34. Advection–diffusion equations ——— *asymmetry and boundary layers*

In fluid mechanics, meteorology, semiconductor device modelling, ecology, oil reservoir simulation, finance, and other fields, partial differential equations arise which combine advection and diffusion ($\rightarrow$ *refs*). The prototypical linear advection–diffusion equation takes the form

$$\boxed{u_t + \mathbf{a}\cdot\nabla u = \varepsilon\,\Delta u.} \tag{1}$$

for $u = u(\mathbf{x})$, $\mathbf{x}\in\mathbb{R}^n$. For example, the n-vector function $\mathbf{a}(\mathbf{x},t)$ might describe the velocity of a fluid, and $u(\mathbf{x},t)$ might represent the temperature of the fluid or the concentration of a pollutant. For $\mathbf{a}=\mathbf{0}$, (1) reduces to the heat equation ($\rightarrow$ *ref*), while in the other extreme case, by setting $\varepsilon = 0$, we arrive at the first-order linear transport equation $u_t + \mathbf{a}\cdot\nabla u = 0$.

Suppose for simplicity that $\mathbf{a}$ is independent of $\mathbf{x}$ and t and that an initial condition $u(\mathbf{x},0)=u_0(\mathbf{x})$ is prescribed. A brief calculation based on the Fourier transform leads to the solution

$$u(\mathbf{x},t) = \frac{1}{(4\pi t\varepsilon)^{n/2}}\int_{\mathbb{R}^n} \mathrm{e}^{|(\mathbf{x}-t\,\mathbf{a})-\mathbf{y}|^2/(4t\varepsilon)}u_0(\mathbf{y})\,\mathrm{d}\mathbf{y},$$

or equivalently,

$$u(\mathbf{x}+t\mathbf{a},t) = \frac{1}{(4\pi t\varepsilon)^{n/2}}\int_{\mathbb{R}^n} \mathrm{e}^{|\mathbf{x}-\mathbf{y}|^2/(4t\varepsilon)}u_0(\mathbf{y})\,\mathrm{d}\mathbf{y}. \tag{2}$$

This last formula is nothing more than the solution to the heat equation shifted from $\mathbf{x}$ to $\mathbf{x}+t\mathbf{a}$. In other words, instead of having advection and diffusion take place simultaneously over a given finite time interval, we could get the same answer by applying one of these physical processes over the given interval and then the other. (We could also have derived this conclusion without the Fourier transform.) This principle of *splitting* applies generally to constant-coefficient linear PDEs on unbounded domains ($\rightarrow$ *ref*). It also applies approximately, for small time intervals, when the coefficients are variable or nonlinearities are present, a fact of great importance for numerical methods.

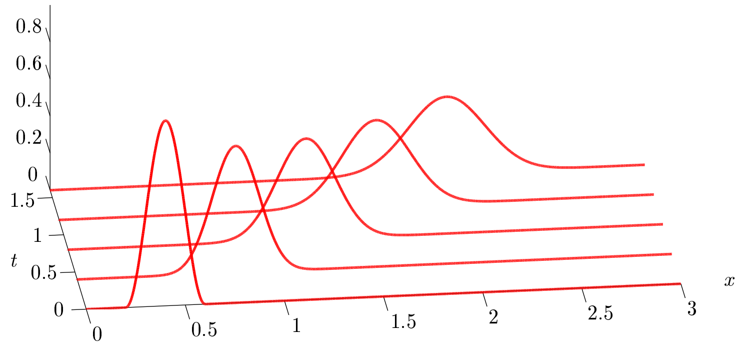

Fig. 1: Solution with $a(x,t)=1$, $\varepsilon = 10^{-2}$

For example, Figure 1 shows the solution to (1) in one dimension with $a(x,t)=1$, $\varepsilon=10^{-2}$, and $u_0(x)=\sin(\pi(x-0.6)/0.4)$ for $x\in[0.2,0.4]$ and $u_0(x)=0$ otherwise. Note that the translation and the diffusion are independent.

In the absence of boundary conditions, the evolution of an initial function u_0 is thus quite simple. However, when (1) is considered in a bounded domain $\Omega\subset\mathbb{R}^n$, conflicts may arise between the prescribed boundary condition and the structure of the free-space solution (2). Such conflicts are resolved by the appearance of *boundary layers* and/or *internal layers*. Since Ludwig Prandtl introduced the term in 1904, it has been appreciated that boundary layers are one of the fundamental phenomena of fluid mechanics, appearing wherever viscous fluids move near solid walls, and perhaps the most important application of advection-diffusion equations is to the elucidation of these boundary effects in fluids. They are also used throughout applied mathematics and numerical analysis as examples of non-hermitian processes, for which the most effective techniques usually depend on the exploitation of asymmetry.

To illustrate boundary and internal layers, consider the case in which $\mathbf{a}$ is constant. By the method of matched asymptotic expansions, it can be shown that under typical circumstances the thickness of a boundary layer is $\mathcal{O}(\varepsilon)$ as $\varepsilon\to 0$ while the thickness of an internal layer is $\mathcal{O}(\sqrt{\varepsilon})$. Figure 2 shows these different layers thicknesses for a problem with $\mathbf{a}=(2,1)$. In the second plot, the boundary layer is still present on the right, but too narrow to be visible on this scale.

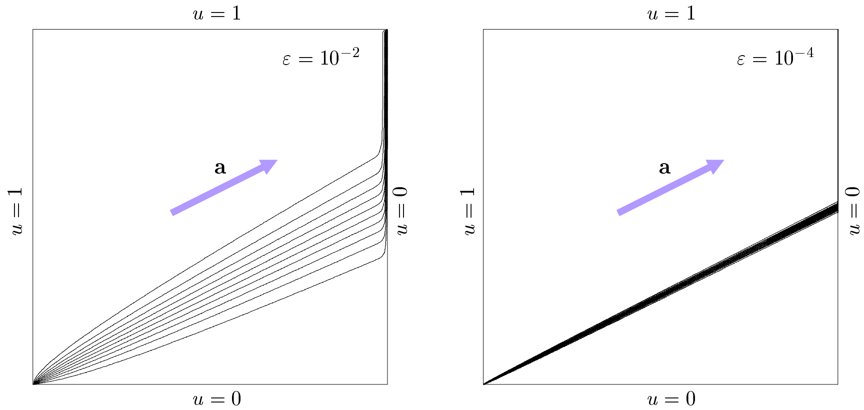

Fig. 2: Internal and boundary layers of thickness $O(\sqrt{\varepsilon})$, $O(\varepsilon)$

References

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