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## NEWSLETTER

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ROBOTS & ALGEBRAIC  
GEOMETRY

PREVENTING THE  
QUANTUM CRYPTO  
APOCALYPSE

NOTES OF  
A NUMERICAL  
ANALYST

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Cover image: a pair of images of punts under Magdalen Bridge, Oxford, illustrating the theme of discrete and continuous in the *Notes of a Numerical Analyst* column (page 32). On the left, 32x48 pixels, and on the right, 512x768.

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# Notes of a Numerical Analyst

## Discrete and Continuous

NICK TREFETHEN FRS

There are many parallels between phenomena of linear algebra (discrete) and differential equations (continuous). Here are two of my favourites.

*Sturm–Liouville.* Consider Wilkinson’s  $(2n+1) \times (2n+1)$  tridiagonal matrix of the form

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad (1)$$

with the diagonal entries running from  $n$  down to 0 and up to  $n$  again. Theory tells us that the eigenvalues of  $A$  are distinct, though there’s no such theorem for a pentadiagonal matrix. Yet the eigenvalues fall in nearly degenerate pairs, like these largest two for  $n = 4$  and 8:

$$n = 4 : 4.745, 4.747,$$

$$n = 8 : 8.7461941826, 8.7461941832.$$

A continuous analogue is the Sturm–Liouville problem

$$y'' + |x|y = \lambda y, \quad -L < x < L$$

with  $L > 0$  and  $y(\pm L) = 0$ . Again, theory tells us that the eigenvalues are distinct, though there’s no such theorem for a fourth-order equation. Here are the largest eigenvalues for  $L = 4$  and 8:

$$L = 4 : 1.645, 1.682,$$

$$L = 8 : 5.661892585, 5.661892595$$

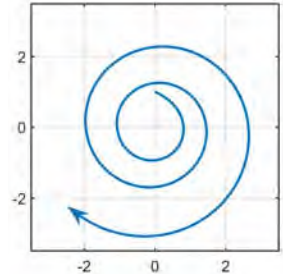
These near-degeneracies are related to line-splitting effects in quantum mechanics.

*Frozen coefficients.* Suppose you have a family of matrices that are individually power-bounded, with all eigenvalues in the unit disk, like this pair:

$$A_1 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

Must products like  $A_1 A_2 A_1 A_2 \cdots$  converge to zero? Certainly not, since  $A_1 A_2$  has eigenvalues 0 and 4.

Figure 1. Solutions to a variable-coefficient linear ODE may diverge even though all the frozen-coefficient problems are stable.



A continuous analogue is the ODE

$$u' = Bu, \quad B = \begin{pmatrix} -1 & m \\ 0 & -1 \end{pmatrix},$$

which is stable since the eigenvalues are in the left half-plane. For  $m > 2$  and  $u(0) = [0, 1]^T$ , however, the solution  $u(t)$  will grow before decaying. If we now define  $A(t)$  to be  $B$  “rotated by  $t$ ”,

$$A(t) = S(t)BS(-t), \quad S(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix},$$

solutions to  $u' = A(t)u$  grow exponentially, as shown in the figure (with  $m = 2.2$ ). Such effects were investigated by Lyapunov, Poincaré, Perron, and Vinograd, and this example comes from a celebrated paper by Kreiss about discrete-continuous analogues [1]. This effect is related to why flow in a pipe becomes turbulent even though the linearized problem is stable.

### FURTHER READING

[1] H.-O. Kreiss, Über die Stabilitätsdefinition für Differenzgleichungen die partielle Differenzialgleichungen approximieren, *BIT*, 2 (1962), 153–181.



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