Lecture 1. Series solution of Laplace problems Nick Trefethen, 4 Sept. 2018

Ref: T, "Series solution of Laplace problems," *ANZIAM Journal*, 2018.

1. Series solutions

How would you solve a 2D Laplace problem?

- (Adaptive) Finite elements
- Fast Multipole Method
- Integral equations
- Conformal mapping
- Method of fundamental solutions = charge simulation method
- Series expansion: Expand in a finite series with coeffs *dependent on n* + Find coeffs via least-squares on boundary

2. Demonstrations

Remarkably simple codes suffice. See paper above and also my 2005 essay "Ten digit algorithms".
Also Chapman-Hewett-T, "Mathematics of the Faraday cage," *SIREV* 2015.

3. Mathematical foundations of approximation by nonconvergent series expansions

1885: Weierstrass approximation theorem (polynomials on an interval)
1885: Runge's theorem (rational functions on a domain with holes, poles in each hole)
1929: Walsh extends Runge from analytic to harmonic functions, with an additional log term in each hole
Theme of these results: geometric convergence if the boundaries and data are analytic. (If not, see #5.)

4. History and philosophy

Various factors have kept series methods from being well known.

1. Most people are far more familiar with convergent series (i.e., coeffs independent of n) — much too finicky in practice.
2. We don't even have a name! "Nonconvergent series" sounds like asymptotic series, which are different.
3. Mathematicians in this area got distracted by difficulties of square matrices and interpolation (Walsh, Curtiss,…).
4. Mathematicians more generally regard analyticity as a special case, preferring big regularity settings and associated methods.
5. Numerical analysts were drawn to series methods too early (50s, 60s), before matrix computations became easy (70s, 80s).
6. So paradoxically, there's a vast field of approximation theory, but it's rarely applied to the problems that engendered it.
7. Finite elements is such a universal tool that it has pushed other tools to the sidelines.
8. Conformal mapping exploits snazzier mathematics (but needlessly — the extra effort serves no purpose).
9. Focus on asymptotic complexity has made FMM and relatives more exciting ("fundamental principle of computer science"?)

A theme in several of the above is that series solutions are, well, too easy. Easy methods don't make for research buzz.

5. Singularities and redundant bases

In practice, singularities usually appear at special points such as corners. Add extra terms to the series accordingly.
Don't try to resolve singularities globally (this is why conf. mapping is bad). Let the linear algebra handle global connections.
Reentrant vs. salient corners. All are singular, except salient π/k angles may not be (proof: Schwarz reflection).
You're now working with bases that may be exponentially ill-conditioned. Often they work fine anyway (cf. theory of *frames*).

6. Beyond Laplace

The series method applies to elliptic problems more generally, e,g, Helmholtz and biharmonic equations.
For Helmholtz, this leads to the closely related *method of plane waves*. More generally one speaks of *Trefftz* methods.
Methods of *radial basis functions*, however, are different; the expansions don't normally satisfy the PDE in the interior.
For eigenvalue problems, put a Bessel series at each corner. See Fox-Henrici-Moler 1967 (a paper that came "too early").
3D? In principle, everything extends, though singularities are no longer confined to points. I doubt much has been done.

7. Eigenmodes of polygonal drums

[Refs: Betcke & T, Reviving the method of particular solutions, *SIREV* 2005; Betcke & T, Computed eigenmodes of planar regions, *Contemp. Math*. 2006]