The Helmholtz equation, or reduced wave equation, has the form

$$\boxed{\Delta u + k^2 u = 0.} \tag{1}$$

It takes its name from the German physicist Hermann von Helmholtz (1821–1894), a pioneer in acoustics, electromagnetism and physiology. The equation arises naturally when one is looking for mono-frequency or *time-harmonic* solutions to the wave equation (→ *ref*). If

$$u(x,t) = v(x)\,\mathrm{e}^{-\mathrm{i}\omega t}$$

satisfies $u_{tt} = c^2 \Delta u$, then v satisfies (1) with $k = \omega/c$, hence the name *reduced wave equation*. Time-harmonic waves are of fundamental importance in applications as diverse as noise scattering, radar and sonar technology and seismology.

The quantity k is the *wave number*. It is often real and constant, but it can be complex if the medium of propagation is energy absorbing, or a function of space if the medium is inhomogeneous. At low wave numbers, (1) behaves very much like the Laplace equation (→ *ref*). However, solutions at large wave number are highly oscillatory, and this causes a great increase in complexity of analytical and numerical methods. Special approximation theories (e.g. Kirchhoff theory, geometrical optics) have been developed for such cases that yield good results at sufficiently high wave numbers.

Many applications of (1) involve unbounded domains. As an example, consider scattering from a bounded obstacle $D \subset \mathbb{R}^d$. Given an incident field u^i that is a solution to (1) in $\mathbb{R}^d$, the problem is to find the scattered field u^s such that the total field $u^t = u^i + u^s$ satisfies (1) in $\mathbb{R}^d \setminus \overline{D}$ as well as a boundary condition on ∂D. Most often one of the boundary conditions

$$u^t = 0 \quad \text{or} \quad \frac{\partial u^t}{\partial n} = 0 \tag{2}$$

on ∂D, i.e. either Dirichlet or Neumann boundary conditions, is appropriate (n denotes the outward normal on ∂D). However, this boundary value problem is not yet well-posed. There is no control of energy propagating inward from infinity, and thus solutions are non-unique. To make the problem well-posed we impose the *Sommerfield radiation condition*, introduced by Arnold Sommerfeld in 1896,

$$\frac{\partial u^s}{\partial r} - \mathrm{i}k\,u^s = O(r^{-(1+d)/2}) \text{ as } r \to \infty. \tag{3}$$

This ensures that the scattered wave is *outgoing*, that is, propagates away from the obstacle. Figure 1 shows an example of this kind with a Dirichlet boundary condition and a plane wave incident from the left.

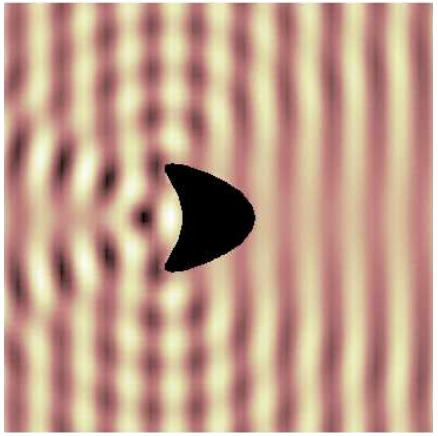

Fig. 1: Scattering of a plane wave

The boundary value problem (1)–(3) involves the infinite d-dimensional domain $\mathbb{R}^d \setminus \overline{D}$ and also the $(d-1)$-dimensional surface ∂D. *Boundary integral equations*, which reduce the boundary value problem to an integral equation on ∂D, are popular analytical and numerical tools because they reduce the complexity of the problem, provided k is independent of x. Figure 1 was computed in this way.

The scattering problem just described is a direct problem: given the obstacle and the incident field, find the scattered field. Of possibly even greater importance for applications and mathematically more challenging is the corresponding *inverse problem:* given the scattered fields for a number of incident fields, deduce the shape of the obstacle.

Another application of (1) is to the study of *waveguides*. Waveguides are devices that transmit acoustic or electromagnetic energy, but they typically do so only at certain discrete frequencies. As an example, consider a simple acoustic waveguide, a long straight pipe with cross-section $D \subset \mathbb{R}^2$. To analyse its behavior we have to find k and the corresponding mode u such that $\Delta u + k^2 u = 0$ in D and $\partial u/\partial n = 0$ on ∂D. In other words we have an eigenvalue problem for the Laplace equation (→ *ref*). In the case where D is a circular disk, we can solve this problem analytically by introducing polar coordinates (r,φ) and using separation of variables. This yields

$$u(r,\varphi) = \mathrm{e}^{\mathrm{i}m\varphi}\, J_m(k_{m,n} r),$$

with $m = 0, 1, 2, \ldots$ and $k = k_{m,n}$, the n-th positive zero of the derivative of the m-th order Bessel function of the first kind, J_m. Some of these solutions are plotted in Figure 2.

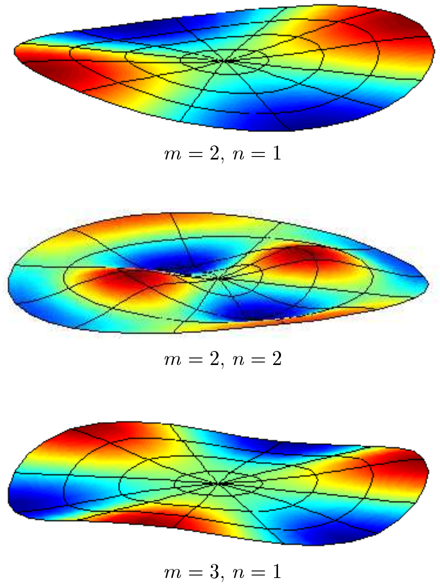

Fig. 2: Modes of a circular waveguide

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