Slides from talk at Householder Symposium on Numerical Linear Algebra, May 2005.

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Begun in 1990.

To appear in July.

(Princeton U. Press)
We’ve switched to a strict inequality \[ (z - A)^{-1} \] and to the notation \( \sigma_\varepsilon(A) \).

**First definition of pseudospectra**

Let \( A \in \mathbb{C}^{N \times N} \) and \( \varepsilon > 0 \) be arbitrary. The \( \varepsilon \)-pseudospectrum \( \sigma_\varepsilon(A) \) of \( A \) is the set of \( z \in \mathbb{C} \) such that

\[
\| (z - A)^{-1} \| > \varepsilon^{-1}.
\]  
(2.1)
**TABLE OF CONTENTS**

**I. Introduction**  
1. Eigenvalues  
2. Pseudospectra of matrices  
3. A matrix example  
4. Pseudospectra of linear operators  
5. An operator example  
6. History of pseudospectra  

**II. Toeplitz Matrices**  
7. Toeplitz matrices and boundary pseudomodes  
8. Twisted Toeplitz matrices and wave packet pseudomodes  
9. Variations on twisted Toeplitz matrices  

**III. Differential Operators**  
10. Differential operators and boundary pseudomodes  
11. Variable coefficients and wave packet pseudomodes  
12. Advection-diffusion operators  
13. Lewy–Hörmander nonexistence of solutions  

**IV. Transient Effects and Nonnormal Dynamics**  
14. Overview of transients and pseudospectra  
15. Exponentials of matrices and operators  
16. Powers of matrices and operators  
17. Numerical range, abscissa, and radius  
18. The Kreiss Matrix Theorem  
19. Growth bound theorem for semigroups  

**V. Fluid Mechanics**  
20. Stability of fluid flows  
21. A model of transition to turbulence  
22. Orr–Sommerfeld and Airy operators  
23. Further problems in fluid mechanics  

**VI. Matrix Iterations**  
24. Gauss–Seidel and SOR iterations  
25. Upwind effects and SOR convergence  
26. Krylov subspace iterations  
27. Hybrid iterations  
28. Arnoldi and related eigenvalue iterations  
29. The Chebyshev polynomials of a matrix  

**VII. Numerical Solution of Differential Equations**  
30. Spectral differentiation matrices  
31. Nonmodal instability of PDE discretizations  
32. Stability of the method of lines  
33. Stiffness of ODEs  
34. GKS-stability of boundary conditions  

**VIII. Random Matrices**  
35. Random dense matrices  
36. Hatano–Nelson matrices and localization  
37. Random Fibonacci matrices  
38. Random triangular matrices  

**IX. Computation of Pseudospectra**  
39. Computation of matrix pseudospectra  
40. Projection for large-scale matrices  
41. Other computational techniques  
42. Pseudospectral abscissae and radii  
43. Discretization of continuous operators  
44. A flow chart of pseudospectra algorithms  

**X. Further Mathematical Issues**  
45. Generalized eigenvalue problems  
46. Pseudospectra of rectangular matrices  
47. Do pseudospectra determine behavior?  
48. Scalar measures of nonnormality  
49. Distance to singularity and instability  
50. Structured pseudospectra  
51. Similarity transformations and canonical forms  
52. Eigenvalue perturbation theory  
53. Backward error analysis  
54. Group velocity and pseudospectra  

**XI. Further Examples and Applications**  
55. Companion matrices and zeros of polynomials  
56. Markov chains and the cutoff phenomenon  
57. Card shuffling  
58. Population ecology  
59. The Pupkovich–Fadde operator  
60. Lasers
Some highlights before 2000

| (1) | Toeplitz matrices and operators | 7 |
| (2) | Constant-coefficient differential operators | 10 |
| (3) | Advection-diffusion problems | 12 |
| (4) | Kreiss matrix theorem | 18 |
| (5) | Gearhart-Prüss theorem | 19 |
| (6) | Transition to turbulence | 20,21 |
| (7) | Upwind-downwind ordering effects | 25 |
| (8) | Krylov subspace iterations | 26,27,28 |
| (9) | Chebyshev & ideal GMRES polynomials | 29 |
| (10) | Spectral methods | 30 |
| (11) | Stability of the method of lines | 31,32 |
| (12) | Stiffness of ODEs | 33 |
| (13) | Random matrices | 35,37,38 |
| (14) | Speed-up via preliminary triangularization | 39 |
| (15) | Backward error analysis | 53 |
| (16) | Companion matrices and polynomial rootfinding | 55 |
| (17) | Markov chains, card shuffling | 57,58 |
(1) Banach spaces, semigroups, numerical range \(4,17\)
(2) Variable coeffs & wave packet pseudomodes \(8,9,11\)
(3) Lewy-Hörmander nonexistence \(13\)
(4) Theorems on transients \(14,15,16\)
(5) Changes in fluid mechanics \(23\)
(6) GKS-stability of boundary conditions \(34\)
(7) Nonhermitian quantum mechanics \(36\)
(8) Lasers \(60\)

<table>
<thead>
<tr>
<th>(9) New Arnoldi theorems</th>
<th>26,28</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10) EigTool</td>
<td>39</td>
</tr>
<tr>
<td>(11) Large-scale matrices via impl. restarted Arnoldi</td>
<td>40</td>
</tr>
<tr>
<td>(12) Criss-cross algorithms</td>
<td>42</td>
</tr>
<tr>
<td>(13) Rectangular/generalized/higher-order problems</td>
<td>45,46</td>
</tr>
<tr>
<td>(14) Links with group velocity</td>
<td>54</td>
</tr>
</tbody>
</table>

Some highlights since 2000

WRIGHT, WRIGHT, BURKE, LEWIS, OVERTON, MENGI, HIGHAM, TISSEUR, RUHE, RIEDEL, WRIGHT, TOUMAZOU, VAN DORSSELAER, FRAYSSÉ

\[ Mx'' + Cx' + Kx = 0 \] — see pp. 426-428
(1) Banach spaces, semigroups, numerical range
We were surprised: almost everything works for closed operators in Banach space.

<table>
<thead>
<tr>
<th>Three equivalent definitions of pseudospectra</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $A \in \mathcal{C}(X)$ and $\varepsilon &gt; 0$ be arbitrary. The $\varepsilon$-pseudospectrum $\sigma_\varepsilon(A)$ of $A$ is the set of $z \in \mathbb{C}$ defined equivalently by any of the conditions</td>
</tr>
<tr>
<td>$| (z - A)^{-1} | &gt; \varepsilon^{-1}$, \hspace{1cm} (4.3)</td>
</tr>
<tr>
<td>$z \in \sigma(A + E)$ for some $E \in \mathcal{B}(X)$ with $| E | &lt; \varepsilon$, \hspace{1cm} (4.4)</td>
</tr>
<tr>
<td>$z \in \sigma(A)$ or $| (z - A)u | &lt; \varepsilon$ for some $u \in \mathcal{D}(A)$ with $| u | = 1$. \hspace{1cm} (4.5)</td>
</tr>
</tbody>
</table>

If $\| (z - A)u \| < \varepsilon$ as in (4.5), then $z$ is an $\varepsilon$-pseudoeigenvalue of $A$ and $u$ is a corresponding $\varepsilon$-pseudoeigenvector (or pseudoeigenfunction or pseudomode).
Numerical range in Banach space (Lumer 1961, Bauer 1962):

\[ W(A) = \{(f, Au): u \in \mathcal{D}(A), f \in X^*, \|u\| = \|f\| = 1, (f, u) = 1\} \]

\( W(A) \) need not be convex. Example: \( A = \begin{pmatrix} -i & i \\ i & i \end{pmatrix} \), \( \infty \)-norm

The significance of \( W(A) \) is the same as in Hilbert space: \textit{the rightmost point determines the initial slope of} \( \| e^{tA} \| \).

All this also works if \( A \) is the generator of a \( C_0 \) semigroup (modulo technicalities).
(2) Variable coeffs & wave packet pseudomodes
It started with Davies’s complex harmonic oscillator in 1999.

\[ Au = -u_{xx} + ix^2u \text{ on } (-\infty, \infty) \]

**Symbol**: \( f(x, k) = k^2 + ix^2 \)
Pseudomodes for such operators have the form of wave packets.

Davies harmonic oscillator (1999)

\[ Au = -u_{xx} + ix^2 u, \quad x \in \mathbb{R} \]

Bender “non-Hermitian Hamiltonian” (1998)

\[ Au = -u_{xx} + ix^3 u, \quad x \in \mathbb{R} \]
Cossu & Chomaz advection-diffusion operator (1997)

\[ Au = u_{xx} + u_x + (\frac{1}{4} - x^2)u, \quad x \in \mathbb{R} \]


\[ Au = \sin(x)u_{xx} + u_x, \quad x \in [-\pi, \pi] \]
Our general theory of such effects is based on a space-dependent “symbol curve” and its winding numbers about values $\lambda$.

Figure 11.2: Winding number interpretation of Theorem 11.2. If the symbol curve $f(x, \mathbb{R})$ crosses $\lambda$ as $x$ increases through $x_*$ in such a way that the winding number about $\lambda$ decreases, then there is an exponentially good wave packet pseudomode localized at $x_*$ with pseudoeigenvalue $\lambda$. Compare Figure 8.6.
This is one quarter of a very satisfying bigger picture. All the theorems involve winding numbers of symbol curves.

<table>
<thead>
<tr>
<th>const. coeff.</th>
<th>Toeplitz matrices</th>
<th>boundary pseudomodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>variable coeff.</td>
<td>Differential operators</td>
<td>wave packet pseudomodes</td>
</tr>
</tbody>
</table>

\[ \text{Wave packet pseudomodes} \]
<table>
<thead>
<tr>
<th>Toeplitz matrices</th>
<th>Differential operators</th>
</tr>
</thead>
<tbody>
<tr>
<td>Landau 75</td>
<td>Reddy 93</td>
</tr>
<tr>
<td>Reichel + T. 92</td>
<td>Davies 00</td>
</tr>
<tr>
<td>Böttcher et al. ≥94</td>
<td></td>
</tr>
<tr>
<td>T. + Chapman 04</td>
<td>Davies 99</td>
</tr>
<tr>
<td>Borthwick + Uribe 04</td>
<td>Dencker-Sjöstrand-Zworski 04</td>
</tr>
<tr>
<td></td>
<td>T. 05</td>
</tr>
</tbody>
</table>

**SOME APPLICATIONS**

Nonhermitian quantum mechanics
Lewy-Hörmander nonexistence
“ghost solutions” of ODEs
exponential dichotomy theory
Orr-Sommerfeld, hydrodyn. stability
many other variable coeff problems?
(3) Lewy-Hörmander nonexistence
Lewy 1957: example of $C^\infty$ linear PDE with no solns., even locally

Hörmander soon after: general theory and “commutator condition”

Many other developments by leading mathematicians
   (Garabedian, Nirenberg, Treves, Beals, Fefferman, Dencker, Lerner,…)

Zworski 2001: points out connection with pseudospectra

Idea: for PDE satisfying commutator condition, can construct arbitrarily good wave packet pseudo-solutions of homogeneous problem.

By a kind of Fredholm alternative, this implies nonexistence for the adjoint problem.

Simplest example: $u_x + i x u_y = f(x,y)$ has no soln near $(0,0)$.

The commutator condition is the same as the twist condition in the theory of wave packet pseudospectra.
(4) Theorems on transients
EXAMPLE: THE 50×50 GRCAR MATRIX, TIMES 0.4

Eigenvalue spectra:

Norms of powers:

unit circle
Theorem.
If $\| (z-A)^{-1} \| = K / (|z| - 1)$ for $|z| > 1$, then $\sup_k \| A^k \| > K$.

This “easy half of the Kreiss Matrix Thm.” was known to Kreiss in the early 1960s.

Proof.: Straightforward estimation of the power series

$$(z-A)^{-1} = z^{-1} (1 + z^{-1}A + (z^{-1}A)^2 + \ldots)$$

If this is large… …then some of these must be big.
10^{-2} pseudospectral radius = 1.185 \Rightarrow \text{transient growth at least 18.5}

10^{-3} pseudospectral radius = 1.102 \Rightarrow \text{transient growth at least 102}

10^{-4} pseudospectral radius = 1.0417 \Rightarrow \text{transient growth at least 417}
What’s missing in such bounds is information about the time scales of these transient effects. The book contains four new multi-part theorems that contain such information. Here is one of these theorems.

These results are new and we are eager to see what uses they find in the years ahead.

**Theorem 15.4** Let $A$ be a matrix or closed linear operator generating a $C_0$ semigroup. If $\|(z - A)^{-1}\| = K/\text{Re}z$ for some $z$ with $\text{Re}z > 0$ and $K > 1$, then

$$\sup_{t \geq 0} \|e^{tA}\| \geq K. \quad (15.7)$$

The $\varepsilon$-pseudospectral abscissa $\alpha_\varepsilon(A)$ is finite for each $\varepsilon > 0$. Taking the rightmost value of $z$ in the complex plane with the same value of $\|(z - A)^{-1}\|$ gives

$$\sup_{t \geq 0} \|e^{tA}\| \geq \alpha_\varepsilon(A)/\varepsilon \quad \forall \varepsilon > 0, \quad (15.8)$$

and maximizing over $\varepsilon$ gives

$$\sup_{t \geq 0} \|e^{tA}\| \geq \mathcal{K}(A), \quad (15.9)$$

where the Kreiss constant is defined by

$$\mathcal{K}(A) \equiv \sup_{\varepsilon > 0} \alpha_\varepsilon(A)/\varepsilon = \sup_{\text{Re}z > 0} (\text{Re}z)\|(z - A)^{-1}\|. \quad (15.10)$$

If $a = \text{Re}z$, then for any $\tau > 0$,

$$\sup_{0 < t \leq \tau} \|e^{tA}\| \geq e^{a\tau}/\left(1 + \frac{e^{a\tau} - 1}{K}\right), \quad (15.11)$$

and if $\|e^{tA}\| \leq M$ for all $t \geq 0$, then for any $\tau \geq 0$, with $K$ defined as before but now with $a < K$ permitted and $-\infty < K/M \leq 1$,

$$\|e^{\tau A}\| \geq e^{\tau a} - \frac{e^{\tau a} - 1}{K/M} = 1 - \frac{(e^{\tau a} - 1)(1 - K/M)}{K/M}. \quad (15.12)$$

In the particular case $a = K = 0$, (15.12) reduces by l'Hôpital's rule to

$$\|e^{\tau A}\| \geq 1 - \frac{\tau M}{\|(z - A)^{-1}\|}. \quad (15.13)$$
transient growth \( >10^4 \)
on time scale \( >1 \)

BOEING 767 FLUTTER MATRICES FROM SLICOT AND MICHAEL OVERTON
unstable 55x55 Boeing 767 flutter matrix

spectral abscissa = 0.102

tThanks to Michael Overton

○ = lower bound from pseudospectra
(somewhere for $0 < \tau < t$)
stable 55x55 Boeing 767 flutter matrix

spectral abscissa = -0.079

thanks to Michael Overton

○ = lower bound from pseudospectra
(somewhere for $0 < \tau < t$)
It is complex pseudospectra, not real ones, that shed light on behaviour of the unperturbed system, even if you only care about behaviour for real data.
Example. Random perturbations of norm 0.04 of \( A = \begin{bmatrix} -1 & 10000 \\ -1 & -1 \end{bmatrix} \).

The point of all these theorems is to relate pseudospectra like this and transient effects like these.
(5) Changes in fluid mechanics
The “nonnormal wars” in fluid mechanics were waged in the 1990s.

This plot for an Orr-Sommerfeld operator is representative. (Plane Poiseuille flow, linearization about laminar solution.)
In early 2004 I spent a few weeks in the U. Queensland library to write the section “Further problems in fluid mechanics”.

What I found was remarkable. The wars are winding down and papers on nonnormality and transients have become ubiquitous.

Some examples:
Papers like these are all about nonnormality (but usually not pseudospectra)
(6) GKS-stability of boundary conditions
Numerical boundary conditions for hyperbolic systems of PDE

Classic theory from 60s & 70s due to Strang, Osher, Kreiss and others

GKS-stability theory
Gustafsson, Kreiss & Sundström 1972
GKS-instability ↔ bulge in the pseudospectra outside unit disk

(a) $v_{0}^{n+1} = v_{1}^{n}$

(b) $v_{0}^{n+1} = v_{1}^{n+1}$

stable

unstable
(7) Nonhermitian quantum mechanics
“Nonhermitian quantum mechanics” has emerged in the last decade. One strand comes from Hatano & Nelson (1996+), who introduced the NONSYMMETRIC ANDERSON MODEL

**Tridiagonal matrix of large dimension**

Main diagonal: random (e.g. uniform \([-2,2]\))

Superdiagonal: \(a\)  Subdiagonal: \(1/a\)  (e.g. \(a=2\))

Physics: issues of localization and “delocalization” in, e.g., quantum system with transverse magnetic field.
Figure 36.3: Eigenvalues and ε-traces (36.3) of dimensions $N$.

Though the eigenvalues are referred to as ‘bubble’.

Behavior as $N \to \infty$
(8) Lasers
Ordinary laser resonant cavity

\[ A u_0(x) = \sqrt{\frac{F}{\pi}} \int_{-1}^{1} e^{-iF(x-s)^2} u_0(s) \, ds \]

Fox-Li operator describing bouncing wave packets (close to normal)

Spectra and pseudospectra

“Unstable oscillator” high-powered laser resonant cavity

\[ A u_0(x) = \sqrt{\frac{F}{\pi}} \int_{-1}^{1} e^{-iFM(x/M-s)^2} u_0(s) \, ds \]

Fox-Li operator (far from normal)

Transients

(H. J. Landau, 1970s)

A. Siegman
If eigenvalue analysis were perfect, ordinary laser light would be perfectly coherent. In fact, random fluctuations cause finite line widths given by the \textit{Schawlow-Townes formula} (1958).

For unstable oscillators, the lines widths may be thousands of times greater: the \textit{Petermann excess noise factor} (1979).

What is this factor? Numerical linear algebraists recognize it instantly:

\[ K = \kappa(\lambda_1)^2 \]
Looking ahead

• Fundamental theory of pseudospectra
  (strangely much is still missing)

• Plasma physics, magnetohydrodynamics
  (fusion power? earth’s dynamo?)