Lecture 3. Chebyshev series Nick Trefethen, 20 Sept. 2017

General reference: T, *"ATAP" = Approximation Theory and Approximation Practice*, SIAM, 2013.

1. Fourier, Laurent, and Chebyshev series (*ATAP* chap 3)

The following are equivalent via $z = e^{i\theta}$, $x = ½(z+z^{-1}) = \cos(\theta)$:

- Fourier series for a periodic function $\mathbb{F}(\theta)$ with $\mathbb{F}(-\theta) = \mathbb{F}(\theta)$, $\theta \in [-\pi,\pi]$,
- Laurent series for a function $F(z)$ with $F(z^{-1}) = F(z)$, $z \in$ unit circle ,
- Chebyshev series for a function $f(x)$, $x \in [-1,1]$.

The Chebyshev series are based on Chebyshev polynomials $\mathrm{T}_k(x) = ½(z+z^{-k}) = \cos(k\cos^{-1}(x))$.

Lipschitz continuity is enough to ensure absolute and uniform convergence of the series.

Series in x^k as opposed to $\mathrm{T}_k(x)$ should never be used for numerical computation on an interval: monomial bases are ill-conditioned, with condition numbers growing exponentially with k .

2. Smoothness and convergence theorems (*ATAP* chaps. 7 & 8)

Central dogma of approximation theory:

smoothness of a function f $\leftrightarrow$ rate of convergence of approximants as $n \to \infty$

The following results apply both to truncated Chebyshev series and to interpolation in Chebyshev points.

Theorem (proof by integration by parts $\nu+1$ times):

f has a νth derivative in BV $\Rightarrow$ Cheb. coeffs. $O(k^{-\nu-1})$, approx. errors $O(n^{-\nu})$.

Theorem (proof by Cauchy integrals in x or equivalently z):

f is analytic in the closed Bernstein ρ-ellipse $\Rightarrow$ Cheb. coeffs. $O(\rho^{-k-1})$, approx. errors $O(\rho^{-k})$.

3. Faber's theorem and inverse Yogiisms

See T, "Inverse Yogiisms," *Notices of the AMS,* Dec.2016.

Yogi Berra: statements that are literally tautological or nonsensical, yet convey something true.
The inverse, common in mathematics: statements that are literally true, yet convey something false.

Faber's theorem (1914): *for any system of polynomial interpolation points, the interpolants diverge as $n \to \infty$ for some continuous function f.*

This result has been one reason experts have warned against polynomial interpolation for generations.
Why it is misleading: the result does not apply if f is even slightly smooth, e.g. Lipschitz continuous.
(An analogous theorem holds for trigonometric interpolation, but nobody warns against Fourier series.)

4. Chopping a Chebyshev series

See Aurentz & T, "Chopping a Chebyshev series", *Transactions on Mathematical Software,* 2016.

An engineering problem at the heart of Chebfun-style computing.
An analogue for functions of the (much simpler) problem of rounding real numbers to floating point.