15. Wave equation — *propagation of light and sound*

One of the "holy Trinity" of partial differential equations is the second–order wave equation, the canonical example of a hyperbolic PDE. In n dimensions the equation takes the form

$$\boxed{u_{tt} = \Delta u,} \tag{1}$$

where Δ is the Laplacian operator, $\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$. A wave speed c can be included by a factor c^2 on the right-hand side. Since (1) is of second order in t, a well-posed initial-value problem for this equation would normally involve two initial conditions such as $u(x,0)$ and $u_t(x,0)$.

The wave equation describes linear, nondispersive wave propagation. For example, Figure 1 presents a pair of images that show the outward spread of a circular pulse in 2D. At $t = 0$ we begin with a cone of radius 0.1 with $u_t(0) = 0$. At $t = 2$, the cone has spread to a concentric ring of outer radius exactly 2.1.

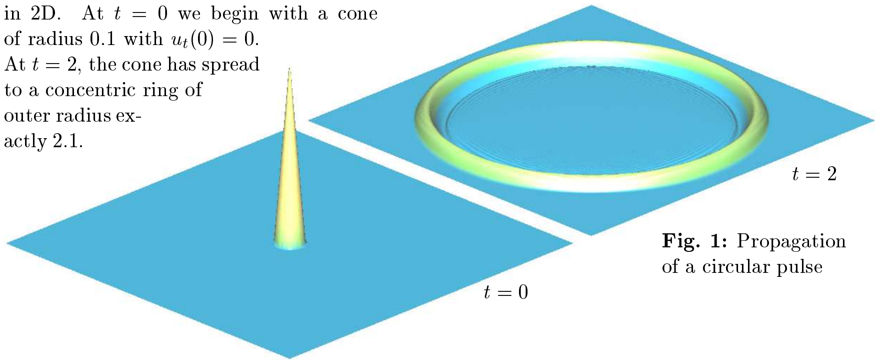

Fig. 1: Propagation of a circular pulse

The wave equation arises in numerous applications. The classical 1D example is the vibration of an ideal string (→ *ref*), and in 2D this becomes the vibration of an ideal membrane or drum (→ *ref*). In 3D, the most famous example is the propagation of sound waves in a gas or liquid. Indeed, equation (1) is often called the *acoustic wave equation* to distinguish it from the more complicated *elastic wave equation* (→ *ref*), where the presence of stiffness as well as compressibility leads to the appearance of two distinct kinds of waves.

Being hyperbolic, the wave equation has finite speed of propagation for all information—namely 1, for the equation as written in (1). A curious property known as *Huygens' principle* is as follows. In dimensions $n = 3, 5, 7, 9, \ldots$, all information propagates under (1) at speed exactly 1, never slower. Thus, the light from a bulb flashed at $t = 0$ passes the observer at a later time as a pure delta function. In dimensions $n = 1, 2, 4, 6, 8, \ldots$, on the other hand, a finite fraction of the energy may travel more slowly than at speed 1, so the observer sees a delta function flash followed by a decaying tail. To illustrate this phenomenon, Figure 2 shows the result at time $t = 1$ of the initial condition $u_t(x,0) = \max\{0, 1 - 10|x|\}$ in dimensions $1, 2, 3, 4, 5, 6$, where $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$.

In an unbounded domain, the wave equation is readily investigated by Fourier analysis. Separation of variables leads to the observation that for any n-vector k, known as the *wave number*, there are

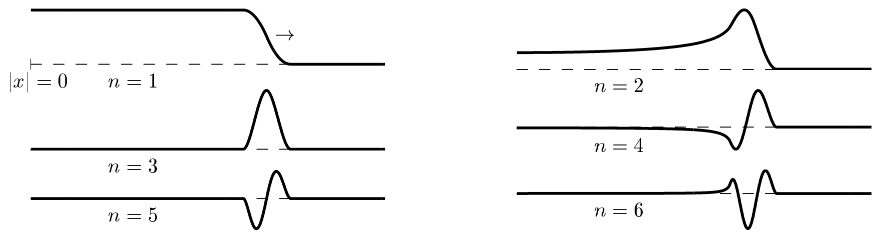

Fig. 2: Huygens' principle: zero tails in odd dimensions $n \geq 3$

plane wave solutions of (1) of the form

$$u(x,t) = e^{\mathrm{i}(\omega t + k \cdot x)}, \tag{2}$$

where $k \cdot x = k_1 x_1 + \cdots + k_n x_n$, so long as $\omega = \pm |k|$. This condition relating the frequency to the wave number is the *dispersion relation* for (1). By a Fourier integral, general solutions to (1) can be obtained by the superposition of plane waves (2), and under suitable technical assumptions, all solutions can be written this way.

In a bounded domain Ω, separation of variables in (1) leads to oscillatory solutions of the form $e^{\mathrm{i}\omega_j t}\phi_j(x)$, where the functions $\phi_j(x)$ are eigenfunctions of the Laplacian operator for Ω (→ *ref*). The allowed frequencies ω_j now belong to a discrete set, and general solutions can be obtained via superpositions as series rather than integrals. If Ω is a rectangle, a disk, or a ball, the eigenfunctions are trigonometric functions, Bessel functions, or spherical harmonics, respectively.

Another technique in the study of the wave equation is *Hadamard's method of descent*. The idea here is that any solution in dimension n can be thought of as a solution in dimension $n + 1$ that happens to be invariant with respect to one coordinate. In particular, solutions in even dimensions can be obtained from solutions in the odd dimension one higher, which are relatively elementary superpositions of expanding spheres thanks to Huygens' principle.

In applications of the wave equation, boundaries and variable coefficients are important, including discontinuities in the sound speed. Among the phenomena that arise are reflection, refraction, and diffraction. Just as the field of fluid mechanics can be described without too much exaggeration as the study of the Navier–Stokes equations (→ *ref*), so the field of acoustics is more or less the study of the wave equation. There are enough subtleties here to fill books, and careers—even if we confine our attention to the fascinating subfield of the physics of musical instruments.

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