19. One-way wave equations — *pseudodifferential operators & absorbing boundaries*

The second-order wave equation (→ *ref*) is isotropic: invariant with respect to rotation in space. In two dimensions, for example, the equation $u_{tt} = u_{xx} + u_{yy}$ admits plane wave solutions $u(x, y, t) = \exp(\mathrm{i}(\omega t + k_x x + k_y y))$ for any wave numbers k_x, k_y and frequency ω that satisfy $\omega^2 = k_x^2 + k_y^2$, and the curves $\omega =$ constant of this dispersion relation are concentric circles in the k_x-k_y plane.

What if we want a PDE that behaves like just half of the wave equation? To be specific, suppose we want an equation that propagates a plane wave exactly like $u_{tt} = u_{xx} + u_{yy}$ if the x component of the velocity is negative, but does not propagate waves at all in any positive x direction. This idea has proved fruitful in underwater acoustics, geophysical imaging ('migration'), and the design of numerical 'radiation' or 'absorbing' boundary conditions.

We can formulate the mathematical problem as follows. The dispersion relation for $u_{tt} = u_{xx} + u_{yy}$ can be written

$$k_x = \pm\omega\sqrt{1 - s^2}, \tag{1}$$

with $s = k_y/\omega$. The *ideal one-way equation* would have exactly half of this dispersion relation:

$$k_x = +\omega\sqrt{1 - s^2}. \tag{2}$$

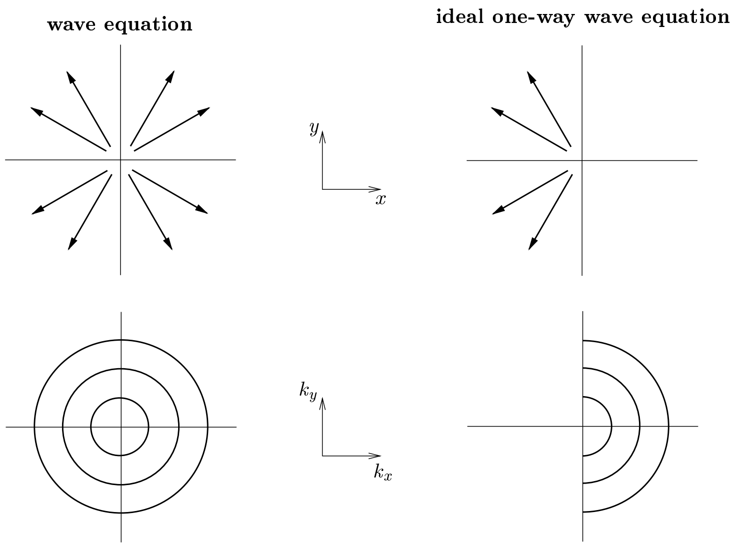

Fig. 1: wave propagation directions (top) and dispersion relations (bottom)

The problem with (2) is that it does not correspond to any partial differential equation. In fact, it is the dispersion relation of a *pseudodifferential equation* which is non-local and not easily worked with. This is the origin of the idea of approximate one-way wave equations, or *one-way wave equations* for short. Such equations were proposed in the 1970s by Tappert and by Lindman and made famous by Engquist and Majda.

To get a PDE with approximate one-way wave behaviour, we replace the square root in (2) by a rational function $r(s)$, i.e., a quotient of polynomials of degrees m and n, to obtain

$$k_x = \omega\, r(s). \tag{3}$$

Specifically, let us take $r(s)$ to be a Padé approximant of $\sqrt{1 - s^2}$, a rational function of specified type whose Taylor series about $s = 0$ matches that of $\sqrt{1 - s^2}$ as far as possible. For example, the Padé approximants of types $(0, 0)$, $(2, 0)$, and $(2, 2)$ are $r_{0,0}(s) = 1$, $r_{2,0}(s) = 1 - \frac{1}{2}s^2$, and $r_{2,2}(s) = (1 - \frac{3}{4}s^2)/(1 - \frac{1}{4}s^2)$. Setting $k_x = r(k_y/\omega)$ and clearing denominators gives the dispersion relations $k_x = \omega$, $k_x\omega = \omega^2 - \frac{1}{2}k_y^2$, and $k_x\omega^2 - \frac{1}{4}k_x k_y^2 = \omega^3 - \frac{3}{4}\omega k_y^2$. These correspond to the PDEs $u_x = u_t$, $u_{xt} = u_{tt} - \frac{1}{2}u_{yy}$ (known as the *paraxial equation*), and

$$\boxed{u_{xtt} - \tfrac{1}{4}u_{xyy} = u_{ttt} - \tfrac{3}{4}u_{tyy}.} \tag{4}$$

We put this third-order example in a box as a representative of an infinite sequence of one-way wave equations derived from approximations along the 'staircase' of the Padé table of types (n, n) and $(n+2, n)$. It is known that in various senses, approximants from this staircase are well-posed, whereas those from off the staircase are ill-posed.

Figure 2 illustrates the use of one-way wave equations as absorbing boundary conditions. To the eye, the second order equation is already excellent, so there seems little need to go to the third-order equation (4), though looking at numbers rather than pictures would reveal that (4) provides a further improvement.

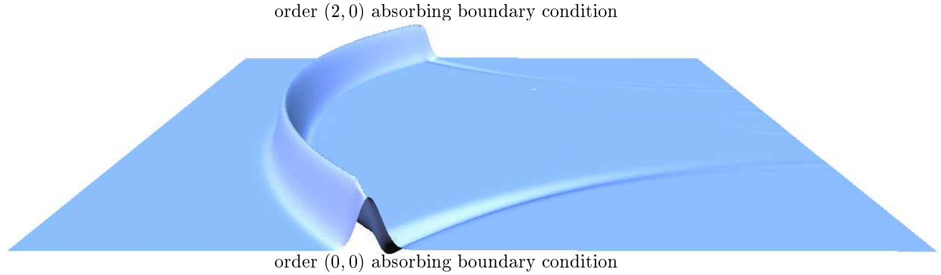

Fig. 2: One-way wave equations as absorbing boundaries on $[0, 1.4] \times [-0.7, 0.7]$

References

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