Notes of a Numerical Analyst

Nonsmooth Landscapes

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Scrambling through the brush on a Welsh hillside last summer, Kate and I ran into a problem. You might expect the ground to be dry high up, but here in the ravine between two little slopes, the water was a foot or two deep! How could we get across without soaking our boots?

Somehow we did, and for the next few miles, I found myself thinking about water and topography. I am sure most of us have considered maxima, minima and saddle points in outdoor landscapes, and these concepts make good sense if the surface $f(x,y)$ is smooth. What I hadn't noticed is that to understand creeks and streams, you need to let f be nonsmooth.

In the simplest mathematical idealization, infinitesimal rivulets flow down paths of steepest descent. If the surface is smooth, as in the left image of Figure 1, though the rivulets grow denser in the bottom of a valley, they remain distinct. This does not explain the water that troubled us on our hike. For that, you need a nonsmooth surface, as in the image on the right. Now, thanks to the singularity, multiple rivulets can coalesce. The flow is irreversible in the sense that in reversed time, the trajectories are nonunique.

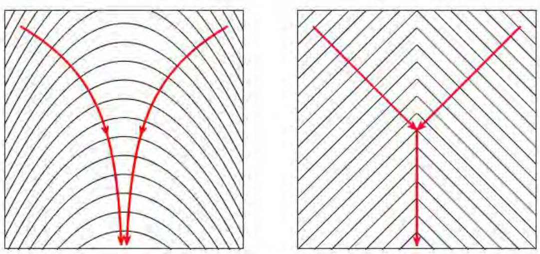

Figure 1. Contours of landscapes $f(x,y) = x^2 + y$ (left) and $|x| + y$ (right). The singularity on the right enables steepest descent paths to coalesce. Thus even a small ravine high on a hillside may have a good deal of water at the bottom.

When mathematicians investigate the world, we usually begin with smooth models, but sometimes nonsmoothness turns out to be essential. A familiar example is sonic booms from supersonic aircraft. Somehow, in a lifetime of hiking, I had not noticed that creeks and streams are another example.

Figure 2. It is rather a long way from Figure 1 to what we see outdoors. (Photo from iStock.)

Whenever something is nonsmooth, there is the question of how it got that way—often as a result of nonlinearity. In the case of landscapes and rivers, an obvious nonlinearity is that flowing water carves and changes the surface. The study of river channellization is a well developed topic of geomorphology, towards which Figure 1 represents only the tiniest of steps [1, 2]. I am grateful to Greg Tucker of the University of Colorado for teaching me about this subject.

FURTHER READING

[1] A. Fowler, *Mathematical Geoscience*, Springer 2011.
[2] I. Rodríguez-Iturbe and A. Rinaldo, *Fractal River Basins*, Cambridge, 1997.

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