SIX MYTHS OF POLYNOMIAL INTERPOLATION AND QUADRATURE

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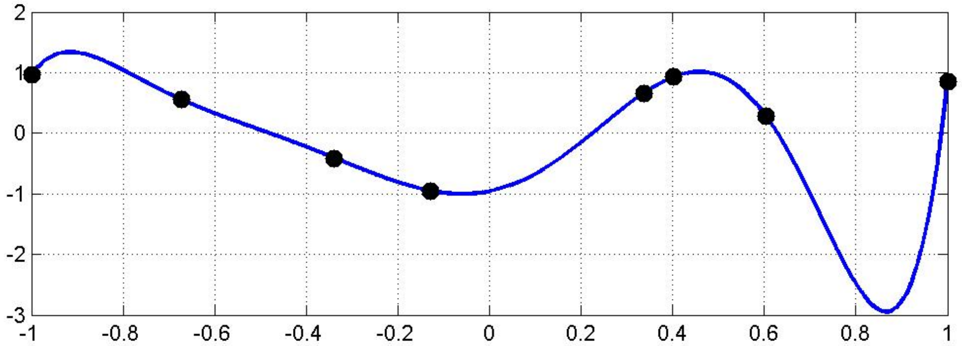
The setting

$n+1$ grid points in $[-1,1]$, and a continuous function f .

There is a unique degree n polynomial p interpolating f at these points.



Equispaced points vs. Chebyshev points, clustered near ± 1.

Other clustered grids like Legendre, Gauss-Jacobi, Gegenbauer have similar properties.

Standard quadrature formulas are derived by integrating the polynomial interpolant.

Equispaced pts: Newton-Cotes. Chebyshev pts: Clenshaw-Curtis. Legendre pts: Gauss.

MYTH 1. POLYNOMIAL INTERPOLANTS DIVERGE AS $n \to \infty$

Quotes

Dahlquist & Björck (1974), p 117: *"But there are many functions which are not at all suited for approximation by a single polynomial in the entire interval which is of interest."*

Kahaner, Moler & Nash (1989), p 94: "Polynomial interpolants rarely converge to a general continuous function.... **Polynomial interpolation is a bad idea.**"

Stewart (1996), p 153: "Unfortunately, there are functions for which interpolation at the Chebyshev points fails to converge. Moreover, better approximations of functions like $1/(1+x^2)$ can be obtained by other interpolants – e.g., cubic splines."

Kincaid & Cheney (2002), p 318: "The surprising state of affairs is that for most continuous functions, the quantity $\| f - p_n \|_\infty$ will not converge to 0."

Burden & Faires (2005), p 137: "The oscillatory nature of high degree polynomials, and the property that a fluctuation over a small portion of the interval can induce large fluctuations over the entire range, restricts their use."

MYTH 1. POLYNOMIAL INTERPOLANTS DIVERGE AS $n \to \infty$

The truth in it

Weierstrass (1885): Every $f \in C[-1,1]$ can be approximated by polynomials.

Runge (1901): In equispaced grids, polynomial interpolants often diverge even if f is analytic.

Faber (1914): No matter what the grids are, polynomial interpolants diverge for some f.

The flaw

If f is even Lipschitz continuous, convergence of interpolants on Chebyshev grids is guaranteed.

The smoother f is, the faster the convergence.

Theorems

For Chebyshev interpolants,

THM. If $f^{(\nu)}$ with $\nu \geq 1$ has bounded variation V, then $\| f - p_n \| = O(Vn^{-\nu})$.

THM. If f is analytic and bounded in the ρ-ellipse about $[-1,1]$, then $\| f - p_n \| = O(\rho^{-n})$.

Further context

The divergence for equispaced grids has been a great distraction.

Polynomial interpolation in Chebyshev points is equivalent to trigonometric interpolation of periodic functions in equispaced points, whose convergence nobody worries about.

Demonstration

Structure of this talk

For each myth:

Quotes

The truth in it

The flaw

Theorem(s)

Further context

Demonstration

Yogi Berra quotes

“It ain’t over till it’s over.”

“Half the lies they tell about me ain’t true.”

“Always go to other people funerals, or they won't come to yours.”

“A nickel ain't worth a dime anymore.”

“Sometimes you can see a lot just by looking.”

“When you come to a fork in the road, take it.” .

Literally false or empty or meaningless, yet suggesting something true.

Our mathematical quotes

Often literally true, yet suggesting something false.

“Polynomial interpolants rarely converge.”

“There are functions for which polynomial interpolation fails to converge.”

“For most continuous functions, polynomial interpolants diverge.” .

Most of our quotes, and all our myths, have some truth in them.

MYTH 2. EVALUATING POLYNOMIAL INTERPOLANTS NUMERICALLY IS PROBLEMATIC

Quotes

Forsythe, Malcolm & Moler (1977), p 68: “Polynomial interpolation has drawbacks in addition to those of global convergence. The determination and evaluation of interpolating polynomials of high degree can be too time-consuming for certain applications. Polynomials of high degree can also lead to difficult problems associated with roundoff error.”

Fröberg (1985), p 234: “Although Lagrangian interpolation is sometimes useful in theoretical investigations, it is rarely used in practical computations.”

Cormen, Leiserson & Rivest (1990), p 780: “Interpolation is a notoriously tricky problem from the point of view of numerical stability. Although the approaches described here are mathematically correct, small differences in the inputs or round-off errors... can cause large differences in the result.”

Stoer & Bulirsch (1993), p 39: “While theoretically important, Lagrange's formula is, in general, not as suitable for actual calculations as some other methods to be described below, particularly for large numbers n of support points.”

Kincaid & Cheney (2002), p 314: “For numerical work, it is probably best to use the Newton form of the interpolation polynomial.”

Parlett (2010), SIAM Review book review: “You do not want to meet a polynomial of degree 1000 on a dark night.”

MYTH 2. EVALUATING POLYNOMIAL INTERPOLANTS NUMERICALLY IS PROBLEMATIC

The truth in it

On an equispaced grid, the interpolation problem is exponentially ill-conditioned (Runge 1901).
Some well-known algorithms take $O(n^2)$ operations (e.g. naïve Lagrange formula)
or are exponentially unstable (e.g. Vandermonde matrices based on monomials x^k).

The flaw

On a Chebyshev grid, interpolation is well-conditioned.
The barycentric formula evaluates stably in $O(n)$ operations (M. Riesz 1916, Salzer 1972).

Theorem

THM (N. Higham 2004). Evaluating polynomial interpolants by the barycentric formula is stable.

Further context

The tangle of two issues has impeded understanding:
(1) Ill-conditioning of the problem (with equispaced grids), (2) Instability of the algorithm.
Cross-talk with a third issue has made matters worse: (3) Polynomial zerofinding (Myth 6).

Demonstration

MYTH 3. BEST APPROXIMATIONS ARE OPTIMAL

Quotes

Ralston (1965), p 272: "The major aim of a computer approximation to a function is to make the maximum error as small as possible."

Hart et al. (1968), p 46: "Since the Remes algorithm, or indeed any other algorithm for producing genuine best approximations, requires rather extensive computations, some interest attaches to other more convenient procedures... to give good, if not optimal, polynomial approximations."

Fike (1968), p 65: "Minimax polynomial approximations are clearly suitable for use in function evaluation routines, where it is advantageous to use as few terms as possible in an approximation."

Atkinson (1978), p 204: "Because of the difficulty in calculating the minimax approximation, we often go to an intermediate approximation called the *least squares approximation."*

Conte & de Boor (1980), p 235: "Ideally, we would want a **best uniform approximation from π_n**."

Powell (1981), p 7: "Our next theorem shows that, if we succeed in finding an approximation $a \in A$ such that the ∞-norm distance function $d(f,a)$ is small, then the 2-norm and 1-norm distance functions are small also."

MYTH 3. BEST APPROXIMATIONS ARE OPTIMAL

The truth in it

True by definition!

The flaw

Comparing true best approximations against Chebyshev interpolants, we find:

(1) The improvement in ∞-norm error is never more than $O(\log n)$.

(2) For most functions it is only $O(1)$.

(3) The price paid is global: the ∞-norm error is achieved globally; the 2-norm is often worse.

Theorems

EQUIOSCILLATION THM. The best approximation p^* achieves maximal error at $\geq n+2$ points.

THM (~Bernstein 1919). For Chebyshev interpolation, $\| f - p_n \| \leq (2+(2/\pi)\log(n+1)) \| f - p^* \|$.

THM (R.-C. Li 2004). For wide classes of functions, this bound sharpens to $\| f - p_n \| \leq 2 \| f - p^* \|$. .

Further context

The fact that best approxs are hard to compute has made it easy to suppose they are superior. .

Demonstration

MYTH 4. GAUSS QUADRATURE HAS TWICE THE ORDER OF ACCURACY OF CLENSHAW-CURTIS

Quotes

Ueberhuber (1997), p 102: "However, the degree of accuracy [for Clenshaw-Curtis quadrature] is only D = N − 1."

Trefethen (2000), p 130: "Gauss quadrature has genuine advantages over Clenshaw-Curtis."

Heath (2002), p 351: "[Clenshaw-Curtis rules are] not optimal in that the degree of an *n*-point rule is only *n* − 1, which is well below the maximum possible."

It's hard to find quotes because most books don't mention Clenshaw-Curtis at all.

MYTH 4. GAUSS QUADRATURE HAS TWICE THE ORDER OF ACCURACY OF CLENSHAW-CURTIS

The truth in it

(n+1)-point Gauss quadrature is exact for polynomials of degree 2n+1; for Clenshaw-Curtis it is only n.

The flaw

Though ($n+1$)-point Clenshaw-Curtis gets the integrals of the Chebyshev polys. T_{n+1}, T_{n+2}, … wrong, the errors are only $O(n^{-3})$. So Gauss and Clenshaw-Curtis often differ little in practice.

Theorem

THM (T. 2008). If $f^{(\nu)}$ with $\nu \geq 1$ has variation V, accuracy of C-C is $O(\, V(2n)^{-\nu}\,)$, same as for Gauss.

Further context

Clenshaw-Curtis is almost invisible in numerical analysis textbooks. One reason may be that Gauss quadrature had 150 years' head start (Gauss 1814, Clenshaw & Curtis 1960).

The impression that Gauss must be better than C-C has probably also been enhanced by the fact that Gauss nodes and weights are more difficult to compute.

Books often suggest that the advantage of Gauss over Newton-Cotes comes from its higher order of exactness. In fact, Newton-Cotes diverges exponentially because it uses equispaced points.

Demonstration

MYTH 5. GAUSS QUADRATURE IS OPTIMAL

Quotes

Isaacson & Keller (1966), p 327: "Gaussian quadrature; maximum degree of precision"

Johnson & Riess (1982), p 330: "The precision is maximized when the quadrature is Gaussian."

Kahaner, Moler & Nash (1989), p 146: "In fact, it can be shown that among all rules using *n* function evaluations, the *n*-point Gaussian rule is likely to produce the most accurate estimate, at least if the integrand is smooth."

Gautschi (1997), p 159: "This optimal formula is called the *Gaussian quadrature formula.*" .

MYTH 5. GAUSS QUADRATURE IS OPTIMAL

The truth in it

Gauss quadrature has the maximal polynomial order of accuracy.

The flaw

Polynomial order of accuracy is a skewed measure, because it's nonuniform across the interval.
In a uniform measure, other quadrature formulas may converge up to π/2 times faster.
E.G. transplanted Gauss quadrature based on ellipse→strip conformal map – Hale & T., SINUM 2008.

Theorem

THM. f analytic in ε-nbhd of $[-1,1]$ $\Rightarrow$ Gauss error $O(\,(1+\varepsilon)^{-2n}\,)$, above method $O(\,(1+\varepsilon)^{-3n}\,)$.

Further context

In practice, one doesn't beat Gauss by much; the main point is conceptual.
Polynomial order *is* the right measure for fixed n on an interval shrinking to a point; but not for $n \to \infty$. .

Demonstration

MYTH 6. POLYNOMIAL ROOTFINDING IS DANGEROUS

Quotes

Wilkinson (1963), p 78: ”Our main object in this chapter has been to focus attention on the severe inherent limitations of all numerical methods for finding the zeros of polynomials.”

Wilkinson (1984): “The perfidious polynomial”

Numerical Recipes in C (1992), p 348: “Beware: Some polynomials are ill-conditioned!”

Kress (1997), p 113: “The zeros of polynomials can be quite sensitive to small changes in the coefficients even if all the zeros are simple and well separated from each other.”

MYTH 6. POLYNOMIAL ROOTFINDING IS DANGEROUS

The truth in it

Zeros of some polynomials are highly ill-conditioned functions of the coefficients.
Also, some rootfinding algorithms (e.g., based on deflation) are unstable.

The flaw

The conditioning issue is mainly one of choice of basis, notably monomials vs. Chebyshev on $[-1,1]$.
Zeros near $[-1,1]$ are well-conditioned if you use the Chebyshev basis.
As for algorithms, there are stable ones, e.g. based on matrix eigenvalue computation.

Theorem

THM (Specht 1957, Good 1961).

{Roots of polynomials in Chebyshev form} = {eigenvalues of colleague matrix}.

Further context

Wilkinson's discovery of polynomial perfidy around 1950 was a life-changing experience for him,
and his influence was very great.

Demonstration

MYTH 7. LAGRANGE DISCOVERED LAGRANGE INTERPOLATION

It was Waring, 1779. (Euler 1783, Lagrange 1795.)

THE END

```
%% Defaults: linelinewidth 2, linemarkersize 18,   plot_numpts 6000,  font 24,  hold off

%% Myth 1: interpolants diverge
runge = @(x) 1./(1+25*x.^2);
p = chebfun(runge,10); plot(p,'.-')   % 100, 1000, 10000
f = chebfun('sin(x) + sin(x.^3)',[0,10])
plot(f), length(f)

%% Myth 2: computation is problematic
xx = linspace(-1,1,1000);
n = 10; x = chebpts(n); y = runge(x);
plot(polyval(polyfit(x,y,n-1),xx))    % 50, 100

%% Myth 3: Best approximations are optimal
x = chebfun('x');
f = abs(sin(3*exp(x)));
plot(f)
pbest = remez(f,100);
plot(f-pbest)
pinterp = chebfun(f,101);
hold on, plot(f-pinterp,'r'), ylim(.05*[-1,1]), grid on

%% Myth 4: Gauss has twice the order of Clenshaw-Curtis
cc_vs_gauss
1/(1+25*x^2)
abs(x)
abs(x)^7
exp(-1/x^2)
exp(-30*x^2)

%% Myth 5: Gauss quadrature is optimal
strip_vs_gauss
1/(1+25*x^2)
exp(-30*x^2)
1/(1.5-cos(5*x))

%% Polynomial rootfinding is dangerous
Back to original f
```