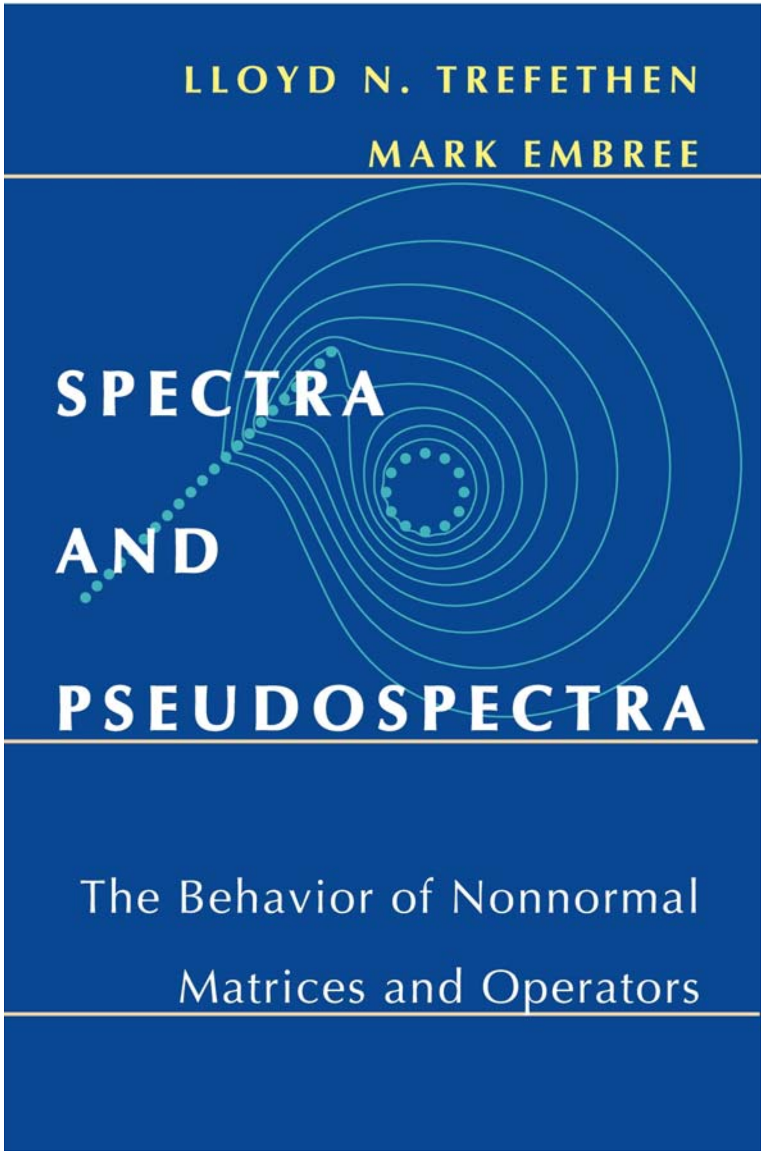



Slides from talk at Householder Symposium on Numerical Linear Algebra, May 2005.

Nick Trefethen, Oxford University

**Begun in 1990.**

**To appear in July.**
**(Princeton U. Press)**

(A reminder)

*We've switched to a strict inequality* *...and to the notation $\sigma_\varepsilon(A)$*

**First definition of pseudospectra**

Let $\mathbf{A} \in \mathbb{C}^{N\times N}$ and $\varepsilon > 0$ be arbitrary. The $\varepsilon$-*pseudospectrum* $\sigma_\varepsilon(\mathbf{A})$ of $\mathbf{A}$ is the set of $z \in \mathbb{C}$ such that

$$\|(z - \mathbf{A})^{-1}\| > \varepsilon^{-1}. \tag{2.1}$$

# TABLE OF CONTENTS

*Some highlights before 2000*

*Some highlights since 2000*

(1) Banach spaces, semigroups, numerical range [4,17]

(2) Variable coeffs & wave packet pseudomodes [8,9,11]

(3) Lewy-Hörmander nonexistence [13]

(4) Theorems on transients [14,15,16]

(5) Changes in fluid mechanics [23]

(6) GKS-stability of boundary conditions [34]

(7) Nonhermitian quantum mechanics [36]

(8) Lasers [60]

| | |
|---|---|
| (9) New Arnoldi theorems [26,28] | BEATTIE, ROSSI, SORENSEN |
| (10) EigTool [39] | WRIGHT |
| (11) Large-scale matrices via impl. restarted Arnoldi [40] | WRIGHT |
| (12) Criss-cross algorithms [42] | BURKE, LEWIS, OVERTON, MENGI |
| (13) Rectangular/generalized/higher-order problems [45,46] | HIGHAM, TISSEUR, RUHE, RIEDEL, WRIGHT, TOUMAZOU, VAN DORSSELAER, FRAYSSÉ |
| (14) Links with group velocity [54] | |

$Mx'' + Cx' + Kx = 0$ — see pp. 426-428

## (1) Banach spaces, semigroups, numerical range

*We were surprised: almost everything works for closed operators in Banach space.*

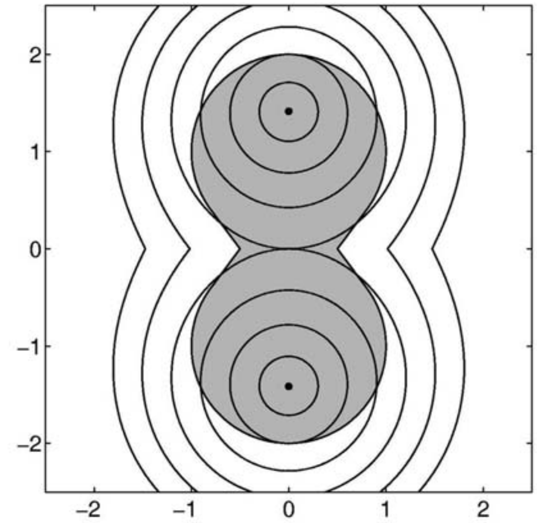
**closed operator on Banach space *X*, not necessarily bounded**

| **Three equivalent definitions of pseudospectra** |
|---|
| Let $\mathbf{A} \in \mathcal{C}(X)$ and $\varepsilon > 0$ be arbitrary. The $\varepsilon$-*pseudospectrum* $\sigma_\varepsilon(\mathbf{A})$ of $\mathbf{A}$ is the set of $z \in \mathbb{C}$ defined equivalently by any of the conditions |
| **resolvent norm** → $\|(z-\mathbf{A})^{-1}\| > \varepsilon^{-1}$, (4.3) |
| **perturbation of spectra** → $z \in \sigma(\mathbf{A}+\mathbf{E})$ for some $\mathbf{E} \in \mathcal{B}(X)$ with $\|\mathbf{E}\| < \varepsilon$, (4.4) |
| **pseudoeigenvectors** → $z \in \sigma(\mathbf{A})$ *or* $\|(z-\mathbf{A})u\| < \varepsilon$ for some $u \in \mathcal{D}(\mathbf{A})$ with $\|u\| = 1$. (4.5) |
| If $\|(z-\mathbf{A})u\| < \varepsilon$ as in (4.5), then $z$ is an $\varepsilon$-*pseudoeigenvalue* of $\mathbf{A}$ and $u$ is a corresponding $\varepsilon$-*pseudoeigenvector* (or *pseudoeigenfunction* or *pseudomode*). |

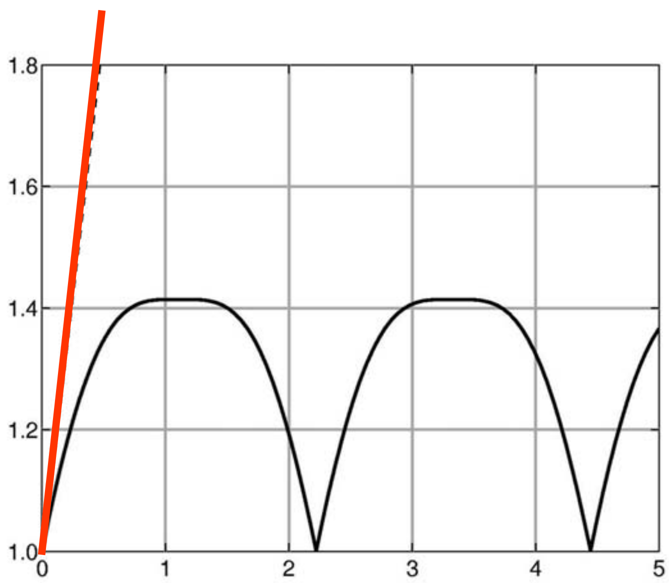
## Numerical range in Banach space (Lumer 1961, Bauer 1962):

$$W(\mathbf{A}) = \{(f, \mathbf{A}u) \colon u \in \mathcal{D}(\mathbf{A}), f \in X^*, \|u\| = \|f\| = 1, (f, u) = 1\}.$$

*W*(*A*) need not be convex. Example: $\mathbf{A} = \begin{pmatrix} -\mathrm{i} & \mathrm{i} \\ \mathrm{i} & \mathrm{i} \end{pmatrix}$ , $\infty$-norm




The significance of *W* (*A*) is the same as in Hilbert space: *the rightmost point determines the initial slope of* $\| e^{tA} \|$ *.*



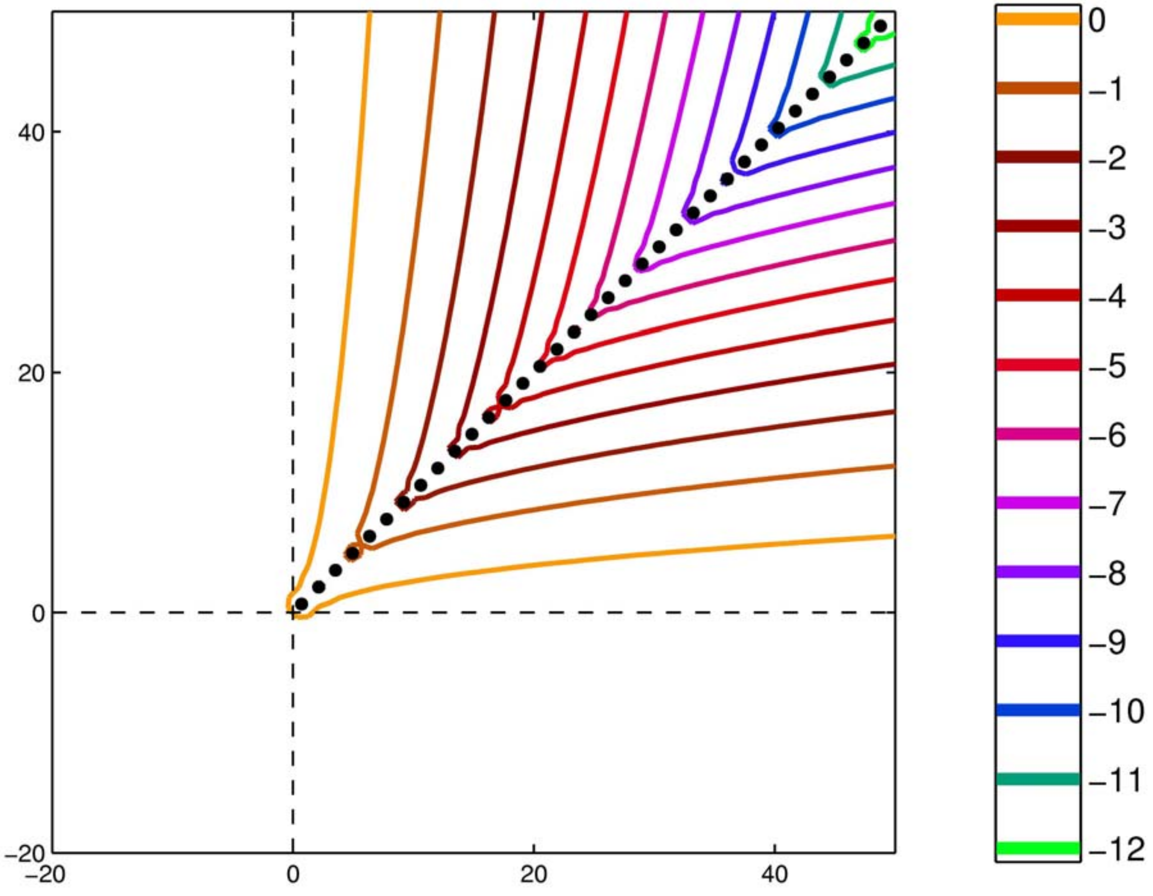

All this also works if *A* is the generator of a $C_0$ semigroup (modulo technicalities).

## (2) Variable coeffs & wave packet pseudomodes

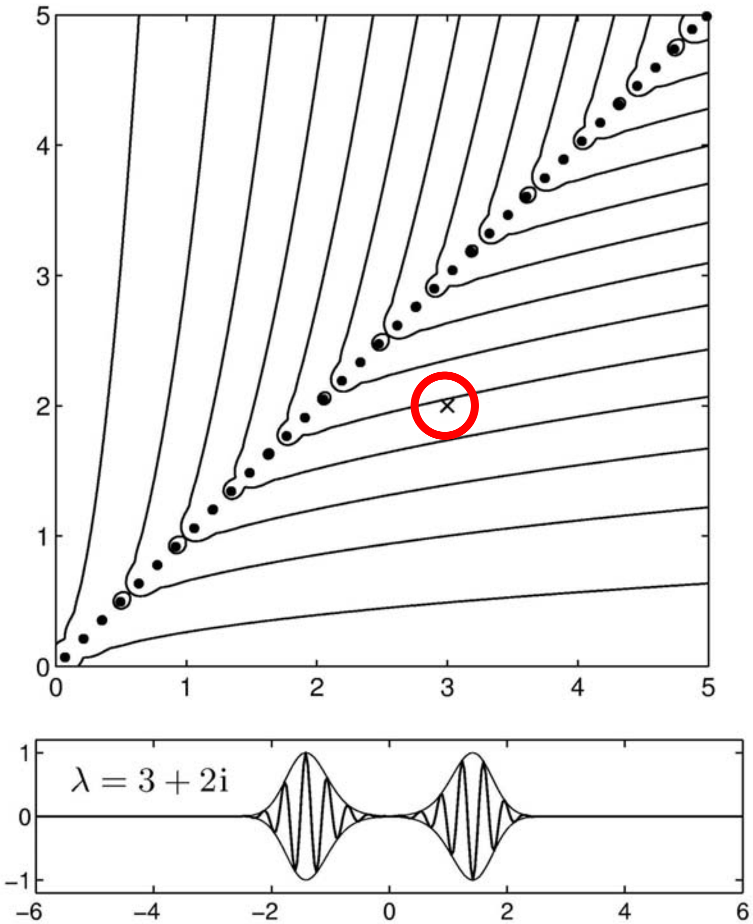
It started with Davies's complex harmonic oscillator in 1999.

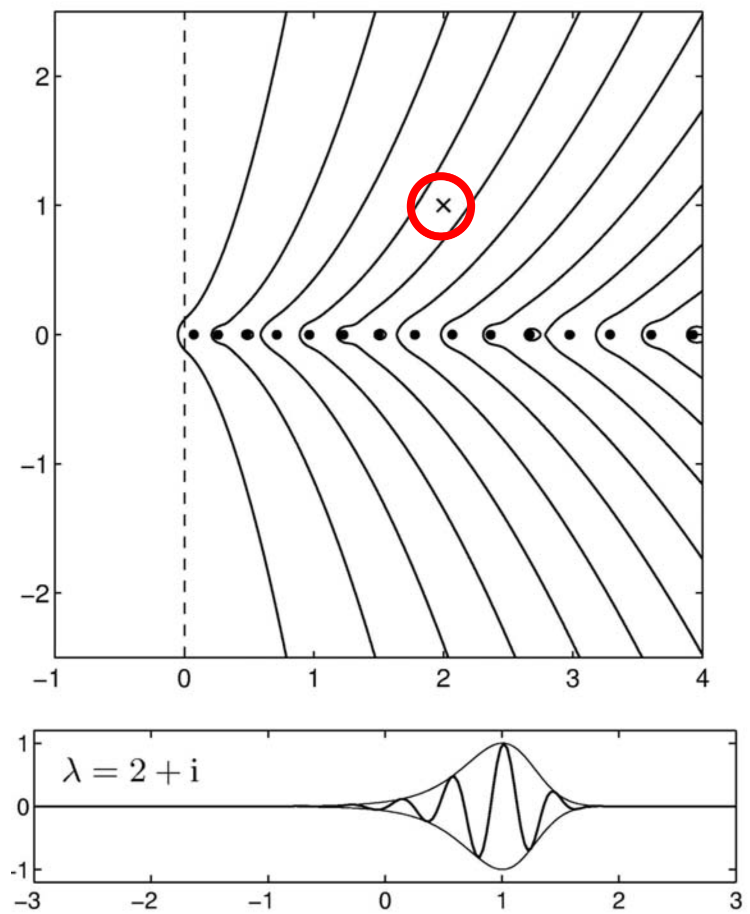
$$\mathsf{Au} \;=\; -\mathsf{u}_{\mathsf{xx}} + \mathsf{i}\mathsf{x}^2\mathsf{u} \;\;\text{on}\;(-\infty,\infty)$$

SYMBOL: $f(x,k) = k^2 + ix^2$



Pseudomodes for such operators have the form of wave packets.



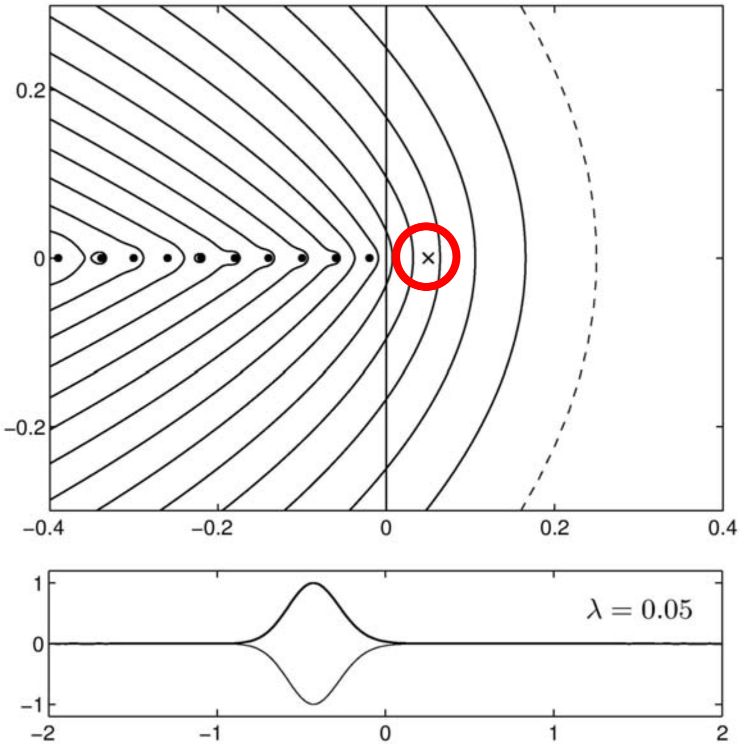



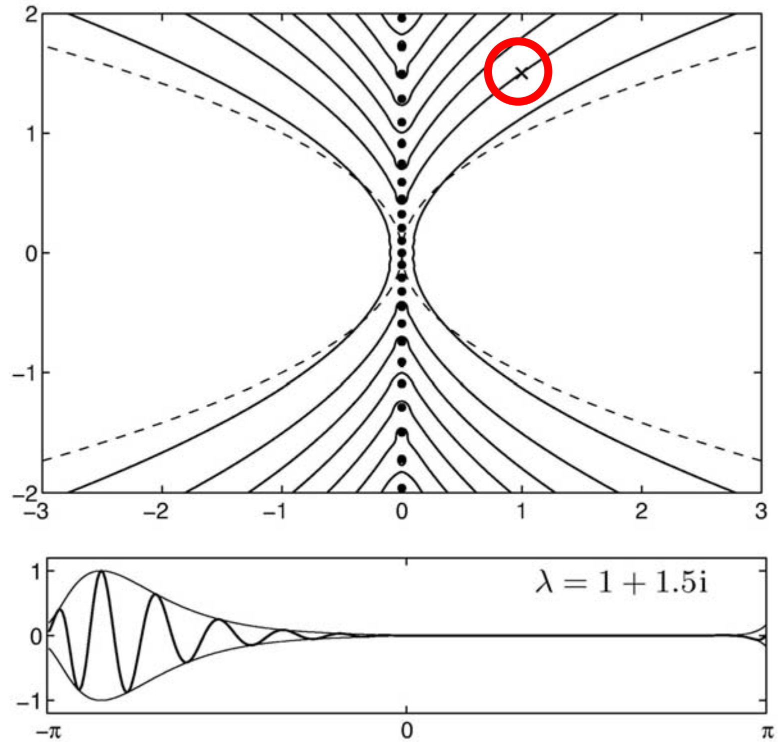

Davies harmonic oscillator (1999)

$$Au = -u_{xx} + ix^2u, \quad x \,\varepsilon\, \mathbf{R}$$

Bender "non-Hermitian Hamiltonian" (1998)

$$Au = -u_{xx} + ix^3u, \quad x \,\varepsilon\, \mathbf{R}$$




Cossu & Chomaz advection-diffusion operator (1997)

$$Au = u_{xx} + u_x + (\tfrac{1}{4}-x^2)u, \quad x \ \varepsilon\ \boldsymbol{R}$$

Benilov, O'Brien & Sazanov model of a liquid film instability (2004)

$$Au = sin(x)u_{xx} + u_x, \quad x \ \varepsilon\ [-\pi,\pi]$$

Our general theory of such effects is based on a space-dependent "symbol curve" and its winding numbers about values λ.

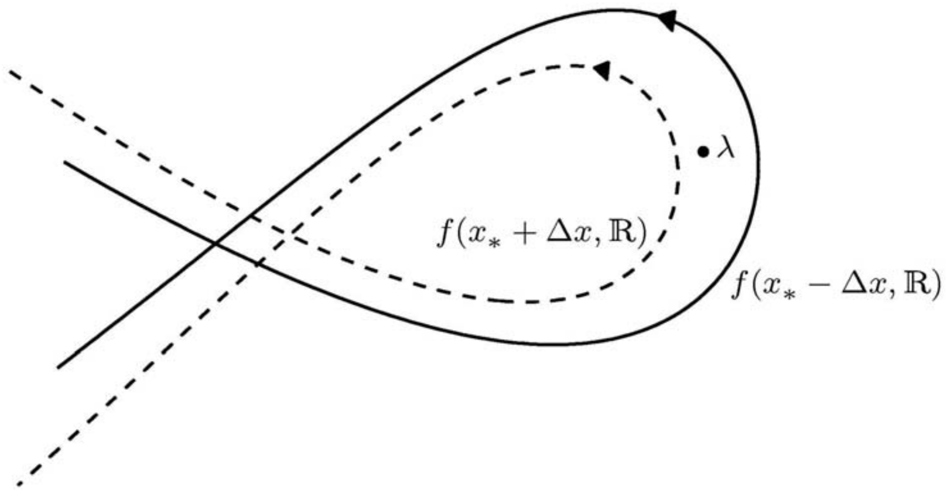
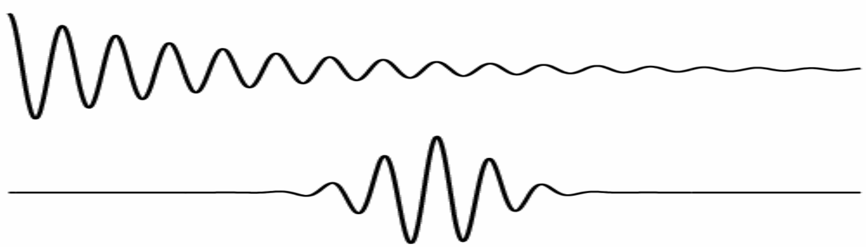



Figure 11.2: Winding number interpretation of Theorem 11.2. If the symbol curve $f(x, \mathbb{R})$ crosses $\lambda$ as $x$ increases through $x_*$ in such a way that the winding number about $\lambda$ decreases, then there is an exponentially good wave packet pseudomode localized at $x_*$ with pseudoeigenvalue $\lambda$. Compare Figure 8.6.

This is one quarter of a very satisfying bigger picture.
All the theorems involve winding numbers of symbol curves.

<table>
<tr><th></th><th>Toeplitz matrices</th><th>Differential operators</th></tr>
<tr><td>const. coeff.</td><td colspan="2">boundary pseudomodes</td></tr>
<tr><td>variable coeff.</td><td colspan="2">wave packet pseudomodes</td></tr>
</table>



| | *Toeplitz matrices* | *Differential operators* |
|---|---|---|
| const. coeff. | Landau 75<br>Reichel + T. 92<br>Böttcher et al. $\geq$ 94 | Reddy 93<br>Davies 00 |
| variable coeff. | T. + Chapman 04<br>Borthwick + Uribe 04 | Davies 99<br>Dencker-Sjöstrand-Zworski 04<br>T. 05 |

**SOME APPLICATIONS**

Nonhermitian quantum mechanics
Lewy-Hörmander nonexistence
"ghost solutions" of ODEs
exponential dichotomy theory
Orr-Sommerfeld, hydrodyn. stability
many other variable coeff problems?

## (3) Lewy-Hörmander nonexistence

**Lewy 1957**: example of $C^\infty$ linear PDE with no solns., even locally

**Hörmander** soon after: general theory and "commutator condition"

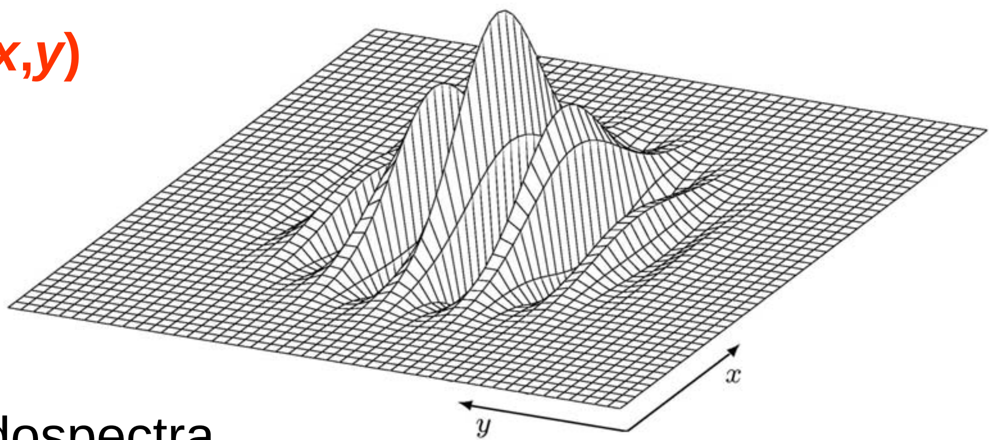
Many other developments by leading mathematicians
(Garabedian, Nirenberg, Treves, Beals, Fefferman, Dencker, Lerner,…)

**Zworski 2001**: points out connection with pseudospectra

Idea: for PDE satisfying commutator condition, can construct arbitrarily good wave packet pseudo-solutions of homogeneous problem.

By a kind of Fredholm alternative, this implies nonexistence for the adjoint problem.

Simplest example: $u_x + ixu_y = f(x,y)$
has no soln near (0,0).



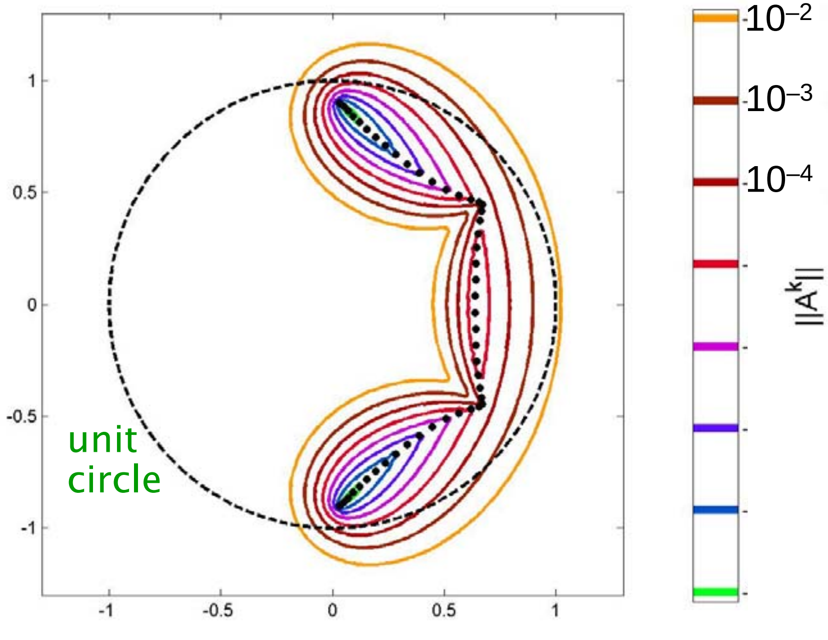

The commutator condition is
the same as the twist condition
in the theory of wave packet pseudospectra.

# (4) Theorems on transients

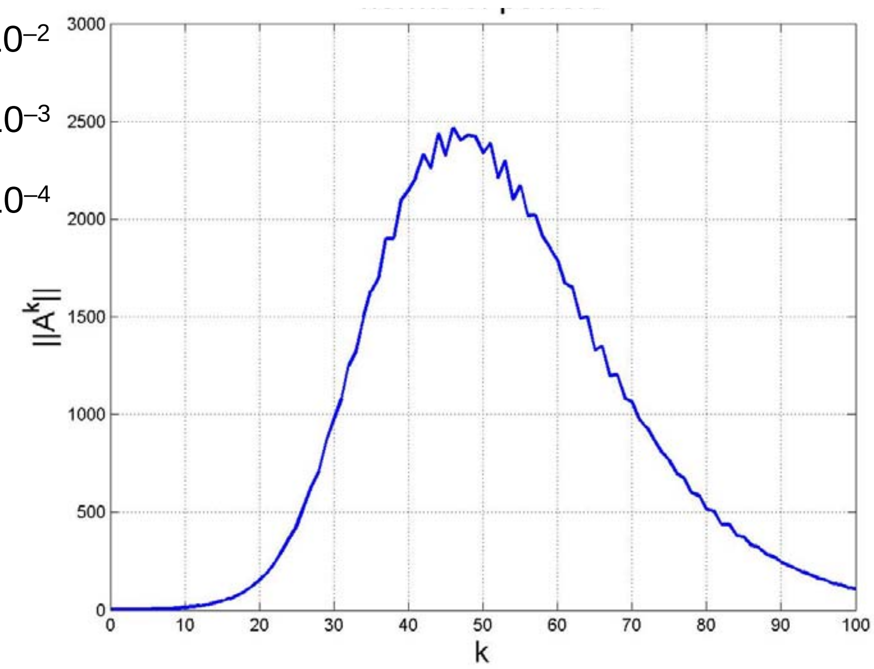
# EXAMPLE: THE 50×50 GRCAR MATRIX, TIMES 0.4

Eigenvalues:

Pseudospectra:




Norms of powers:



# Simplest lower bound for discrete time

**Theorem.**
If $\| (z-A)^{-1} \| = K / (|z| - 1)$ for $|z| > 1$, then $\sup_k \| A^k \| > K$.

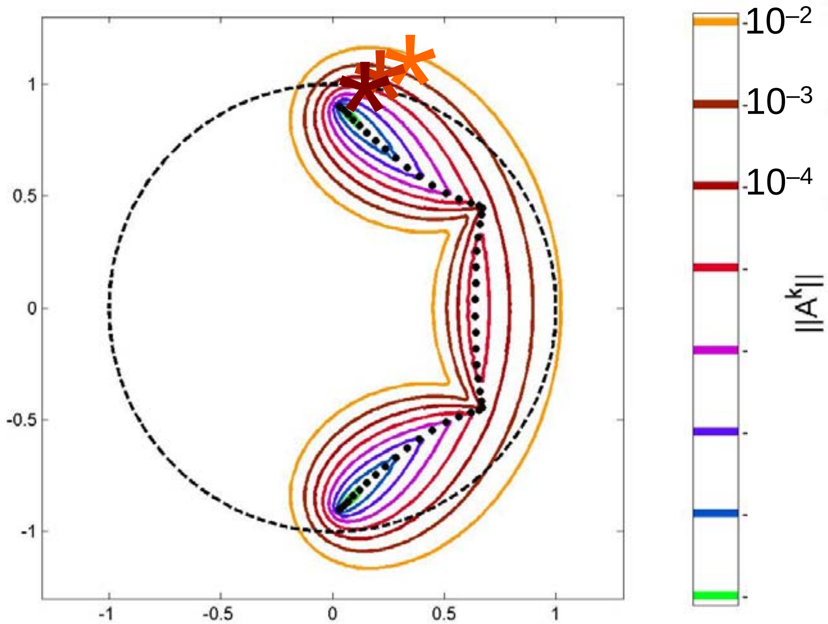
This "easy half of the Kreiss Matrix Thm." was known to Kreiss in the early 1960s.

Proof.: Straightforward estimation of the power series

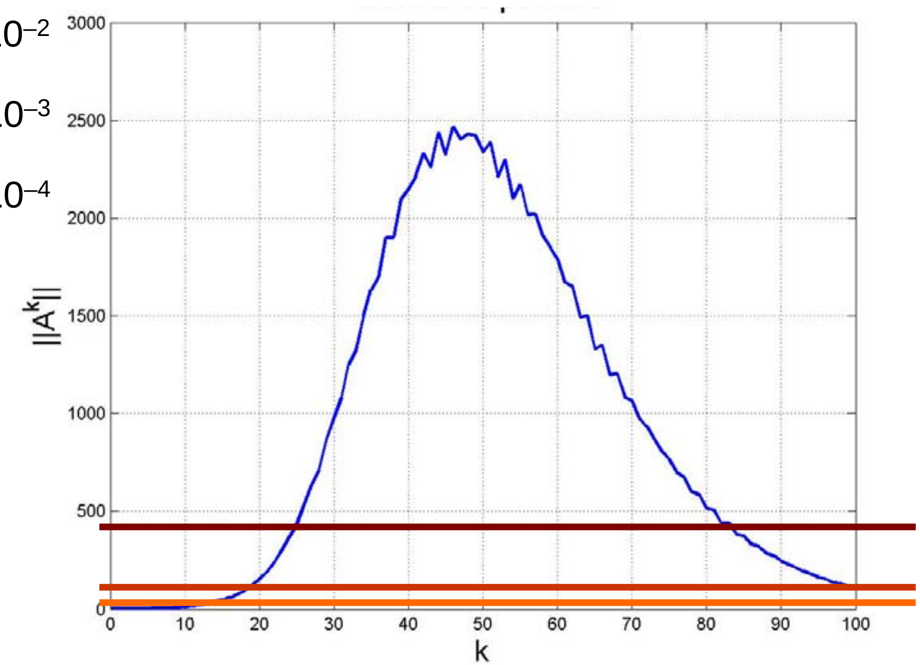
$$(z-A)^{-1} = z^{-1} ( I + z^{-1}A + (z^{-1}A)^2 + \dots )$$

If this is large…        …then some of these must be big.

Pseudospectra:




Norms of powers:




$10^{-2}$ pseudospectral radius = 1.185 $\Rightarrow$ transient growth at least 18.5

$10^{-3}$ pseudospectral radius = 1.102 $\Rightarrow$ transient growth at least 102

$10^{-4}$ pseudospectral radius = 1.0417 $\Rightarrow$ transient growth at least 417

What's missing in such bounds is information about the time scales of these transient effects. The book contains four new multi-part theorems that contain such information. Here is one of these theorems.

These results are new and we are eager to see what uses they find in the years ahead.

**Lower bounds on $\|e^{tA}\|$**

**Theorem 15.4** *Let $\mathbf{A}$ be a matrix or closed linear operator generating a $C_0$ semigroup. If $\|(z-\mathbf{A})^{-1}\| = K/\mathrm{Re}z$ for some $z$ with $\mathrm{Re}z > 0$ and $K > 1$, then*

$$\sup_{t\ge 0} \|e^{t\mathbf{A}}\| \ge K. \tag{15.7}$$

*The $\varepsilon$-pseudospectral abscissa $\alpha_\varepsilon(\mathbf{A})$ is finite for each $\varepsilon > 0$. Taking the rightmost value of $z$ in the complex plane with the same value of $\|(z-\mathbf{A})^{-1}\|$ gives*

$$\sup_{t\ge 0} \|e^{t\mathbf{A}}\| \ge \alpha_\varepsilon(\mathbf{A})/\varepsilon \qquad \forall \varepsilon > 0, \tag{15.8}$$

*and maximizing over $\varepsilon$ gives*

$$\sup_{t\ge 0} \|e^{t\mathbf{A}}\| \ge \mathcal{K}(\mathbf{A}), \tag{15.9}$$

*where the* Kreiss constant *is defined by*

$$\mathcal{K}(\mathbf{A}) \equiv \sup_{\varepsilon>0} \alpha_\varepsilon(\mathbf{A})/\varepsilon = \sup_{\mathrm{Re}z>0} (\mathrm{Re}z)\|(z-\mathbf{A})^{-1}\|. \tag{15.10}$$

*If $a = \mathrm{Re}z$, then for any $\tau > 0$,*

$$\sup_{0<t\le\tau} \|e^{t\mathbf{A}}\| \ge e^{a\tau} \Big/ \left(1 + \frac{e^{a\tau}-1}{K}\right), \tag{15.11}$$

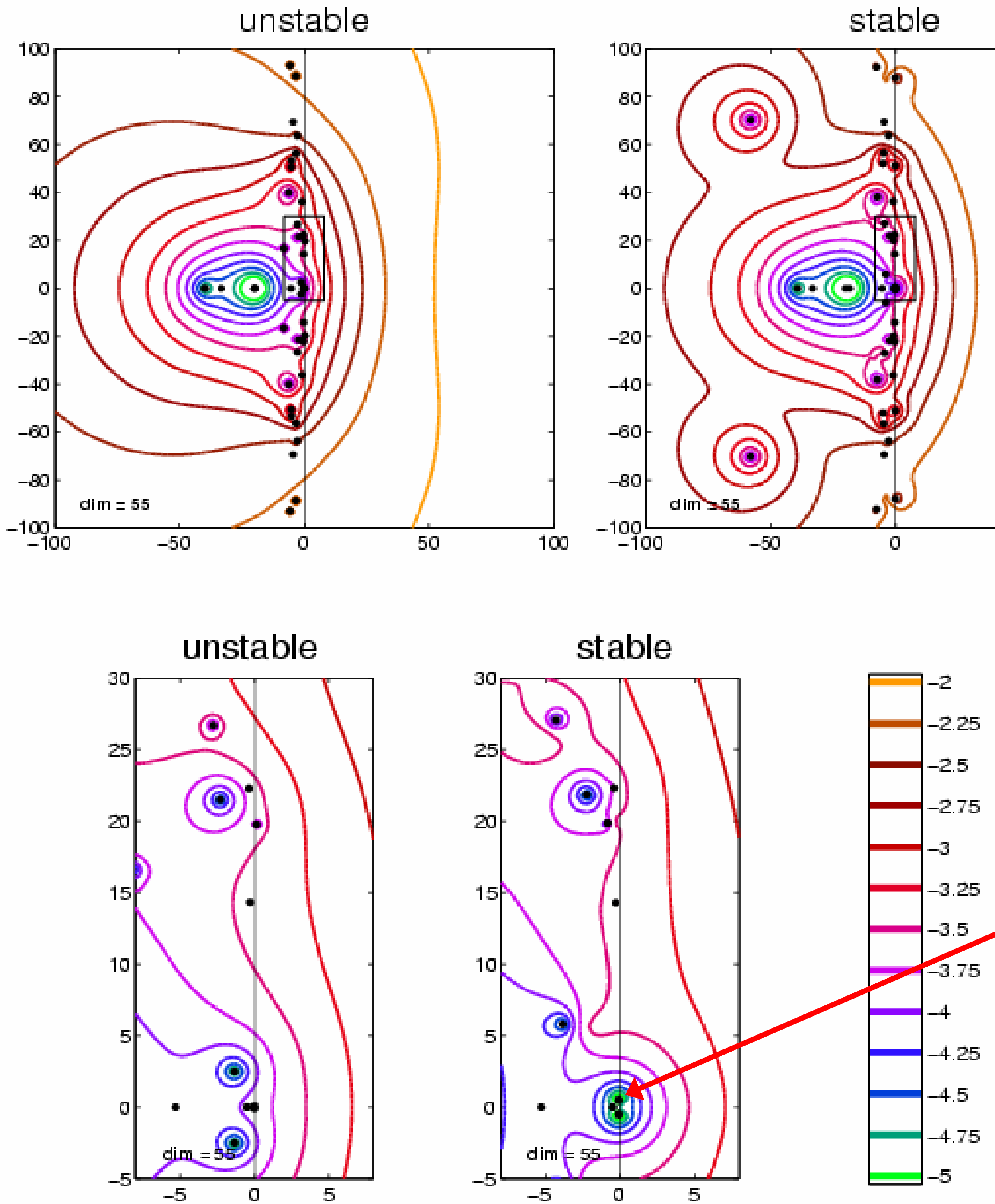
*and if $\|e^{t\mathbf{A}}\| \le M$ for all $t \ge 0$, then for any $\tau \ge 0$, with $K$ defined as before but now with $a < 0$ permitted and $-\infty < K/M \le 1$,*

$$\|e^{\tau\mathbf{A}}\| \ge e^{a\tau} - \frac{e^{a\tau}-1}{K/M} = 1 - \frac{(e^{a\tau}-1)(1-K/M)}{K/M}. \tag{15.12}$$

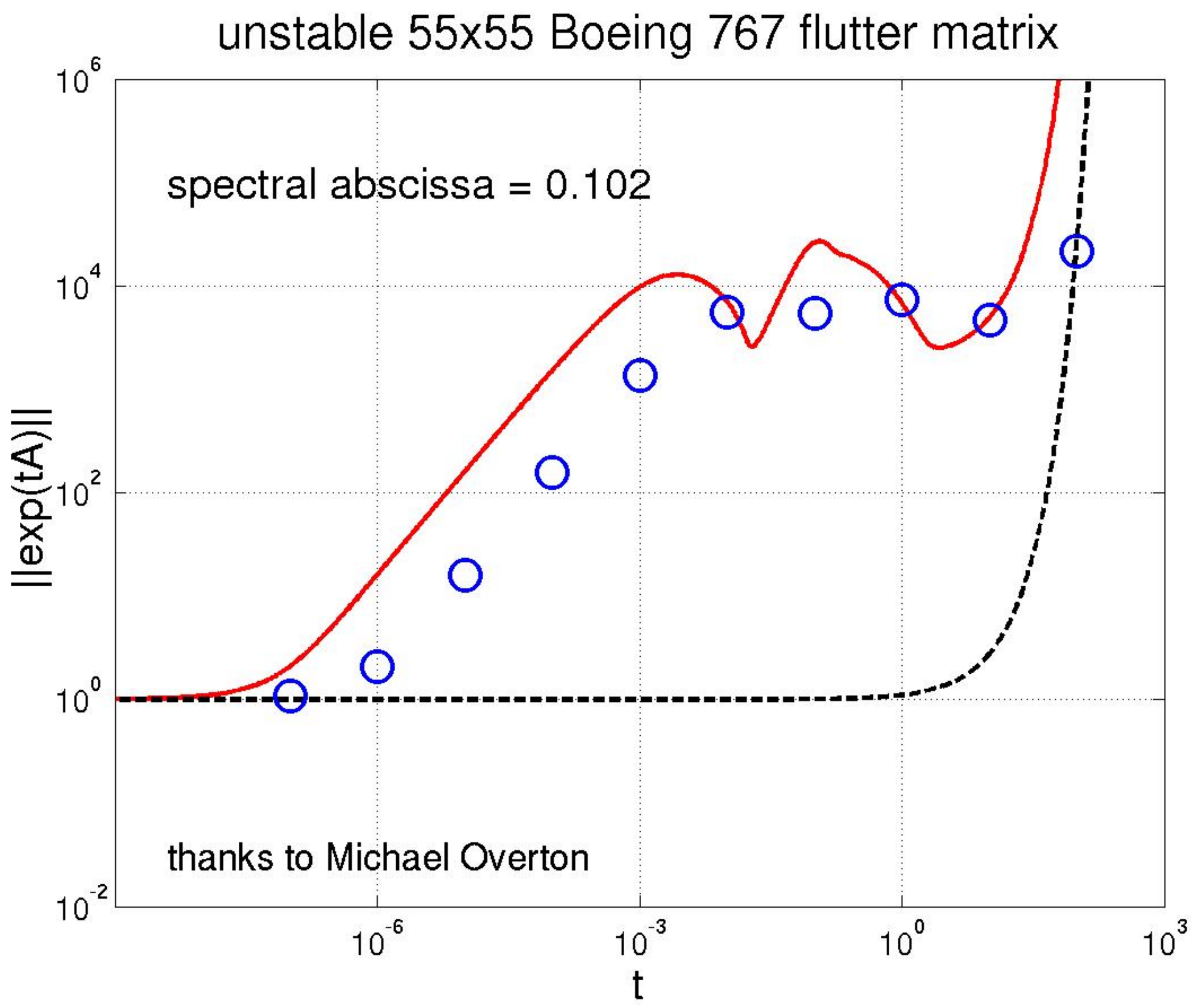
*In the particular case $a = K = 0$, (15.12) reduces by l'Hôpital's rule to*

$$\|e^{\tau\mathbf{A}}\| \ge 1 - \frac{\tau M}{\|(z-\mathbf{A})^{-1}\|}. \tag{15.13}$$

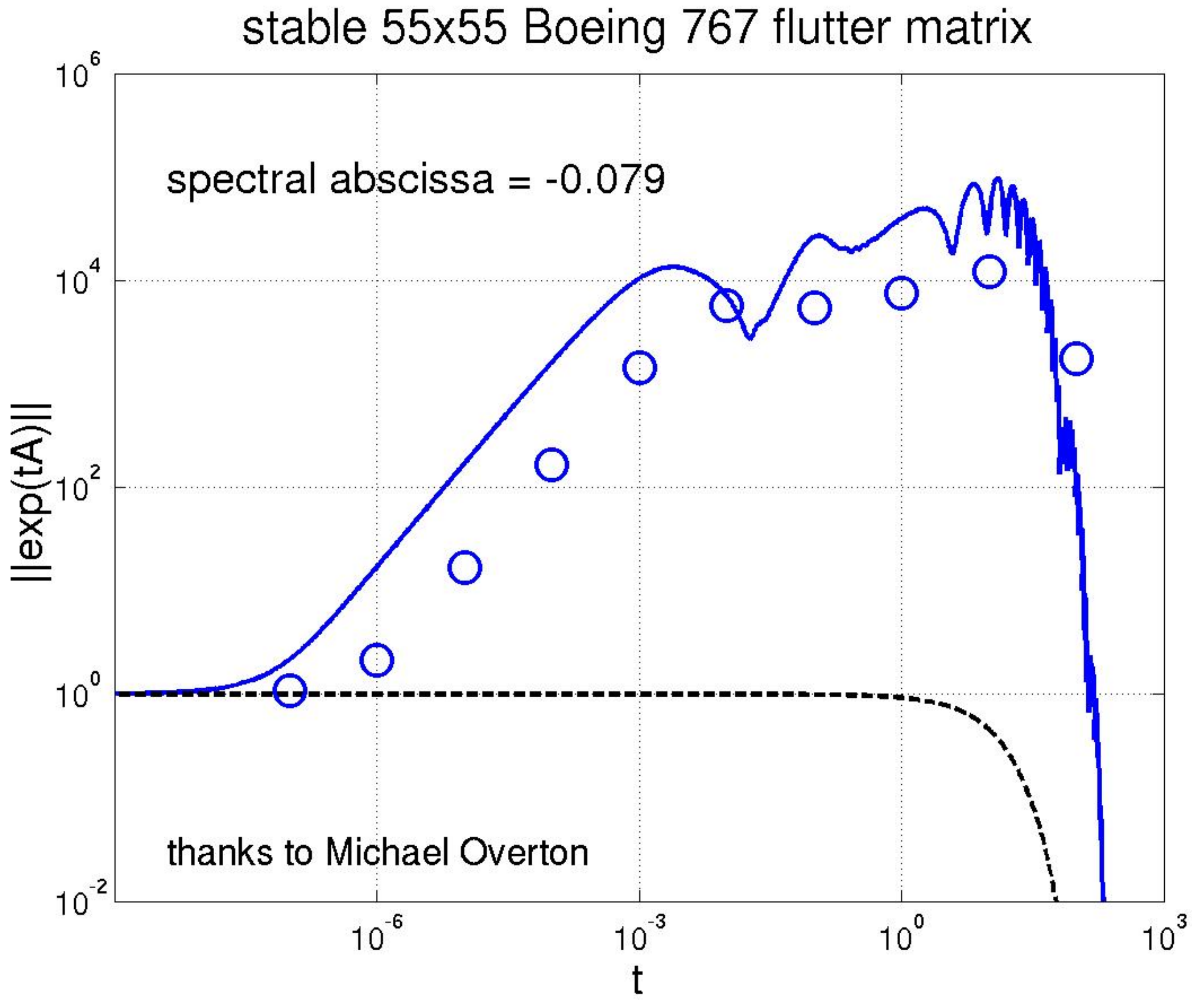
**BOEING 767 FLUTTER MATRICES FROM SLICOT AND MICHAEL OVERTON**




transient growth $>10^4$ on time scale $>1$



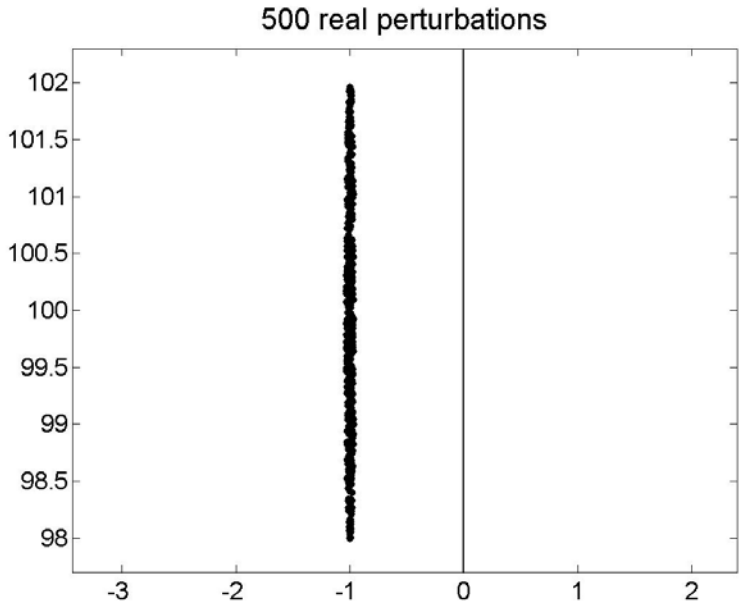

○ = lower bound from pseudospectra
(somewhere for $0 < \tau < t$ )

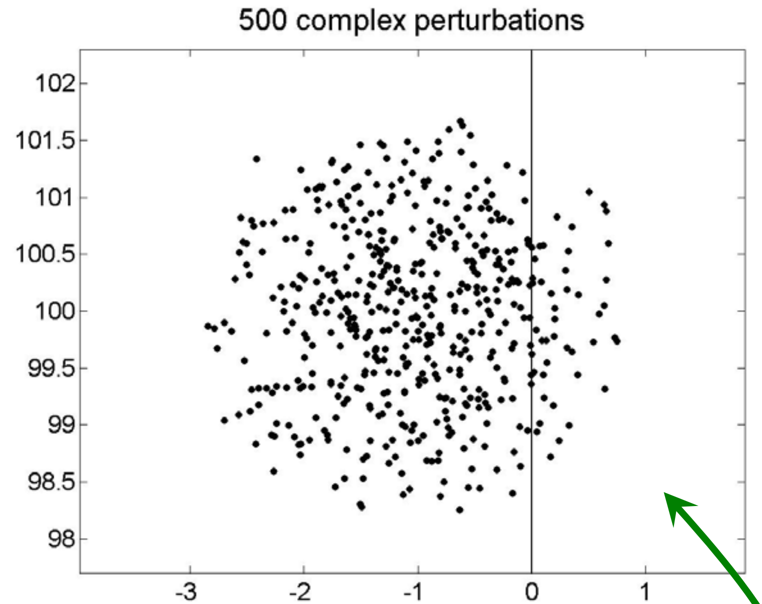



O = lower bound from pseudospectra
(somewhere for $0 < \tau < t$ )

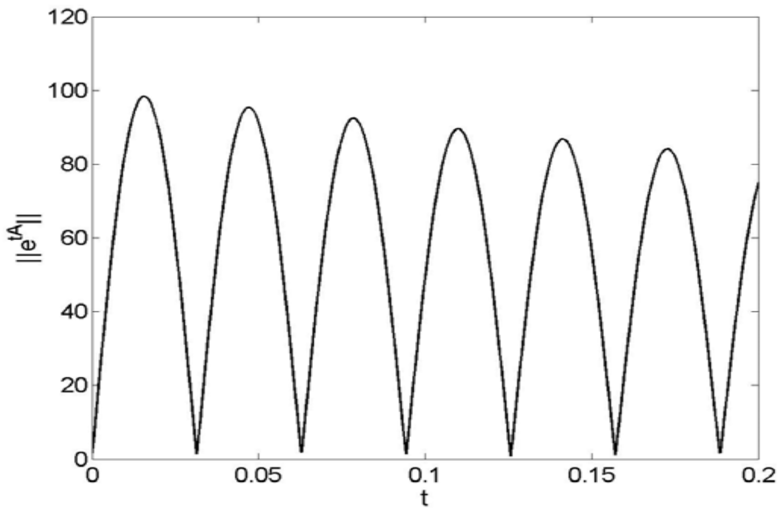
It is complex pseudospectra, not real ones, that shed light on behaviour of the unperturbed system, even if you only care about behaviour for real data.

Example. Random perturbations of norm 0.04 of $A = \begin{bmatrix} -1 & 10000 \\ -1 & -1 \end{bmatrix}$




Norms of $e^{tA}$ – realized even for real initial data.

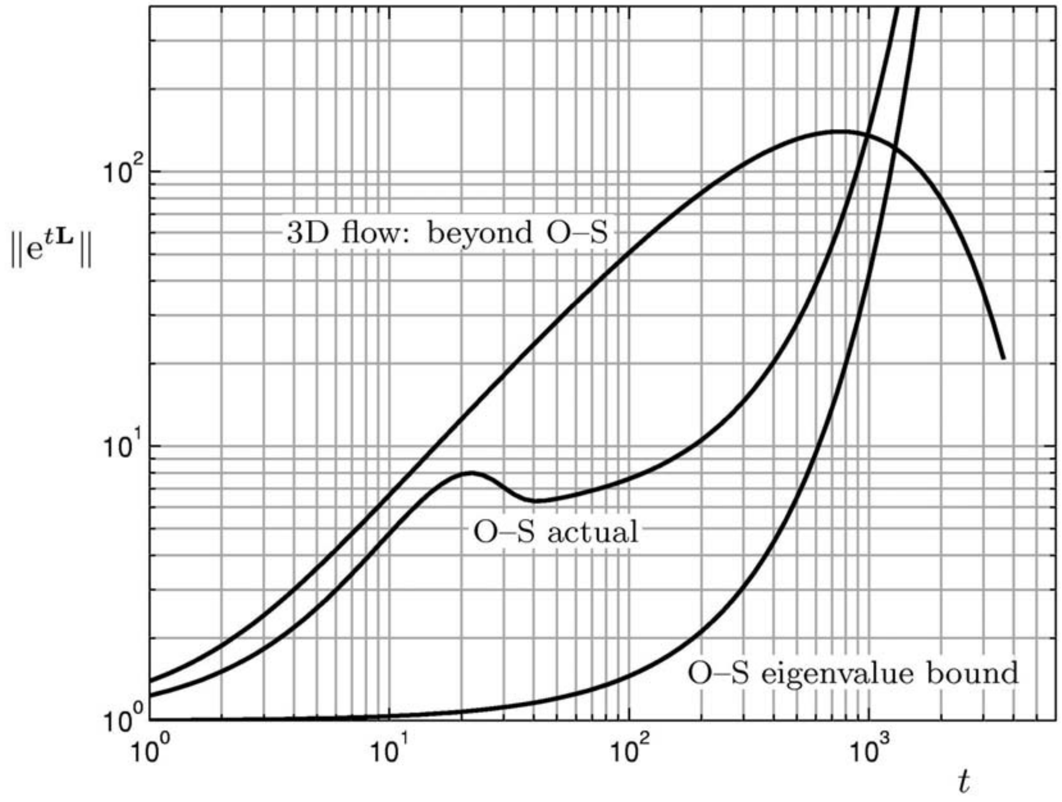



The point of all these theorems is to relate pseudospectra like this and transient effects like these.

## (5) Changes in fluid mechanics

*The “nonnormal wars” in fluid mechanics were waged in the 1990s.*

This plot for an Orr-Sommerfeld operator is representative.
(Plane Poiseuille flow, linearization about laminar solution.)



In early 2004 I spent a few weeks in the U. Queensland library to write the section “Further problems in fluid mechanics”.

What I found was remarkable.  The wars are winding down and papers on nonnormality and transients have become ubiquitous.

Some examples:

*J. Climate* 1999: "The nonnormal nature of El Niño and intraseasonal variability"

*Phys. Plasmas 2000:* "Linear stability theory and bypass transition in shear flows"

*Flow Turb. Combust. 2000:* "Maximum spatial growth of Görtler vortices"

*Prog. Aero. Sci. 2001:* "Flow control: new challenges for a new Renaissance"

*Theor. Comp. Fluid Dyn. 2001:* "Optimal control of nonmodal disturbances in boundary layers"

*J. Fluid Mech. 2001:* "Optimal linear growth in swept boundary layers"

*J. Fluid Mech. 2001:* "Simulations of bypass transition"

*J. Fluid Mech. 2001:* "Disturbance growth in boundary layers subjected to free-stream turbulence"

*Phys. Fluids 2001:* "Transient growth: a factor in bypass transition"

*Astron. Astrophys. 2002:* "Does spiral galacy IC 343 exhibit shear induced wave transformations!?"

*J. Phys. Oceanography 2002:* "The nonnormality of coastal ocean flows…"

*Phys. Fluids 2002:* "Energy growth of initial perturbations in two-dimensional gravitational jets"

*J. Fluid Mech. 2002:* "Linear optimal control applied to instabilities in spatially developing boundary layers"

*Phys. Fluids 2002:* "Transient growth in Taylor-Couette flow"

*Phys. Fluids 2003:* "Experimental study of non-normal nonlinear transition in a rotating… flow"

*Dyn. Atmos. Oceans 2003:* "Non-normal perturbation growth in idealized island and headland wakes"

*Phys. Fluids 2004:* "Transient energy growth for the Lamb-Oseen vortex"

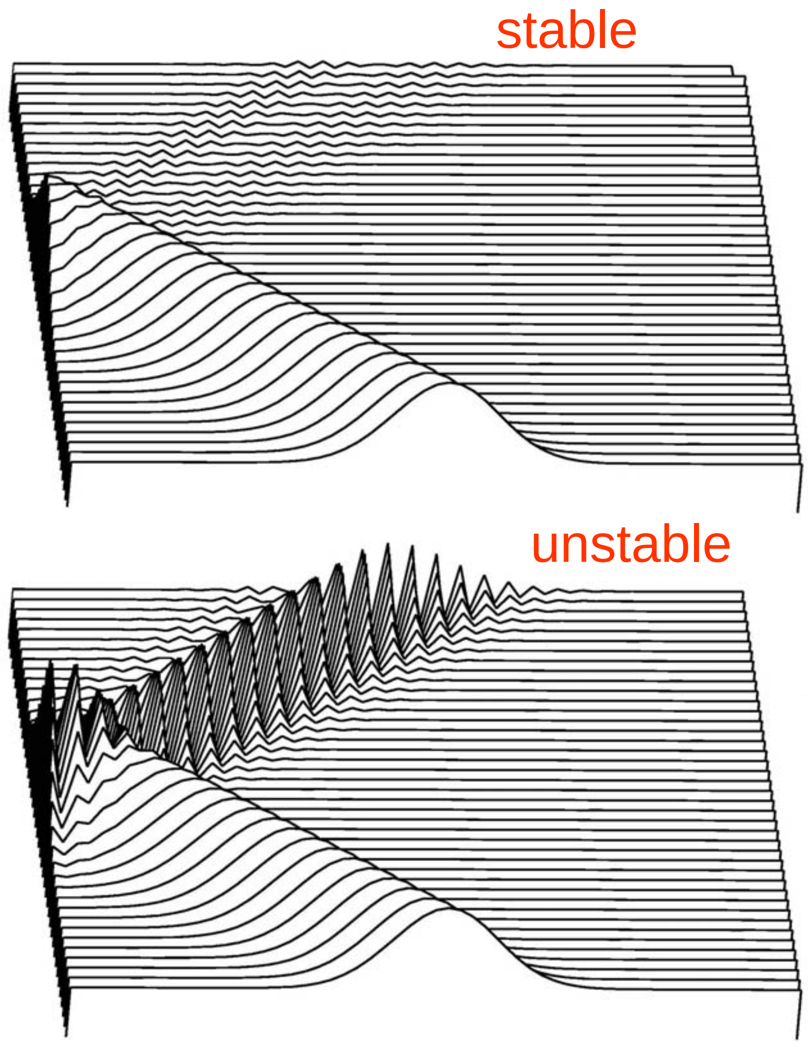
Papers like these are all about nonnormality (but usually not pseudospectra)

## (6) GKS-stability of boundary conditions

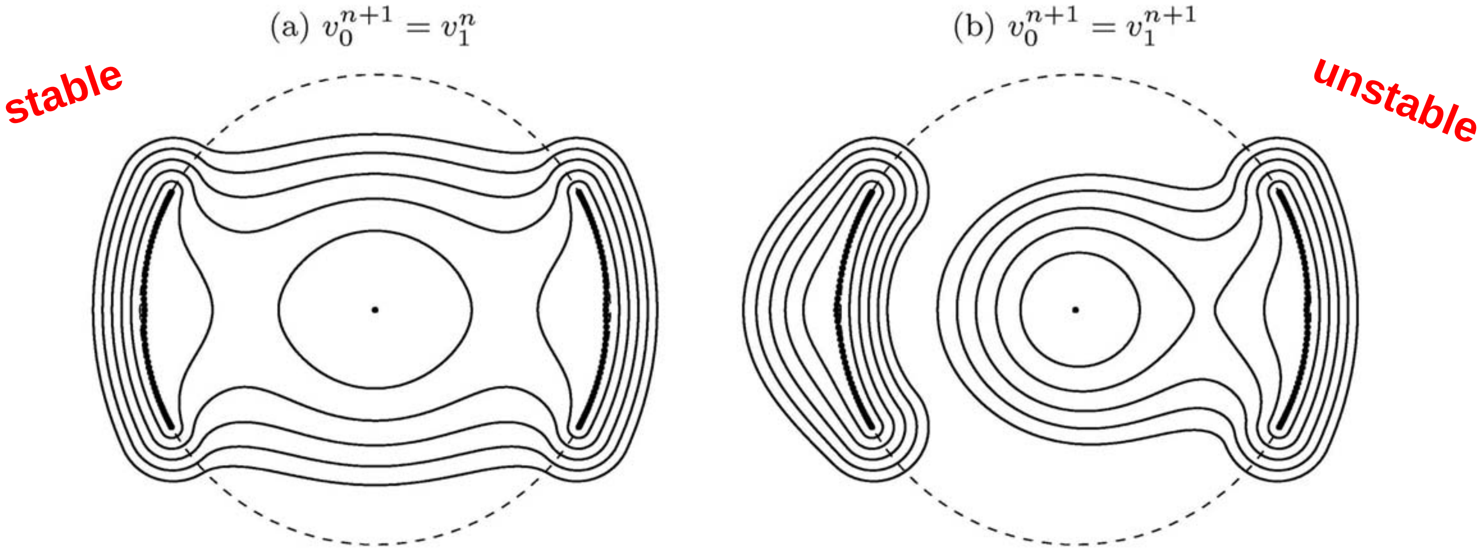
Numerical boundary conditions for hyperbolic systems of PDE

Classic theory from 60s & 70s due to Strang, Osher, Kreiss and others

GKS-stability theory
Gustafsson, Kreiss & Sundström 1972



# *GKS-instability ↔ bulge in the pseudospectra outside unit disk*



# (7) Nonhermitian quantum mechanics

“Nonhermitian quantum mechanics” has emerged in the last decade. One strand comes from **Hatano & Nelson** (1996+), who introduced the

## NONSYMMETRIC ANDERSON MODEL

Tridiagonal matrix of large dimension

Main diagonal: random (e.g. uniform $[-2,2]$)

Superdiagonal: $a$    Subdiagonal: $1/a$    (e.g. $a$=2)

Physics: issues of localization and “delocalization” in, e.g., quantum system with transverse magnetic field.

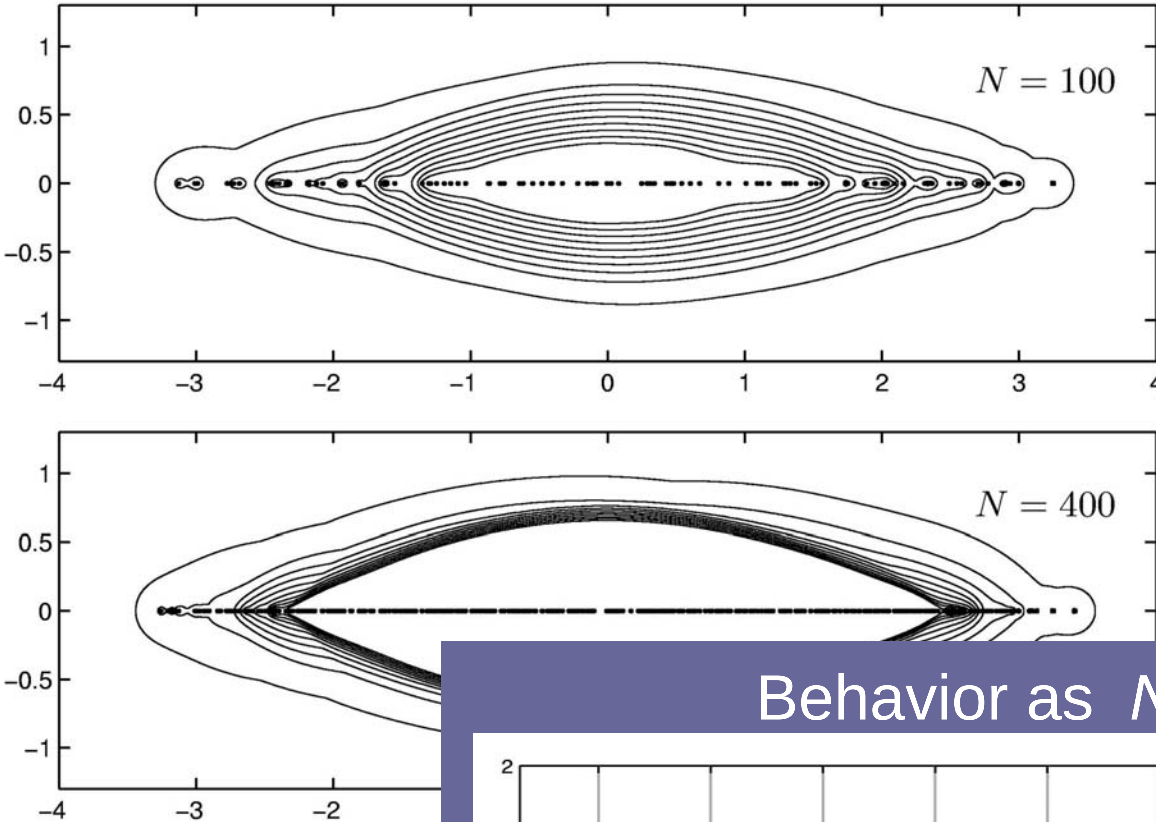
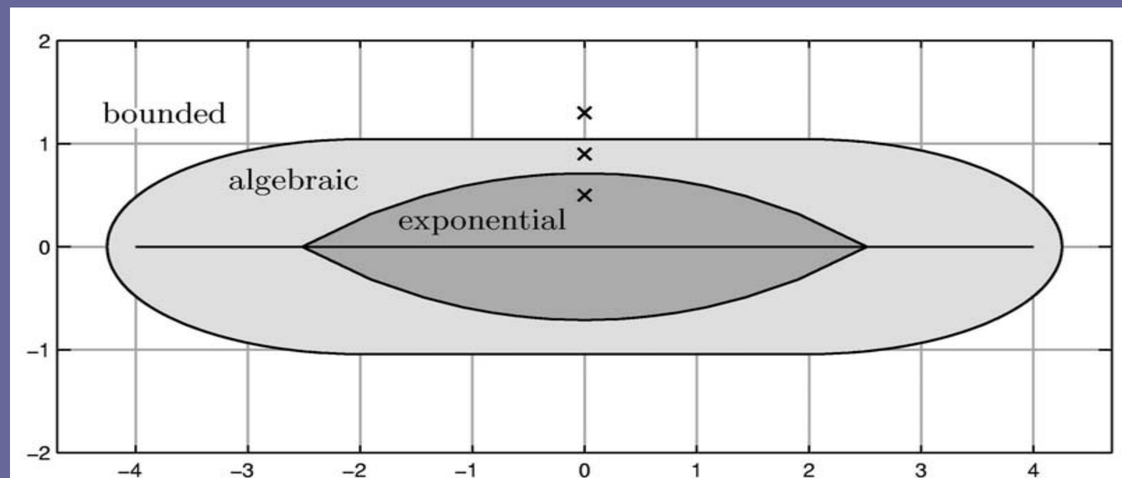


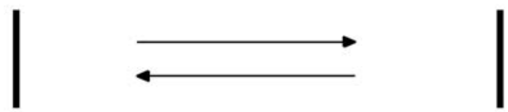

Figure 36.3: Eigenvalues and ε-
trices (36.3) of dimensions $N$
Though the eigenvalues are re
'bubble'.

## Behavior as $N \to \infty$



# (8) Lasers

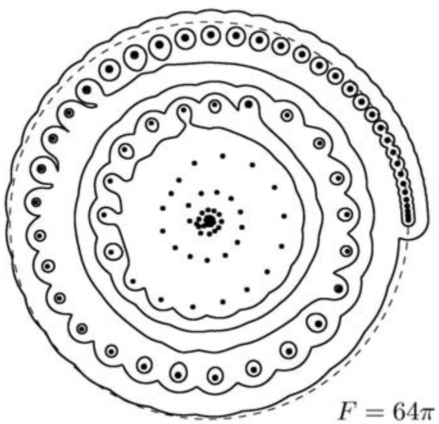
Ordinary laser resonant cavity



Fox-Li operator describing bouncing wave packets (close to normal)

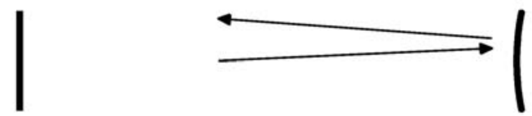
$$\mathbf{A}u_0(x) = \sqrt{\frac{\mathrm{i}F}{\pi}} \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}F(x-s)^2} u_0(s)\,\mathrm{d}s$$

Spectra and pseudospectra




**(H. J. Landau, 1970s)**

"Unstable oscillator" high-powered laser resonant cavity



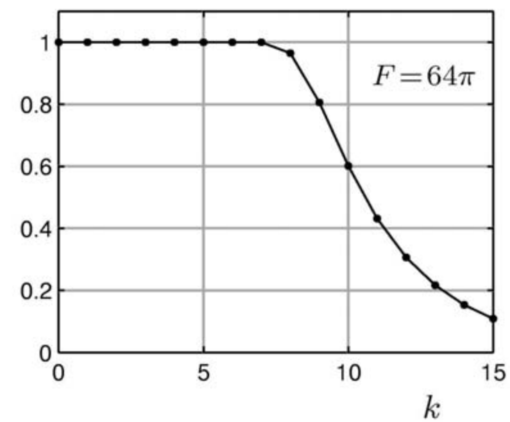
A. Siegman

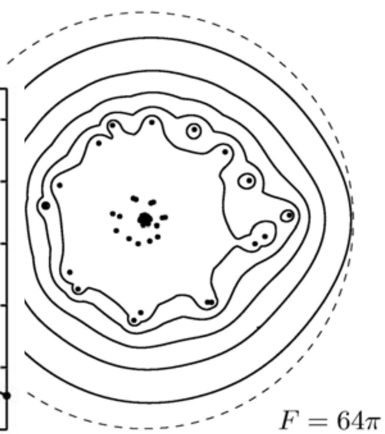
Fox-Li operator
(far from normal)

$$\mathbf{A}u_0(x) = \sqrt{\frac{\mathrm{i}F}{\pi}} \int_{-1}^{1} \mathrm{e}^{-\mathrm{i}FM(x/M-s)^2} u_0(s)\,\mathrm{d}s$$

Spectra and pseudospectra

Transients



If eigenvalue analysis were perfect, ordinary laser light would be perfectly coherent. In fact, random fluctuations cause finite line widths given by the *Schawlow-Townes formula* (1958).

**Nobel Prize**

For unstable oscillators, the lines widths may be thousands of times greater: the *Petermann excess noise factor* (1979).

What is this factor? Numerical linear algebraists recognize it instantly:

$$\boxed{K = \kappa(\lambda_1)^2}$$

**Petermann factor** →

← **square of condition number of leading eigenvalue**

# Looking ahead

- Fundamental theory of pseudospectra
  (strangely much is still missing)

- Plasma physics, magnetohydrodynamics
  (fusion power? earth's dynamo?)