Lecture 2. Nonnormality and pseudospectra

Nick Trefethen, 11 Sept. 2018

Ref: T and Embree, *Spectra and Pseudospectra: the Behavior of Nonnormal Matrices and Operators,* Princeton, 2005.
Software: EigTool (by Thomas Wright, maintained by Mark Embree), available at https://github.com/eigtool/eigtool.

1. Some history: from eigenvalue perturbations to pseudospectra (chaps. 1-6)

ϵ-pseudospectrum $\sigma_\epsilon(A)$
= set of eigenvalues of $A+E$, $\|E\| < \epsilon$
= set of complex numbers z s.t. $\|(zI-A)^{-1}\| > \epsilon^{-1}$ ← resolvent
= set of complex numbers z s.t. $s_{min}(zI-A) < \epsilon$ ← SVD

Near-symmetric/hermitian/normal matrix or operator:

- evals insensitive to perturbations, near-orthogonal evecs
- evals & evecs closely connected with behavior

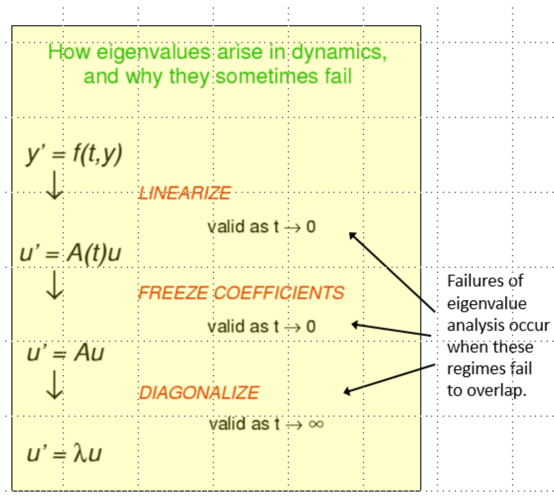
Highly nonnormal matrix or operator:

- evals sensitive to perts, evec basis ill-conditioned ← NOT the main point
- evals & evecs lose their significance ← The main point

Our intuition about eigenvalues has been misled by vibration problems and quantum mechanics, both normal.

2. Some mathematics: transient growth (chaps. 14-19)

Theorem. If $\|(zI-A)^{-1}\| = K/Re\, z$ for some $Re\, z > 0$, then $\sup_t \|e^{tA}\| > K$.



In words: if the resolvent norm is "surprisingly large" ($K > 1$) somewhere in the right half-plane, there must be transient growth.

Proof: Estimate Laplace transform integral $(zI-A)^{-1} = \int_0^\infty e^{-tz}e^{tA}\,dt$.
If the left side is large for some z, then e^{tA} must be large for some t.

Similarly in discrete time if $\|(zI-A)^{-1}\|$ is surprisingly large outside the unit disk.

The book gives more refined theorems, e.g. about time scales.

3. Some physics: hydrodynamic stability (chaps. 20-23)

Hydrodynamic stability theory says a flow is stable if all eigenvalues of the linearized problem are in the left half-plane.

By this theory, flow in a pipe is stable for all values of the Reynolds number (nondimensionalized speed).
Yet in practice, high Reynolds number flows are unstable/turbulent.

This paradox troubled fluid mechanics throughout the 20th century. Particularly troublesome has been *Squire's theorem*, which "proves" based on eigenvalues that the worst instabilities are 2D. See T, "Inverse Yogiisms", *Notices AMS*, 2016.

Pseudospectra point the way to the resolution. Through eigenvalues are in the left half-plane, the 3D pseudospectra go far into the right half-plane, implying great transient amplification of 3D disturbances. Then the nonlinearities kick in.

If you see people analysing an Orr-Sommerfeld problem — inevitably 2D — they are probably missing the real physics.

4. Some more mathematics: wave packet pseudomodes (chaps. 7-13)

Nonsymm. Toeplitz matrices ⟷ non-selfadjoint diff'l operators. Exponentially nonnormal; pseudo-evecs = decaying waves at bndry.

"Twisted" Toeplitz matrices ⟷ diff'l operators with variable coeffs. Pseudo-evecs = wave packets in the interior.

Analysis involves the *symbol*, dependent on wave number and position. Cf. WKBJ ≈ semiclassical analysis ≈ microlocal analysis.

5. Some wisdom? How I view pseudospectra today

Anyone who plots eigenvalues of nonsymmetric/nonhermitian matrices operators in the complex plane should routinely include pseudospectra. This gives an instant check of whether the eigenvalues are likely to be meaningful. Some people do this. Most don't.

In fluid mechanics, reliance on eigenvalues for high Reynolds no. flow is fading, with analysis of transient dynamics now ubiquitous.

The lesson that eigenvalues lose their significance in cases of extreme nonnormality is not yet widely appreciated.
For example, investigation of *structured eigenvalue perturbations* lacks a good justification, but it has become a big field.