51. Inviscid Burgers equation

A PDE of the form $u_t + (f(u))_x = 0$ is called a *conservation law*, with u representing the *density* of some quantity and $f(u)$ the associated rightward *flux*. Conservation laws arise in fluid dynamics and many other fields. By integrating in x, we see that for any a and b, the integral of u over $[a, b]$ changes only because of fluxes through the endpoints:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b u(x)\,\mathrm{d}x \;=\; f(u(a,t)) - f(u(b,t)). \tag{1}$$

The simplest nonlinear example of a conservation law is the *inviscid Burgers equation*,

$$\boxed{u_t + (\tfrac{1}{2}u^2)_x \;=\; 0,} \tag{2}$$

i.e., $u_t + uu_x = 0$. This equation appears in studies of gas dynamics and traffic flow, and it serves as a prototype for nonlinear hyperbolic equations and conservation laws in general. It is the inviscid limit of the *Burgers equation* ($\rightarrow$ *ref*)

$$u_t + (\tfrac{1}{2}u^2)_x \;=\; \epsilon u_{xx}, \tag{3}$$

where $\epsilon > 0$ is a constant. Equations (2) and (3) were perhaps first considered by Bateman in 1915 and they were studied extensively by Burgers, Hopf, Cole, and others beginning in 1948.

A crucial phenomenon that arises with the Burgers equation and other conservation laws is the formation of *shocks*, which are discontinuities that may appear after a certain finite time and then propagate in a regular manner. Figure 1 shows an example.

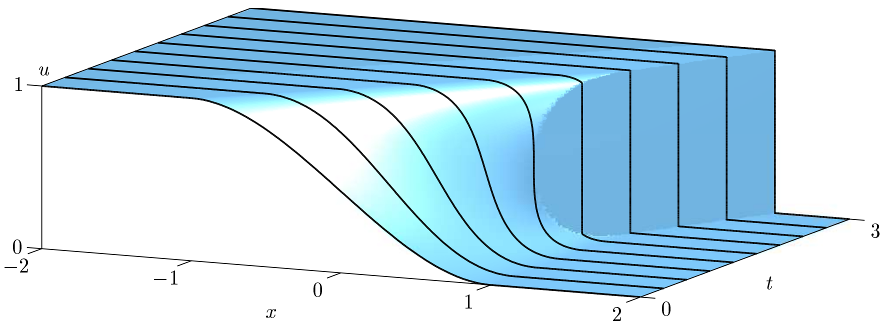

Fig. 1: Formation of a shock

Figure 1 is not as straightforward as it looks. It suggests that a shock simply forms and propagates, and that is all there is to it. But (2) is a PDE, defined by derivatives that do not exist for discontinuous functions. In what sense do these discontinuous curves satisfy the PDE?

One answer can be based on the idea of *vanishing viscosity*. For any $\epsilon > 0$, a unique solution of (3) exists for all time, and it is smooth. The curves of Figure 1 are what one obtains by taking the limit $\epsilon \to 0$. This simple idea is the right one physically in many applications.

Alternatively, we may define *weak* or *generalised solutions* by working from the conservation principle (1) rather than the PDE. If $u(x,t)$ is a smooth solution of (2), then for any rectangle R in the x-t plane, we have $\iint_R [u_t + (\frac{1}{2}u^2)_x]\,\varphi\,\mathrm{d}x\mathrm{d}t = 0$ for any smooth function $\varphi = \varphi(x,t)$, and if in addition φ vanishes on the boundary of R, then integrating by parts gives

$$\iint_R \left[u\,\varphi_t \;+\; (\tfrac{1}{2}u^2)\,\varphi_x\right]\mathrm{d}x\mathrm{d}t \;=\; 0. \tag{4}$$

This equation makes sense regardless of whether u is smooth, and a weak solution of (2) is defined as a function $u(x,t)$, not necessarily continuous, that satisfies (4) for all R and corresponding φ.

From (4) or (1), one can readily derive the velocity s of a shock that separates states u_L and u_R on the left and right of a discontinuity. The result is the *Rankine-Hugoniot* formula

$$s \;=\; \frac{f(u_R) - f(u_L)}{u_R - u_L}, \tag{5}$$

hence for (2), $s = \frac{1}{2}(u_L + u_R)$. Thus in Figure 1, the shock has velocity exactly 1/2.

It may seem that solutions to (4) should be unique. However, this is not so, and we can see why by solving (2) via its *characteristics*, which are the lines $x = x_0 + u(x_0,0)t$, with constant value $u(x,t) = u(x_0,0)$. Figure 2 shows a "backwards shock", a discontinuity at $x = t/2$ separating states $u_L = 0$ and $u_R = 1$. This function $u(x,t)$ satisfies (4) and (5), but it is not the solution obtained by the method of vanishing viscosity. To get that solution, a *rarefaction wave*, one must impose the additional condition that shocks are permitted only if they satisfy $f'(u_L) > s > f'(u_R)$, or for (2), $u_L > s > u_R$. This is called an *entropy condition*, for it is related to the condition that fluid passing through a shock must increase in entropy.

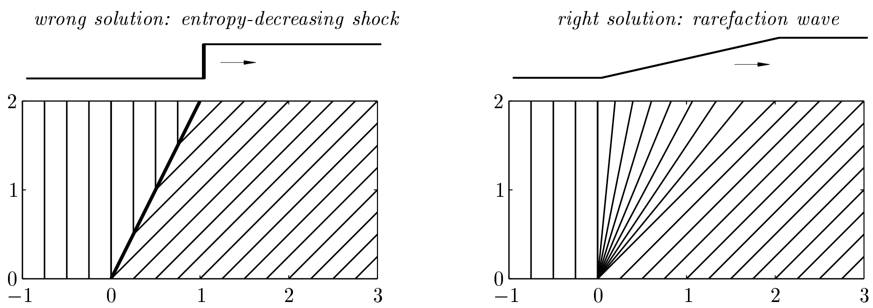

Fig. 2: nonunique weak solutions of (2)

References

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