The equation $u_t = \nabla^2 u + \lambda e^u$ arises in mathematical models of heat transfer in reacting media: in combustion theory and chemical reactor theory, for example. It also arises in the study of fluid flows with strongly temperature-dependent viscosity. The variable u is the temperature, and the diffusion term $\nabla^2 u$ is counteracted by the exponential source term λe^u, which represents an approximation, named after Frank–Kamenetskii, to the Aarhenius function $\exp(-E/RT)$. The number $\lambda > 0$ is a fixed parameter.

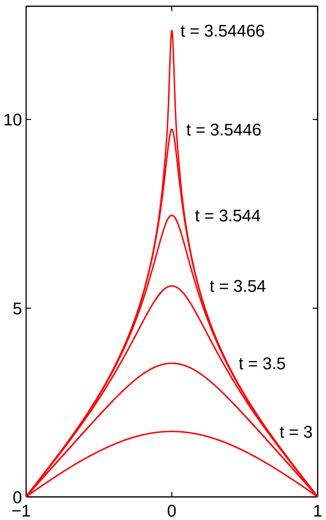


Fig. 1: blowup at $t \approx 3.54466$ with $\lambda = 1$ and zero initial data.

The positive feedback introduced by the e^u term allows for an explosive increase of u. We can estimate the rate of the explosion by noting that on an infinite interval, the equation has the x-independent solution $u = -\ln[\lambda(t^* - t)]$ for any number t^*. As $t \to t^*$, u increases to infinity. When diffusion is included, the same phenomenon occurs, except that in general the blow-up occurs at a single point. To be specific let us consider the 1D equation

$$\boxed{u_t = u_{xx} + \lambda e^u} \tag{1}$$

for $x \in [-1, 1]$ with boundary conditions $u(\pm 1) = 0$ and initial data zero. Figure 1 shows the blowup process for $\lambda = 1$ as $t \to t^* \approx 3.54466$. The logarithmic nature of the blow-up can be seen in the fact that each time we move forward one more digit towards t^*, the height of the curve increases by about a fixed amount. Note also that the curves lie below an envelope whose only infinite point is at $x = 0$. Thus the solution remains bounded at each fixed point $x \neq 0$, in contrast to the form of some blowups in the blow-up equation with u^p nonlinearity ($\to$ *ref*).

The phenomenon of blow-up in finite time, also called *thermal runaway*, is associated with the non-existence of solutions of the steady problem $u_{xx} + \lambda e^u = 0$ for values of λ greater than a critical value λ_c. For the 1D problem (1), $\lambda_c \approx 0.878$, and Figure 2 shows results for various λ above and below this critical value. Similar behaviour occurs in the symmetrically equivalent problems in two and three dimensions, with $\lambda_c = 2$ and $\lambda_c \approx 3.322$, respectively. Below these critical values, multiple solutions of the steady problem exist, two in 1D and 2D and infinitely many in 3D. In the simple 1D case, the upper (warm) branch is unstable, and a sufficiently large initial condition will cause runaway here too.

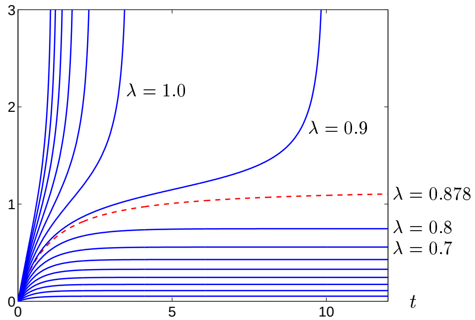


Fig. 2: $\max_x u(x,t)$ as a function of t for $\lambda = 0.1, 0.2, \ldots, 1.4, 1.5$

Dold has shown that the approach to the singularity has a universal but delicate character. For a problem with blow-up at $x = 0$ and $t = t^*$, define new time and space variables by

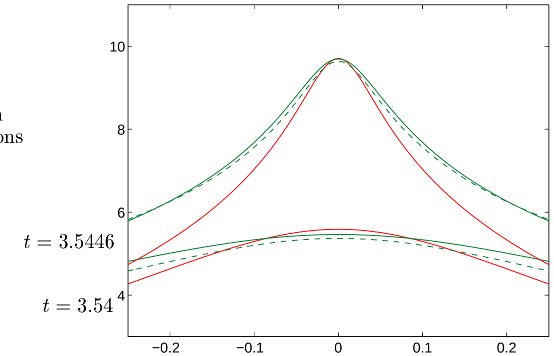
$$\tau = -\ln(t^* - t), \qquad \xi = x \Big/ \sqrt{4\tau(t^* - t)}.$$

Then it can be shown that $u(x,t)$ has an asymptotic expansion as $t \to t^*$ that begins

$$u(x,t) \;\sim\; \tau - \ln\lambda - \ln(1+\xi^2) - \frac{5}{2}\frac{\ln\tau}{\tau}\frac{\xi^2}{1+\xi^2} + \frac{1}{\tau}\left[\frac{1}{2} + \frac{\xi^2}{1+\xi^2}\left\{\alpha - \ln(1+\xi^2)\right\}\right] + \cdots$$

for some constant α. Figure 3 compares exact results with asymptotic approximations for the curves $t = 3.54$ and 3.5446 of Figure 1. The dashed green line represents the approximation through the term $-\ln(1+\xi^2)$, and the solid green line includes the next term involving $\ln\tau/\tau$ too.

Fig. 3: comparison with asymptotic approximations



The time-independent version of (1), in which the left-hand side is replaced by 0, is known as the *Liouville equation* ($\to$ *ref*).

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