Notes of a Numerical Analyst

Prime Gaps and Numerical Eigenvalues

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Despite their similar names, number theory and numerical analysis are about as far apart as you can get in mathematics. To oversimplify outrageously, all that number theorists care about is integers, and all that numerical analysts care about is real numbers. It's chalk and cheese, discrete and continuous, algebra and analysis.

So my interest was just that of an onlooker when I heard a few years ago about big results concerning prime gaps. First, Yitang Zhang proved that there are infinitely many pairs of primes separated by less than $70{,}000{,}000$. Then James Maynard showed that $70{,}000{,}000$ could be improved to 600 [1]. (Nine years later he was awarded a Fields Medal.) The current best result is that there are infinitely many primes separated by gaps no greater than 246 [2].

3,5	**5,7**	**11,13**	**17,19**
29,31	**41,43**	**59,61**	**71,73**

Figure 1. It is not known if there are infinitely many pairs of primes like these that differ by just 2, but it has been proved that there are infinitely many that differ by no more than 246

But who knew that all these theorems depend on calculating eigenvalues of matrices? I learned this during a lunch with Maynard (whom I thank for contributing to this column). "This number 246", I asked him, "it must come from some kind of calculation, right? What is the calculation?"

Maynard surprised me by explaining that it's the numerical calculation of an eigenvalue of a matrix. It turns out that if you can prove that a certain $1{,}780 \times 1{,}780$ matrix has an eigenvalue >4, you have proved that there are infinitely many prime gaps ≤ 246. The number 246 is the smallest gap size for which the relevant matrix has this property.

The number theory details can be found in the papers just cited. I'd like to say a word about this other aspect, the method of proving something rigorously by a real number computation, even though real numbers can only be approximated on our computers.

A famous tool is *interval arithmetic*, and you might imagine this is what Maynard and his collaborators must have used. In interval arithmetic, when two numbers are combined by (say) division, although the exact result is not be stored, upper and lower bounds are retained. As a calculation proceeds, the gap between the upper and lower bounds widens, but the bounds are rigorous.

The trouble is that this can be terribly pessimistic in practice. So for years I had a low opinion of interval arithmetic, until I learned that many results in this area are attained in a cleverer, a posteriori fashion. You compute your result by ordinary numerical means, and then you *validate it* [3]!

Suppose, say, you've computed a numerical eigenvector x and eigenvalue λ of a symmetric matrix A. The computations are not rigorous. But now, you validate your result by rigorously computing a bound $\|Ax - \lambda x\|_2 \leq \varepsilon$ on the residual by interval arithmetic or other means. This implies that A has an eigenvalue within a distance ε of λ. Maynard and his collaborators used a one-sided bound, not intervals, but the a posteriori nature of their rigorous calculation was exactly this.

FURTHER READING

[1] J. Maynard, Small gaps between primes, *Ann. Math.*, 181 (2015) 383–413.
[2] D.H.J. Polymath, Variants of the Selberg sieve, and bounded intervals containing many primes, *Res. Math. Sci.*, 1 (2014) 12.
[3] S. Rump, Verification methods: rigorous results using floating point arithmetic, *Acta Numer.*, 19 (2010) 287–449.

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