The Fisher equation ($\rightarrow$ *ref*), with reaction term $f(u) = u(1-u)$, admits a stable equilibrium state $u = 1$ and an unstable equilibrium state $u = 0$. Regions with $u \approx 1$ grow at the expense of regions with $u \approx 0$, and the result is a traveling wave.

The *Allen–Cahn* or *bistable equation* (also known as the the *Chafee–Infante equation*) is the analogous reaction-diffusion PDE with $f(u) = u(1-u)(1+u)$. In one space dimension,

$$\boxed{u_t = u_{xx} + u - u^3.} \tag{1}$$

Now there are three equilibria: $u = 1$, $u = -1$, and $u = 0$. By checking the sign of $u - u^3$ in various intervals we see quickly that $u = 0$ is an unstable equilibrium, whereas $u = 1$ and $u = -1$ are stable. This introduces fundamentally new behaviour. For solutions with $u \geq 0$, the state $u = -1$ is not relevant and there will be traveling waves as before, but if we allow u to approach both values 1 and -1, then we get a rather interesting competition between equal and opposite stable states.

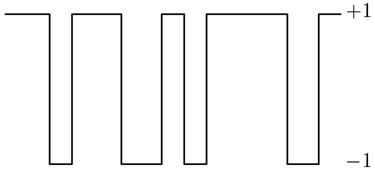

Fig. 1: Metastable fronts (schematic)

Suppose, for example, that initial data are given as in Figure 1, with periodic or Neumann boundary conditions. If the space scale is small, the interfaces will quickly diffuse away, and, typically, the solution will converge to 1 or -1 everywhere. If the space scale is large, however, the interfaces may remain approximately unchanged for a very long time. In fact, features of width L persist for time scales on the order of e^L, a figure that becomes $e^{L/\varepsilon}$ if u_{xx} is replaced in (1) by $\varepsilon^2 u_{xx}$. Only over these exponentially long times do the interfaces between regions slowly move. This is the phenomenon known as *metastability*.

By symmetry, there must be special cases of initial conditions for which the wave fronts move neither left nor right. In particular, there exist periodic waveforms that are steady states of (1) for all t, which can be studied by phase plane analysis as described for the Fisher equation. However, these solutions are unstable (over exponentially long time scales) to small perturbations.

Figure 2 shows an example. At $t = 0$ we have two wells with $u \approx -1$. The first is absorbed at $t \approx 500$, but for the second, which is slightly wider, the absorption time is delayed past $t = 1200$. Doubling the width of the well would increase the time into the millions. Careful analyses of effects like these have been published based on geometrical ideas such as fast and slow manifolds and energy surfaces that are exponentially close to flat at locations far from the global minimum.

In physical applications with small diffusion constants, the slow time scale for the motion of patterns can have significant consequences. Allen and Cahn investigated (1) in connection with the motion of boundaries between phases in alloys. In this and other problems of physics, chemistry, and biology, the Allen–Cahn equation serves as a warning that states that appear to be steady may be, mathematically speaking, merely very long transients. Conversely, the mathematical steady state of a partial differential equation may not always represent what is physically important.

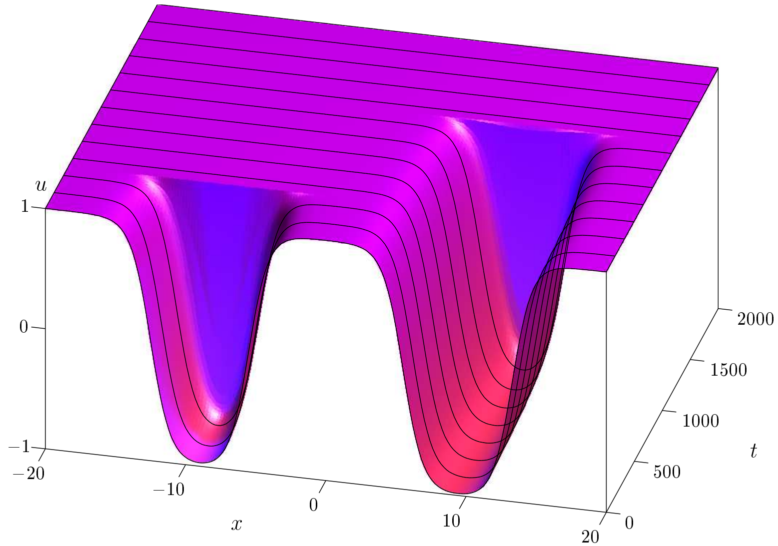

Fig. 2: Computed example

In multiple space dimensions, the Allen–Cahn equation becomes

$$u_t = \Delta u + u - u^3. \tag{2}$$

The same stable and unstable equilibria are present as before, but further dynamical possibilities arise because the fronts separating them may be curved. A front moves locally at a speed proportional to its mean curvature, generating complex patterns ($\rightarrow$ *ref*). Metastability of the 1D problem corresponds to the special case of a multidimensional problem with zero front curvature, so that the velocity, to leading order in an asymptotic expansion (in fact to all orders), is zero.

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