37. Fisher-KPP equation — *nonlinear traveling waves*

A reaction–diffusion equation looks like the heat equation (→ *ref*) with a function $f(u)$ added on,

$$u_t = \Delta u + f(u).$$

Such equations appear in the sciences as models of diverse physical, chemical and biological phenomena. Since f may be non-linear, explicit solutions cannot usually be found. Whereas the linear wave equation (→ *ref*) propagates arbitrary solutions at a fixed speed, reaction-diffusion equations may single out certain wave forms and allow only these to propagate without distortion. A typical problem for an equation of this kind investigates the existence, form, and stability of these *traveling waves*. Such a solution can be written as $u(x,t) = U(z)$, with $z = x - ct$, where c is the wave speed. The existence of traveling waves in chemical reactions was first observed and studied by Luther at the beginning of the twentieth century.

The *Fisher–KPP equation* is one of the simplest examples of a nonlinear reaction-diffusion equation. The equation dates to two independent publications in 1937. Kolmogorov, Petrovsky and Piscounov began with an equation in 2D with a general reaction term, and their work laid the foundation for rigorous analytical study of reaction-diffusion models. Fisher's equation was one-dimensional and had a specific "logistic" reaction term:

$$u_t = D u_{xx} + ru\left(1 - \frac{u}{K}\right).$$

Fisher proposed this equation as a model of diffusion of a species in a 1D habitat; D is the diffusion constant, r is the growth rate of the species, and K is the carrying capacity. A dimensionless version of the equation takes the form

$$\boxed{u_t = u_{xx} + u(1-u).} \qquad (1)$$

Since 1937, (1) has also been used to study flame propagation and nuclear reactors.

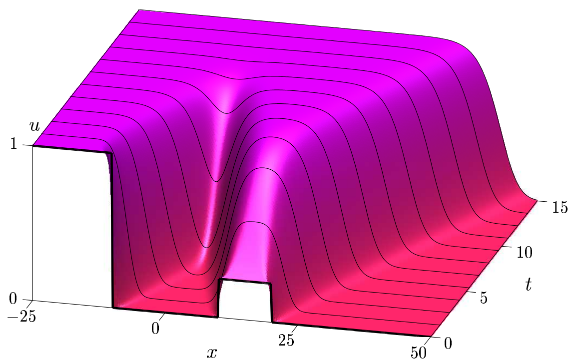

Fig. 1: Formation of traveling wave

The solution to (1) depends on the initial data, $u_0(x)$. Kolmogorov et al. proved that if $u_0(x)$ is monotonic and continuous with $u_0(x) = 1$ for $x < a$ and $u_0(x) = 0$ for $x > b$, where $-\infty < a < b < \infty$, then the solution evolves into a traveling wave with speed $c = 2$. Figure 1 shows the evolution of the initial function $u_0(x) = 1$ for $x < -10$, $1/4$ for $10 < x < 20$ and 0 otherwise. The initial well soon vanishes and a monotonic traveling wave with $c = 2$ remains.

We can begin to understand the behaviour of the Fisher-KPP equation by noting that since the right-hand side of (1) is zero for $u = 0$ or 1, two solutions are obvious: $u(x,t) = 0$ or $u(x,t) = 1$ for all x, t. More generally, it is clear that for constant initial data $u_0(x) = C$, the wave will remain flat for all t, with $u(x,t) = C(t)$ for some function $C(t)$. Moreover, the signs of the term $u - u^2$ are such that positive solutions will be repelled from 0 and attracted to 1. Now the real interest in (1) lies in the behaviour when both $u \approx 0$ and $u \approx 1$ are present: how does the wave get from one of these values to the other, and how does the wave front connecting the two move with time? The general answer is that regions with $u \approx 1$ tend to grow, eating up adjacent regions with $u \approx 0$—and the waves may travel in either direction.

These ideas can be made more precise with the aid of phase plane analysis. If we look for a solution $u(x,t) = U(z) = U(x - ct)$ and set $V = U'$, (1) reduces to the system of ODEs

$$U' = V, \qquad V' = -cV - U(1-U).$$

Without loss of generality let us consider just $c > 0$, i.e., right-going waves. The critical points in the U, V plane are $(1,0)$, a saddle point, and $(0,0)$, a stable node for $c \geq 2$ and a spiral for $c < 2$.

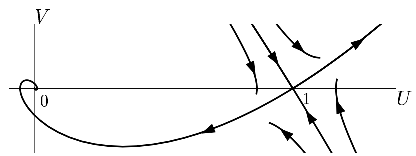

Fig. 2: Phase portrait with $c = 1$

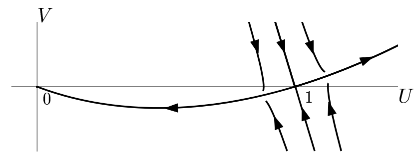

Fig. 3: Phase portrait with $c = 3$

From phase portraits such as those of Figures 2 and 3, we can derive various properties of solutions of the Fisher-KPP equation, which can be proved rigorously by a more detailed analysis. For any $c > 0$, there exists a unique right-going traveling wave with speed c connecting the state $u = 1$, $u_x = 0$ for $x \to -\infty$ to the state $u = 0$, $u_x = 0$ for $x \to \infty$. Thicker fronts correspond to faster waves. For $c \geq 2$, the wave is a monotonically decreasing function of x, while for $c < 2$ it is oscillatory; in some applications the latter may be non-physical. McKean showed by probabilistic methods that under appropriate assumptions, these traveling waves are stable with respect to small perturbations.

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