3 compares convergence of EquiLS and EquiRat by plotting $|I_n - I|$ against $\sqrt{n}$ for six different integrands $f(x)$. As in [13], the results show advantages of each method in different contexts. EquiLS is the winner when f is "highly analytic,"

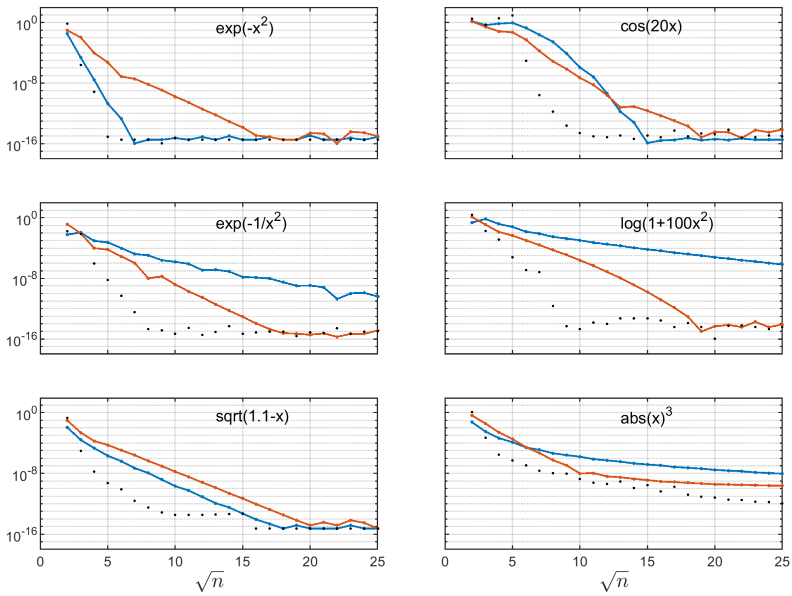

FIG. 6.3. *EquiLS (blue) and EquiRat (red) convergence for six example integrands. Note that the horizontal axis is $\sqrt{n}$, so that root-exponential convergence will show up as a straight line. The black dots show corresponding errors for integration of the AAA approximant, a nonlinear quadrature strategy that can be remarkably effective.*

as with $f(x) = \exp(-x^2)$, or when it has singularities near $x = \pm 1$, as with $f(x) = \sqrt{1.1 - x}$. EquiRat is the winner when f is nonanalytic, as with $f(x) = \exp(-1/x^2)$ and $f(x) = |x|^3$, or when it has singularities near the interior of $[-1, 1]$, as with $f(x) = \log(1 + 100x^2)$.

When $f(x)$ is analytic in a neighborhood of $[-1, 1]$ but not throughout the complex x-plane, we expect both EquiLS and EquiRat to converge root exponentially: a straight line asymptotically on these axes (see section 8). This convergence rate appears to some approximation in every case except $|x|^3$.

In addition to the blue and red curves, each panel of Figure 6.3 includes black dots representing the integration of the AAA rational approximant of the given equispaced data (computed to the usual default relative accuracy 10^{-13}). Just as was found in [16] for function approximation, evidently AAA is highly effective for quadrature too. AAA is a highly nonlinear scheme—data dependent—not a quadrature formula in the sense of (1.2).

7. Comparison with other methods. It is interesting to look at some of the other methods of equispaced quadrature discussed in section 3 from the points of view explored in the last section for EquiLS and EquiRat. We will not attempt anything like an exhaustive comparison, which would quickly get overwhelming, but content ourselves with a pair of plots in a consistent format for five methods, always (apart from one case) with $n = 32$.

Figure 7.1 starts with the trapezoidal rule, where all the weights are equal except those at the endpoints. (This is also the order 2 Gregory formula.) Here we see beautiful behavior in the interior of the Takahasi–Mori plot, with contour lines parallel to $[-1, 1]$ and hugging it very closely, suggesting a very weak analyticity requirement

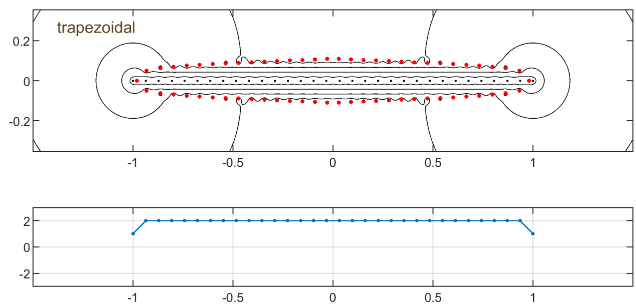

FIG. 7.1. *Above, Takahasi–Mori contours with* $\log_{10}|r_n(s) - 2\pi i\,\mathcal{C}(s)| = -3,\dots,0$ *for the trapezoidal rule with* $n = 32$ *(much of the level* -3 *curve is off-scale and not visible). Red dots mark numerically computed points of interpolation with* $r_n(s) = 2\pi i\,\mathcal{C}(s)$*, in addition to interpolation to 3rd order at* $s = \infty$*. Below, scaled weights* $\{(n-1)w_k\}$*.*

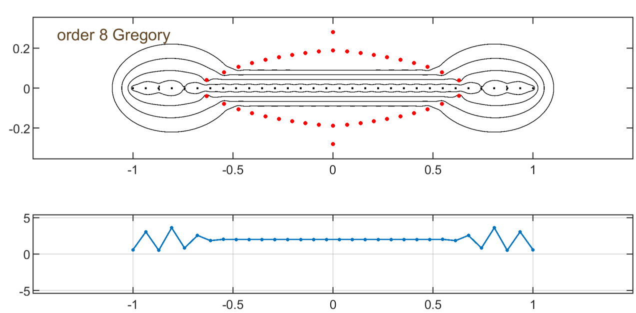

FIG. 7.2. *Repetition of Figure* 7.1 *for the order* 8 *Gregory formula. Higher order Gregory formulas have some negative weights, but Fornberg has shown that this problem can be alleviated with least-squares regularization* [10].

for high accuracy. Unfortunately, this favorable configuration is undone near $x = \pm 1$, where the 10^{-3} contour projects far beyond the axes. From these curves one can see that not much analyticity of the integrand $f(x)$ will be needed to get a digit or two of accuracy, but there is little hope of 3 or more digits even if f is entire.

The order 8 Gregory formula, shown in Figure 7.2, is qualitatively similar but quantitatively much better. Now the bulbs at the end are a good deal smaller, showing that for good accuracy, less regularity near $x = \pm 1$ is needed. However, the mismatch between $x \approx \pm 1$ and $|x| \ll 1$ remains, revealing graphically the hybrid nature of these formulas.

The unstable Newton–Cotes quadrature formula with $n = 32$, shown in Figure 7.3, is very different. Now the weights are exponentially large, and the error contours no longer hug $[-1, 1]$, as explained by Runge [25].

Switching to a composite Newton–Cotes formula tames much of the instability, as shown in Figure 7.4. For this computation we increase n to 33 so that $[-1, 1]$ splits into four subintervals.

Figure 7.5 extends the comparison to an order 8 Floater–Hormann formula. As described in section 3, the mathematical derivation has something in common with that of Gregory formulas, but a comparison with Figure 7.2 suggests that the quadrature formulas do not have so much in common after all.

We end this section with Figure 7.6, a convergence comparison of different meth-

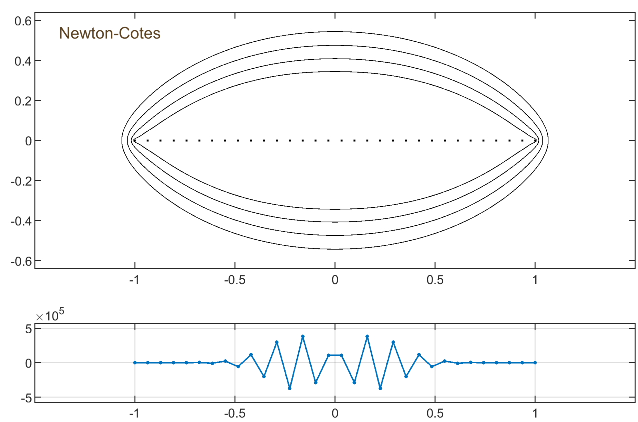

FIG. 7.3. *Repetition of Figure* 7.1 *for Newton–Cotes quadrature. Now even the innermost error contour at level* 0 *is far from* $[-1,1]$*, delineating Runge's "football." Analyticity throughout this domain would be needed for successful quadrature even in the absence of rounding errors. The weights are exponentially large and of oscillating signs (note the vertical scale), causing well-known instability.*

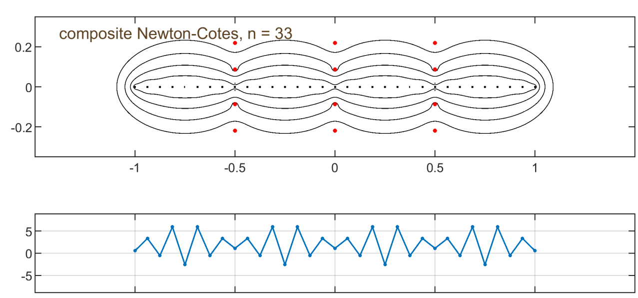

FIG. 7.4. *The order* 8 *composite Newton–Cotes formula, on the other hand, has more reasonable behavior. Here* n *has been increased from* 32 *to* 33 *to enable a clean division into subintervals.*

ods for $f(x) = \exp(-1/x^2)$.

8. Accuracy of EquiRat and EquiLS. Takahasi–Mori curves as in section 6, as well as convergence experiments as in sections 6 and 7, suggest that both EquiRat and EquiLS converge root-exponentially whenever the integrand $f(x)$ is analytic on $[-1,1]$. We believe this is mathematically true regardless of the values of the parameters $A > 0$ and $B > 0$ that define these methods, but choices of A or B much larger than 3 will lead to unstable behavior, which can cause trouble for non-analytic integrands and even for analytic ones on a computer with rounding errors. Stability and instability are discussed in the next section.

The following conjecture is stated in the notation of Theorem 2.1, with E denoting a compact subset of $\mathbb{C}$ containing $[-1,1]$ in its interior and $\mathcal{A}(E)$ the Banach space of functions continuous on E and analytic in the interior with norm $\|f\|_E = \sup_{x\in E}|f(x)|$.

CONJECTURE 8.1. *Both EquiRat and EquiLS converge root-exponentially for all integrands* $f \in \mathcal{A}(E)$*:*

$$|I_n(f) - I(f)| \le MC^{-n^{1/2}}\|f\|_E, \quad C > 1. \tag{8.1}$$

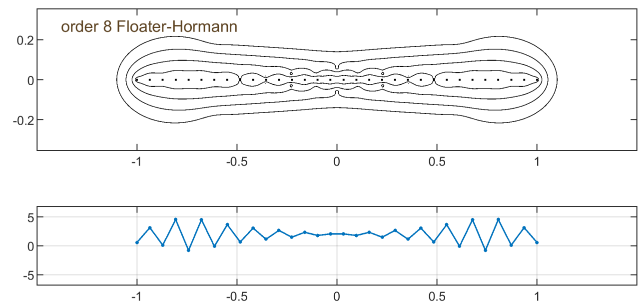

FIG. 7.5. *Although the Floater–Hormann and Gregory formulas have some similarities of derivation, both starting from an adjustment of just a few numbers near the endpoints, a comparison with Figure* 7.2 *suggests that their quadrature properties are quite different.*

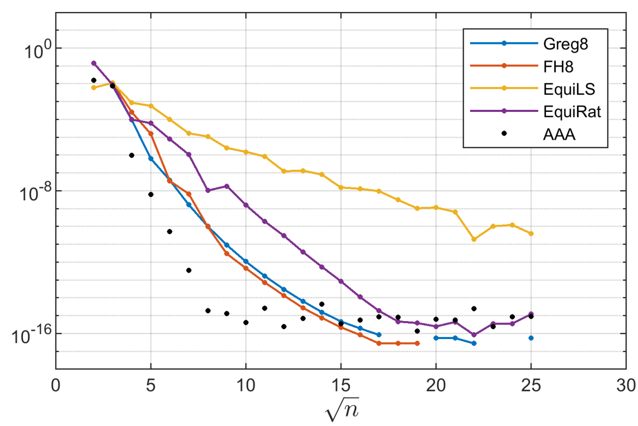

FIG. 7.6. *Convergence of five equispaced quadrature methods for the integrand* $f(x) = \exp(-1/x^2)$.

The constant C depends on E and on the parameters A and B defining EquiRat and EquiLS, respectively, but otherwise not on f.

It is interesting to note that a variant of EquiRat probably achieves exponential convergence for all analytic integrands, though inevitably with exponential instability. This is in contrast to the Newton–Cotes formulas, which require analyticity over a larger region than $[-1, 1]$ for exponential convergence. The variant, which we call *Fat EquiRat*, is defined by placing the $n/2$ interpolation points $\{s_j\}$ along the sides of a strip with fixed width A rather than diminishing width $A/\sqrt{n}$ as $n \to \infty$.

CONJECTURE 8.2. *The Fat EquiRat scheme just described converges exponentially for all integrands $f \in \mathcal{A}(E)$,*

$$|I_n(f) - I(f)| \le MC^{-n}\|f\|_E, \quad C > 1, \tag{8.2}$$

with the constant C depending only on E and A. However, as implied by Theorem 2.1 *and Corollary* 2.2*, Fat EquiRat is exponentially unstable, and in particular, it does not converge in general if f is just continuous.*

9. Stability of EquiRat and EquiLS. As a practical matter, both EquiRat and EquiLS will work well if their parameters A and B are on the order of 3 or 4 or

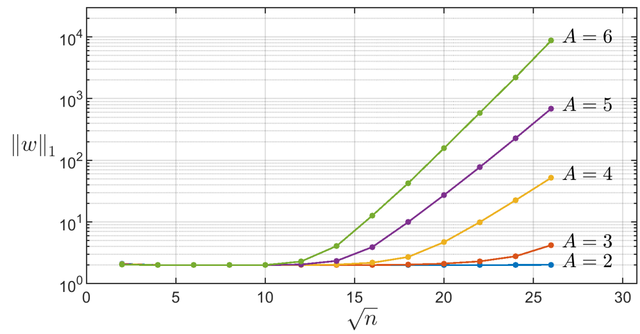

FIG. 9.1. *1-norms of weight vectors for EquiRat with various parameters A, plotted against $\sqrt{n}$. It appears that for any A, the method is root-exponentially unstable, but with constants small enough to make the method stable in practice for $A \leq 3$, as suggested by the empirical approximation* (9.1).

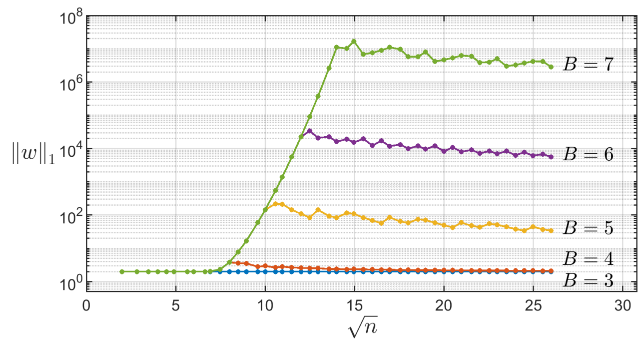

FIG. 9.2. *1-norms of weight vectors as in Figure* 9.1, *but for EquiLS rather than EquiRat. The boundedness of each curve suggests that EquiLS is mathematically stable for any B, though unstable in practice for larger values of B. Note that for $B = 3$, since it is smaller than the positivity threshold identified by Huybrechs* [15], *we have $\|w\|_1 = 2$ for all n.*

smaller, and both will work badly for much larger values of A and B. The mathematical details, however, differ interestingly between the two methods. In this section we summarize the situation as we understand it based on numerical experiments. We do not have proofs of these results.

Consider EquiRat, defined by a strip about $[-1, 1]$ of half-width $A/\sqrt{n}$. Numerical experiments like that of Figure 9.1 suggest that for all $A > 0$, EquiRat is unstable in the sense that the 1-norms of its weight vectors are unbounded as $n \to \infty$. However, the divergence is just root-exponential, and the constants involved are small when A is small. An empirical fit looks something like this:

$$\|w\|_1 \approx 0.005 \exp(0.11A(\sqrt{n} - 4)). \tag{9.1}$$

(This approximation is rough; we are making no precise claim.) For $A = 3$, $\|w\|_1 = 10$ is not achieved until $n = 838$ and $\|w\|_1 = 100$ until $n = 1326$, so instability is unlikely to cost more than a digit or two in any practical calculation. (Newton–Cotes passes these thresholds at $n = 13$ and 17, and for $n = 838$ and 1326 it has $\|w\|_1 \approx 10^{246}$ and 10^{393}, respectively.)

EquiLS appears to be very different in theory and not so different in practice. As mentioned in section 4, Huybrechs found that all the quadrature weights are positive if EquiLS is defined with $B \leq 3.3$ [15]. This would seem to suggest that for larger values of B, the method may be unstable. However, numerical experiments indicate that it may be stable for all values of B in the sense of having weights with bounded 1-norm as $n \to \infty$. Note how different the curves of Figure 9.2 are from those of Figure 9.1. For each value of B, $\|w\|_1$ appears to be bounded as $n \to \infty$. (The parabolic envelope on the left corresponds to smaller values of n, where $B\sqrt{n} > n/2$. Here we arbitrarily set d to the integer nearest to $n/2$.) The bounded levels of these curves, however, are very high, appearing to grow faster than exponentially with B. So for large B, EquiLS would appear to be stable in mathematical theory, implying convergence for continuous integrands in exact arithmetic, but unstable in practice.

In sum, both EquiRat with $A = 3$ and EquiLS with $B = 3.3$ will perform well in practice, whereas doubling or tripling those values of A and B will make them perform badly, even though EquiRat appears to be mathematically unstable for all A and EquiLS appears to be mathematically stable for all B.

10. Summary and discussion. We summarize the contributions of this paper.

Review and comparison of methods (sections 3 and 7). We reviewed existing methods for equispaced quadrature: the trapezoidal rule, Newton–Cotes and composite Newton–Cotes, Gregory and regularized Gregory [10], Floater–Hormann [18, 9], and least-squares or EquiLS [15, 30, 31], as well as the nonlinear methods of adaptive Floater–Hormann and AAA [13].

Impossibility theorem (section 2). Theorem 2.1 extended the "impossibility" theorem of [23] from function approximation to quadrature, generalizing Pólya's extension of Runge's theorem about polynomial interpolation to Newton–Cotes quadrature [24, 25]. This establishes that any equispaced quadrature formula that converges faster than root-exponentially for analytic integrands must be unstable.

Optimality of root-exponential convergence (section 2). As an instance due to Pólya [24] of the Fundamental Theorem of Numerical Analysis [26], Corollary 2.3 showed that no family of quadrature formulas that converges faster than root-exponentially for analytic integrands can converge at all for continuous integrands.

Duality between integrand and Cauchy transform approximation (section 4). We reviewed the duality introduced in [14] that any quadrature formula can be interpreted equivalently either as exact integration of a rational approximant to the integrand or as application of a rational approximation of the Cauchy transform (poles $\leftrightarrow$ nodes, residues $\leftrightarrow$ weights).

EquiRat (section 4). We introduced a new family of formulas, EquiRat, which is analogous to EquiLS but based on rational rather than polynomial approximation of the Cauchy transform. This leads to uniform resolution along $[-1, 1]$ rather than the familiar polynomial pattern of extra resolution for $x \approx \pm 1$ at the expense of less resolution over the rest of the interval.

Tall-skinny/short-fat equivalence (section 4). We showed that for both the EquiLS and EquiRat formulas, one can equivalently regard the method as integrating a least-squares approximant to the integrand (tall-skinny matrix) or as applying a weight vector obtained as the ℓ^2-minimal solution of an underdetermined moment problem (short-fat matrix).

Stable calculation of EquiRat weights (section 5). We showed how EquiRat weights can be stably computed by Cauchy with Arnoldi orthogonalization, and provided a MATLAB code (Figure 5.1). We have confirmed the accuracy of this code by comparisons with the Advanpix multiple precision package (details not given).

Root-exponential convergence (Conjecture 8.1). We conjectured that both the EquiLS and EquiRat formulas converge root-exponentially for analytic integrands.

Exponential convergence of an unstable formula (Conjecture 8.2). We noted a "Fat EquiRat" variant of EquiRat which is unstable but is conjectured to converge exponentially for analytic integrands, based on an approximation scheme that likewise converges exponentially. By contrast, Newton–Cotes formulas and their underlying polynomial interpolants require more than just analyticity for exponential convergence.

Stability (section 9). We showed numerically that EquiRat is weakly unstable, with $\|w\|_1$ growing at a rate approximately $O(\exp(0.1A\sqrt{n}))$ as $n \to \infty$. This rate of growth is so slow as to be of little importance in practice for $A \leq 3$. Curiously, EquiLS appears to be mathematically stable for all parameters B, though unstable in practice for $B \gg 3.3$.

We finish with a comment about the significance of this work. In practice, all the methods we have considered are effective, apart from Newton–Cotes. We would not argue that if one really has to evaluate an integral based on equispaced samples, it is important to use EquiRat rather than EquiLS or, say, a 6th-order Gregory or composite Newton-Cotes formula. In fact, we would likely start simply with AAA approximation, which is so easy and usually converges faster than the other methods even if it is nonlinear and comes with no guarantees. Conceptually, on the other hand, we think the contributions of this paper are a significant addition to a mathematical problem going back 300 years.

Acknowledgments. We have benefited from very substantial discussions with Yuji Nakatsukasa, Endre Süli, and Kieran Vlahakis.

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