29. Boussinesq Equation

can an ill-posed equation be physical?

Russell's discovery of solitary waves in the 1840s ($\rightarrow$ *ref*) raised a challenge: to devise a mathematical theory for water waves that would admit a wave solution that did not disperse with time. In the 1870s, the French hydrodynamicist Joseph Boussinesq proposed what he believed was a suitable model (Fig. 1). By assuming that the wave amplitude was small compared to the canal depth, he arrived at the *Boussinesq equation*,

$$\boxed{u_{tt} - u_{xx} = u_{xxxx} + (u^2)_{xx},} \qquad (1)$$

posed for $x \in \mathbb{R}$ and $t \geq 0$ with initial conditions for $u(x,0)$ and $u'(x,0)$. This equation has something of the flavour of a "square of the KdV equation" (derived two decades later) in that it is symmetric with respect to x and $-x$. It consists of the 1D wave equation ($\rightarrow$ *ref*) plus two dispersive terms, one linear and one nonlinear.

The Boussinesq model remained of interest mainly to hydrodynamicists until the discovery of solitons in the 1970s, which sparked worldwide interest in nonlinear wave equations. Then it was found that, like the KdV equation, (1) admits soliton solutions—locally supported waves which interact with one another with a phase shift but no change in shape. The solitons travel in both directions, allowing for head-on collisions. Hirota developed analytic expressions for solitons; a typical interaction is shown in Figure 2. Other researchers devised inverse scattering techniques. As with the Sine–Gordon equation ($\rightarrow$ *ref*), one can generate a series of solutions to the Boussinesq equation from a single given solution using the Bäcklund transform.

These remarkable developments could be carried out despite a basic fact about (1): it is explosively ill-posed. This can be seen by considering low amplitude solutions (so that the nonlinear term is negligible) of the form $u(x,t) = \varepsilon \mathrm{e}^{\mathrm{i}(kx+\omega t)}$. The dispersion relation is $\omega^2 = k^2 - k^4$, so for $|k| > 1$ we have an imaginary frequency ω and exponential growth in t at a rate approximately $\mathrm{e}^{k^2 t}$, which is unbounded since k is unbounded. Because of this behaviour, results like that of Figure 2 are rather theoretical. The slightest perturbation of the initial data would change the behaviour utterly—and indeed, this figure was not generated by a general-purpose discretisation of the PDE applicable to arbitrary initial conditions, for such a computation would be impossible.

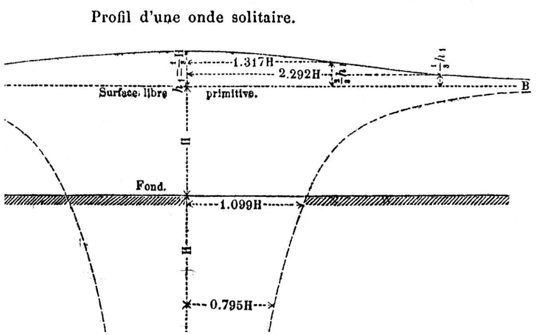

Fig. 1: Illustration from Boussinesq (1872)

The literature on the Boussinesq equation, especially in earlier years, is not very clear on the matter of its ill-posedness. (Some authors tried to cope with its difficult behaviour numerically, as if ill-posedness of a PDE could be neutralised by sufficiently clever discretisation.) Gradually, however, it became clear that perhaps attention should be directed to other similar equations that are well-posed. One is the "improved Boussinesq" equation,

$$u_{tt} - u_{xx} = u_{xxtt} + (u^2)_{xx}, \qquad (2)$$

where the x-derivative on the fourth order term has been replaced by a mixed derivative. This gives rise to the dispersion relation $\omega^2 = k^2/(1+k^2)$. Dropping the non-linear term, we get the "linear Boussinesq" equation, $u_{tt} - u_{xx} = u_{xxtt}$. Equation (1) can also be altered by flipping the sign on the fourth order derivative, yielding the "good Boussinesq" equation,

$$u_{tt} - u_{xx} = (u^2)_{xx} - u_{xxxx}. \qquad (3)$$

Now the dispersion relation is $\omega^2 = k^2 + k^4$, and again we have a well-posed problem. Both the good and improved Boussinesq equations exhibit many attractive properties of the classical Boussinesq model, including solitons and analytic solutions via the inverse scattering transform.

Research on the Boussinesq equation and its variants flourishes, with interesting questions concerning existence and uniqueness of solutions, blow-up in finite time, multidimensional analogues, and unusual *breather* solitons, more familiarly associated with the Sine–Gordon equation ($\rightarrow$ *ref*).

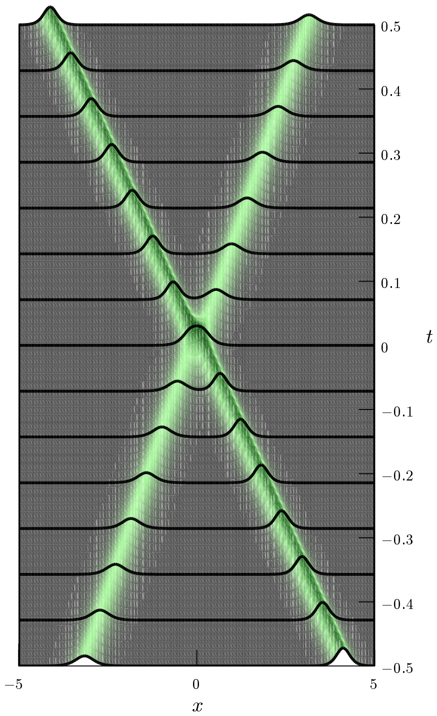

Fig. 2: soliton interactions

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