15. Wave equation in 1D

In one dimension the wave equation ($\rightarrow$ *ref*) takes the form

$$u_{tt} = u_{xx}, \tag{1}$$

the simplest second order hyperbolic PDE. The standard example of a physical system governed by the wave equation is a vibrating ideal elastic string (such as a guitar string) fixed at both ends. If the string is distorted, or plucked, at some initial time and then allowed to vibrate, the displacement of the resulting transverse wave will be a solution of (1). This equation also models many other physical problems, such as propagation of sound waves in a tube.

An initial value problem can be posed by combining (1) with initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

The unique solution to this problem can be expressed by d'Alembert's formula,

$$u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(y)\,\mathrm{d}y.$$

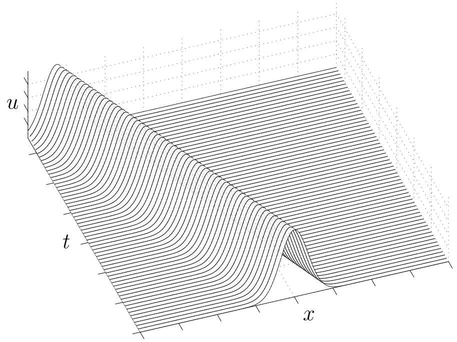

Fig. 1: Propagation in a single direction

Alternatively, for any initial data, solutions to (1) can be written as a linear combination

$$u(x,t) = F(x+t) + G(x-t),$$

where F represents a left-going and G a right-going wave. D'Alembert's solution is the special case in which the left-going and right-going waves are

$$F(x) = \frac{1}{2}f(x) + \frac{1}{2}\int_0^x g(y)\,\mathrm{d}y,$$

$$G(x) = \frac{1}{2}f(x) - \frac{1}{2}\int_0^x g(y)\,\mathrm{d}y.$$

Consider an example with initial conditions $F(x) = e^{-x^2}$ and $G(x) = 0$, i.e., a left-going Gaussian pulse. In Figure 1 we do not specify any boundary conditions but just observe the propagation of the wave as time progresses. In Figure 2, on the other hand, we restrict the same problem to the interval $[-L, L]$ and specify boundary conditions

$$u(-L,t) = 0 \text{ (Dirichlet)},$$
$$u_x(L,t) = 0 \text{ (Neumann)}.$$

Darker and lighter lines indicate positive and negative wave heights, respectively. The Dirichlet condition asserts that the solution vanishes at the left boundary. Thus as the wave approaches and hits the wall, the crest diminishes and is reflected back along the boundary. The Neumann condition requires the normal derivative to be zero at the right end. Here the wave approaches the wall and is reflected back just as it came. This wave pattern can be interpreted as an idealisation of sound waves in a clarinet, where the pressure deviation from ambient is zero at the open end while its derivative is zero at the reed.

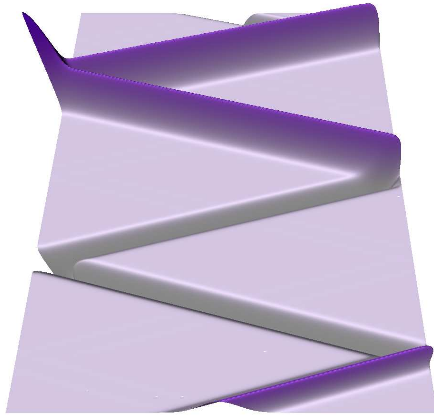

Fig. 2: Effect of Dirichlet (left) and Neumann (right) boundary conditions

Like its n-dimensional generalisation, the 1D wave equation can be studied by Fourier analysis. For any wave number $k \in \mathbb{R}$, the wave $e^{\mathrm{i}(kx+\omega t)}$, is a solution to (1) provided that the frequency ω satisfies $\omega^2 = k^2$, i.e., $\omega = \pm|k|$. This dispersion relation is sketched in Figure 3. On a bounded domain $[0, L]$ with Dirichlet boundary conditions, the eigenfunctions of the Laplacian are just the sine functions $\sin(j\pi x/L)$, so Fourier analysis remains applicable, with the continuous range of wave numbers replaced by the discrete subset, $\pi/L, 2\pi/L, 3\pi/L, \ldots$.

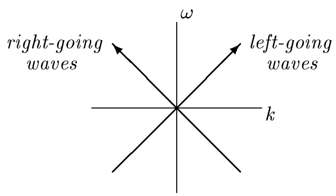

Fig. 3: Dispersion relation

The 1D wave equation is a starting point for more complicated hyperbolic PDEs. For example, a forcing function $\mathcal{F}(x,t)$ can be incorporated which acts as a source or sink for the wave form,

$$u_{tt} = u_{xx} + \mathcal{F}(x,t). \tag{2}$$

Two forcing functions of particular interest give the Klein–Gordon and Sine–Gordon equations ($\rightarrow$ *refs*).

References

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