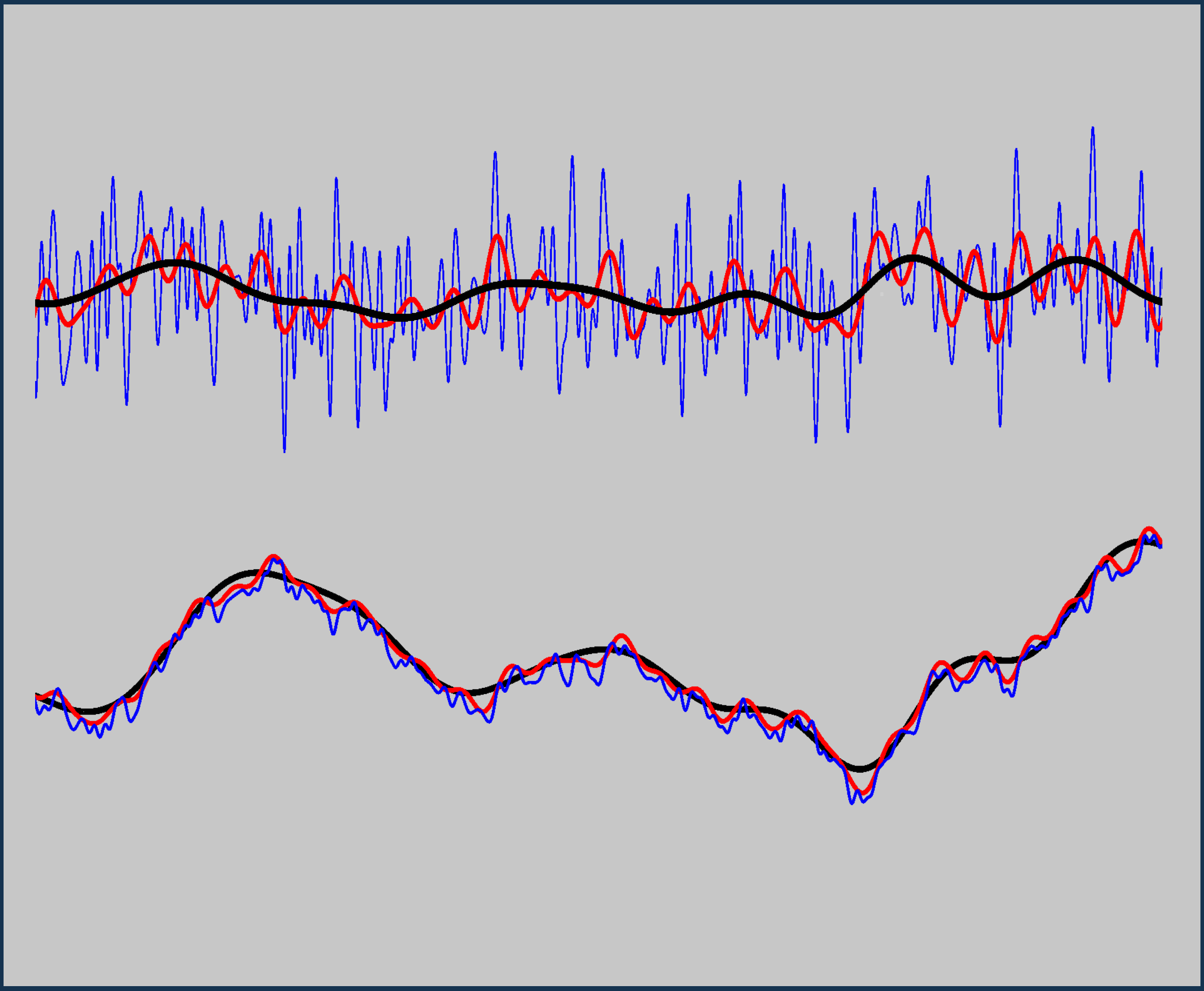
LONDON MATHEMATICAL SOCIETY
EST. 1865

# NEWSLETTER

*Issue: 518 - February 2026*



SHARP RESTRICTION THEORY

NOTES OF A NUMERICAL ANALYST

MATHEMATICS NEWS FLASH

## Notes of a Numerical Analyst

# The White Noise Paradox

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White noise means a random function with equal energy at all wave numbers. On the periodic domain $[0, 2\pi]$, we can define it by the random Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx}, \qquad (1)$$

where the coefficients $a_k$ are samples from the complex normal distribution of mean 0 and variance 1. The paradox is that this function does not exist, because the series does not converge: it would have infinite amplitude and energy. (All statements apply with probability 1.)

So, what makes white noise an indispensable concept? One answer is that its *indefinite integral* makes good sense and is nothing other than a Brownian path (the Wiener process),

$$g(x) = a_0 x + \sum_{k\neq 0} \frac{a_k}{ik}(e^{ikx} - 1). \qquad (2)$$

The integral converges because of cancellation among the random coefficients. (More precisely, we get a Brownian path if we eliminate periodicity by rescaling to intervals $[-L, L]$ with $L \to \infty$.)

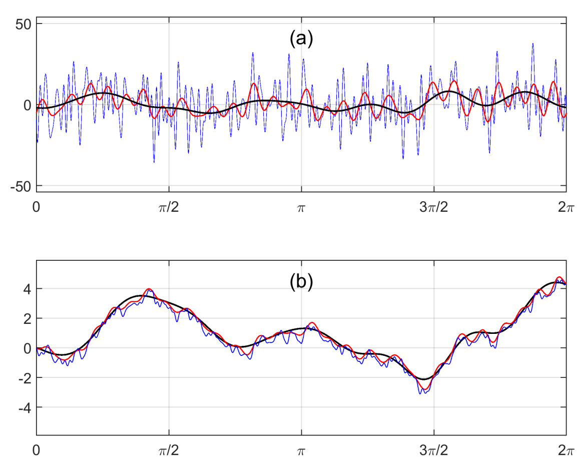


Figure 1. (a) The approximations in Eq. (3) to white noise diverge as $n \to \infty$, but (b) their indefinite integrals converge to Brownian paths. The plots show real parts for $n = 8$, 32, and 128.

I like to understand these effects by truncating Eq. (1) to get a *smooth random function*, illustrated in Fig. 1(a) for $n = 8$, 32, 128:

$$f(x) = \sum_{k=-n}^{n} a_k e^{ikx}. \qquad (3)$$

The indefinite integral is a *smooth random walk*, illustrated in Fig. 1(b), which converges to a Brownian path as $n \to \infty$.

White noise appears in innumerable corners of mathematics, physics and engineering, and one bit of history that fascinates me is that, with hindsight, we can see that it relates to *two* of Einstein's great papers from his *annus mirabilis* 1905. In one, he showed how Brownian motion (the physicist's version) comes about as an integral of white noise. Bouncing molecules are not infinitesimally small, but this argument would work even if they were. In another paper, the one that won the Nobel Prize, he showed how Planck's theory of 1900 leads to photons and an explanation of the photoelectric effect. As perhaps Einstein was first to appreciate, Planck's theory resolved the 'ultraviolet catastrophe', the 19th century prediction that a black body should radiate infinite energy in finite time, by arguing that the noise in question is non-white since quantum effects set a limit on the wave number. This time we are saved by physical reality, not by taking the integral. Einstein's two other great works of 1905 introduced special relativity and the equation $E = mc^2$.

## FURTHER READING

[1] S. Filip, A. Javeed, L.N. Trefethen, Smooth random functions, random ODEs, and Gaussian processes, *SIAM Rev.* 61 (2019) 185–205.
[2] J.-P. Kahane, *Some Random Series of Functions*, 2nd edn, Cambridge University Press, 1993.

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