Notes of a Numerical Analyst

The Meaning of Eigenvalues

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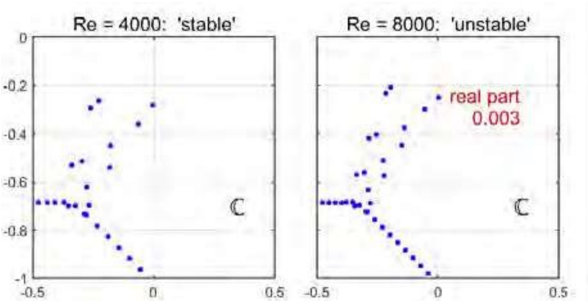
In mathematics, we idealize things. This is an essential and powerful part of what we do. But we must be on our guard to make sure the idealizations don't lose contact with the phenomena they are intended to shed light on.

Eigenvalues of nonsymmetric matrices and operators are a particularly extreme case of this challenge. In many applications, they don't have the significance we are trained to expect.

Let's look at possibly the most extensively studied nonsymmetric eigenvalue problem of all: the Orr-Sommerfeld equation. The topic is stability of fluid flows. The scientific question is, what makes high speed flows go turbulent?



To find answers, for more than a century, fluid mechanicians have investigated eigenvalues. The Navier–Stokes equations are nonlinear, and the eigenvalues are those of the linearization about a smooth, nonturbulent solution. The idea is that if all the eigenvalues are in the left half-plane, the flow is stable, whereas if there is an eigenvalue in the right half-plane, it is unstable and the unstable mode may grow into turbulence.

Consider the barely distinguishable plots above. The Reynolds number $\mathfrak{R}$ is the nondimensionalized speed, and on the left, at $\mathfrak{R} = 4000$, the flow is eigenvalue stable. On the right, at $\mathfrak{R} = 8000$, it is eigenvalue unstable. An incredible amount is known about these eigenvalues, and Steve Orszag got famous for calculating that the critical value at which one of them moves into the right half-plane is Re $= 5772.22$.

So the traditional view is that something suddenly changes when $\mathfrak{R}$ hits 5772.22. And of course, there is a theorem which proves that in a certain sense this is true.

Yet laboratory experiments almost never fit this picture. Actual flows don't show a sharp Reynolds number for transition to turbulence, and turbulence is often observed at both $\mathfrak{R} = 4000$ and 8000. The reason becomes clear if we consider the eigenvalue labeled in red. What exactly does it imply? It implies that flow perturbations can grow at rate $\exp(0.003t)$. By $t = 300$, such a perturbation will be amplified by a factor of e. This corresponds to a channel 300 times as long as it is wide, pretty much the limit of what can be built in the lab. And of course, amplification by e is not going to drive turbulence; you'll need more than that. So if you think about this image quantitatively, it is hardly surprising that the "instability" it represents is not observed.

The actual mechanism of transition to turbulence involves other parts of the spectrum, well in the stable left half-plane, associated with strong nonnormality, not shown in these pictures.

In fluid mechanics and in other applications, if an eigenvalue is in the right half-plane, this implies nothing about local behavior. The implications only concern the potential fate of certain trajectories if the system remains linear and unperturbed as $t \to \infty$. There are systems that behave like that, usually featuring matrices or operators that are symmetric or nearly so. But in plenty of other cases, certainly in high Reynolds number fluid mechanics, eigenvalue analysis has brought confusion. In areas like ecology and food webs, with all their complexities and time-dependencies, the idea of inferring anything precise from whether or not there are eigenvalues in the right half-plane is really very nebulous.

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