Notes of a Numerical Analyst

At the Edge of Infinity

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2^n is bigger than n, and Cantor showed this is true even when n is infinite. The theory is beautiful, and most of us know the basics. But we are easily caught off guard by finite numbers when they are big enough.

Take this equation adapted from [2], with (!) as a warning that equality does not actually hold:

$$10^{10^{428{,}000}} = e^{10^{428{,}000}}. \qquad (!)$$

Of course the two numbers aren't really equal — their difference is enormous. Yet they are indistinguishable if you regard the top exponent as known to just three digits, for the number on the left is equal to $\exp(10^{428{,}000.36\ldots})$. Or consider this one:

$$\left(10^{10^{428{,}000}}\right)^2 = 10^{10^{428{,}000}}. \qquad (!)$$

This time the number on the left is equal to $10\hat{}(10^{428{,}000.30\ldots})$. Evidently the familiar rules of arithmetic break down, in a practical sense, when numbers are huge, giving us principles like

$$n^2 = n, \quad 2^n > n. \qquad (!)$$

Note how these formulae echo Cantor's results for true infinities, which we can write in shorthand as

$$\infty^2 = \infty, \quad 2^\infty > \infty.$$

For another curiosity at the edge of infinity, let A be an infinite "random Fibonacci matrix" with zero entries everywhere except ± 1 (independent coin tosses) on the first two superdiagonals, i.e., entries $a_{j,j+1}$ and $a_{j,j+2}$ [3]. The spectrum Σ of A as an operator on ℓ^2 is the closed disk $|z| \le 2$ (with probability 1), which we can explain by noting that A contains arbitrarily large regions where all the signs are equal. Yet spectral theory is missing something essential about A if we view it as a limit of matrices A_n of finite dimension n. In an inner region $\Sigma_i \subset \Sigma$, roughly the disk $|z| < 1.3$, the resolvent norm $\|(z - A_n)^{-1}\|$ grows exponentially as $n \to \infty$, but in the remainder of Σ it grows only algebraically, as shown by the plot of $\log_{10}(\|(z - A_n)^{-1}\|)$ in Figure 1 for a matrix of dimension 200. The crown of this "witch hat" is very tall (truncated raggedly by floating-point arithmetic), but the brim is flat. If a physical system were governed by such matrices, the spectrum measured in the lab would probably be Σ_i, not Σ.

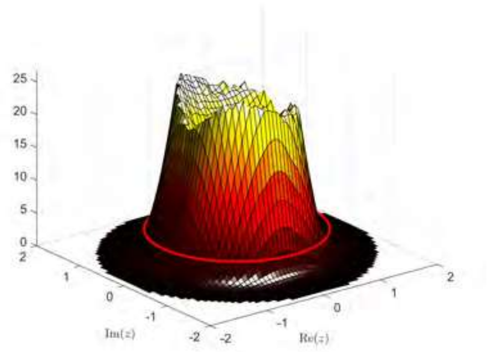

Figure 1. Random Fibonacci witch hat

Mathematics has a wonderful ability to reason rigorously about idealisations. Sometimes it is good to remember, however, that they are idealisations. In moral philosophy, the field of "infinite ethics" draws conclusions based on the supposition that there may be infinitely many worlds with infinitely many sentient beings, including a creature epsilon close to my own self down to the home address and the children's names [1]. Personally, I find it hard to believe that anything is quite that infinite.

FURTHER READING

[1] N. Bostrom, Infinite ethics, *Analysis and Metaphysics,* 10 (2011), 9–59.
[2] S. J. Chapman, J. Lottes, and L. N. Trefethen, Four bugs on a rectangle, *Proc. Roy. Soc. A,* 467 (2010), 881–896.
[3] L. N. Trefethen and M. Embree, *Spectra and Pseudospectra,* Princeton, 2005.

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