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Analysis and design of polygonal resistors by conformal mapping

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1. Introduction

In integrated circuitry and elsewhere in electronics, the electrical resistance of a polygonal circuit element or pathway is often physically important [7]. In the simplest problem of resistor *analysis*, a polygon is given and its resistance must be determined. In the simplest problem of resistor *design*, a desired resistance is given and a corresponding polygon must be selected from a family of candidates parametrized by some geometric quantity. Of course one could devise mathematically equivalent problems involving heat conduction, ideal fluid flow, magnetostatics, etc. Questions connected with electrical capacitance are also closely related. In problem-independent terminology, we are concerned here with the *conformal modulus* of what is sometimes called a polygonal *quadrilateral* [1, 4].

The resistance is a global quantity depending on the solution of a simple boundary value problem for Laplace's equation, and as a result it is invariant under a conformal map. (Intuitively, a conformal map behaves locally like a scale change times a rotation, and both of these preserve resistance.) Therefore one way to perform a calculation of this sort is by constructing a conformal map onto a new domain where the problem is trivial – a rectangle. If the resistor is polygonal, this map can be written as a composition of Schwarz-Christoffel transformations.

Conformal mapping methods for resistor analysis on polygons have been proposed before [13, 14]. Their practical implementation, however, usually requires the numerical solution of a Schwarz-Christoffel "parameter problem," which has only recently become feasible for general polygons [9, 10]. A Fortran package for such computations called SCPACK is available from the author [11]. Section 2 gives examples of high-accuracy resistance computations based on SCPACK.

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Conformal mapping methods for resistor design have perhaps not been discussed previously. The main purpose of this paper, carried out in Sec. 3, is to show that design can be accomplished at roughly the same cost as analysis by formulating an appropriate "generalized parameter problem." For definiteness we consider a particular problem of "resistor trimming," in which the aim is to cut a slit in a resistor of such a length as to increase its resistance to a prescribed value. This idea is of some practical importance in integrated circuit manufacture, for it is difficult to fabricate integrated circuit wafers containing resistance elements accurate to high tolerances. Instead, one can design the wafer so that the resistance is 10% or so too low, and then tune it to the desired value by cutting a slit with a laser. The mathematical problem of predicting the appropriate slit length has been attacked previously by other methods [3, 6].

The Schwarz-Christoffel approach to Laplace-related calculations on polygons has the virtue that unlike methods based on integral equations, finite differences, finite elements, or the superposition of particular solutions, it preserves the property inherent in the problem of having only a finite number of parameters. In particular it automatically obtains the correct singularities at corners. Because of this, the Schwarz-Christoffel method can achieve very high accuracy without much penalty in execution time, as our examples will show. (For low accuracy, other methods may be better; see [5] for a description of a highly effective method based on breaking the domain up into simple subpolygons.) Its chief drawback is that it generalizes poorly to related problems involving more complicated differential equations (e.g. variable coefficients or nonzero righthand side) or geometries (e.g. curved boundaries or multiple connectivity).

The idea of a generalized parameter problem is applicable in a wide variety of Schwarz-Christoffel computations, of which our resistor trimming problem is only the simplest example. Some other examples are mentioned in Sec. 4.

2. Analysis

Let P be a polygon with $N \geq 4$ vertices $w_1, \ldots, w_N$, and let $\beta_k \pi$ denote the external angle at vertex w_k. See Fig. 1. Let a, b, c, d be four distinguished vertices in counterclockwise order, and let Γ_1 and Γ_2 denote the boundary arcs $a-b$ and $c-d$. We wish to calculate the resistance of P, $R(P)$, when voltages V_1 and V_2 are applied on Γ_1 and Γ_2 and the remainder of the boundary is insulated. Of course we are concerned with the idealized problem, not with complications related to finite thickness of slits, inhomogeneity, etc.

Let I be the total current between Γ_1 and Γ_2. Then $R(P) = (V_2 - V_1)/I$. Mathematically, the problem is to solve Laplace's equation, $\nabla^2 V = 0$, subject to Dirichlet boundary conditions $V = V_k$ on Γ_k and Neumann conditions $\partial V/\partial n = 0$ on the remainder of the boundary. Once this is done, $R(P)$ might be computed by evaluating I as a line integral of ∇V along a curve running from one insulated

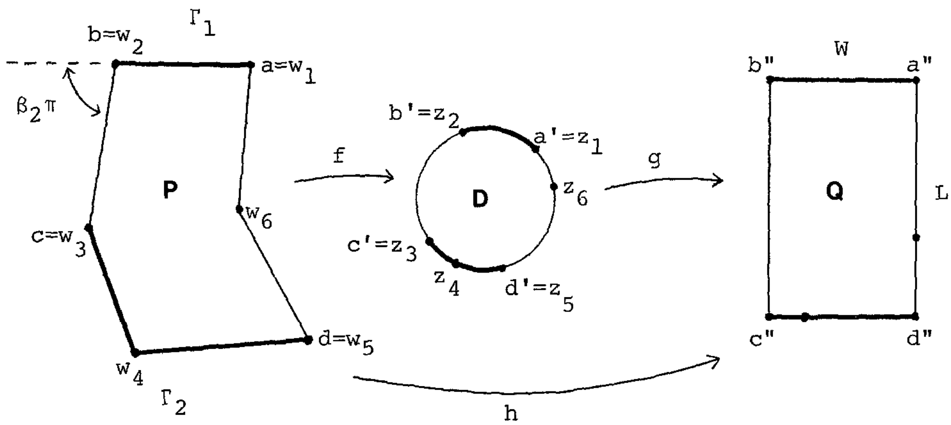

Figure 1
Conformal map of a polygon onto an equivalent rectangle.

boundary component to the other. However, a conformal map will make this unnecessary. Suppose a map h is found that carries P onto a rectangle Q in such a way that $h(\Gamma_1)$ and $h(\Gamma_2)$ are two opposite edges of Q, as in Fig. 1. Let these edges be considered the ends of Q, with length W, and let the other two edges be considered its sides, with length L. If we normalize by letting a square have resistance 1, then obviously Q has resistance L/W. Therefore $R(P) = L/W$ also.

Since $R(P)$ is a unique number, it cannot be possible to map P in this fashion onto an arbitrary rectangle. In fact, P can be mapped onto precisely those rectangles with length-to-width ratio L/W. This is consistent with the Riemann mapping theorem, which states that a given simply connected region can be mapped onto any other (provided neither is the entire plane), but than the images of no more than three boundary points can be chosen arbitrarily in the process. Here we are prescribing the images of four rather than three boundary points, and this extra condition gives rise to the L/W restriction.

The map h can be constructed as the composition of one (inverse) Schwarz-Christoffel map f from P to the unit disk D and another Schwarz-Christoffel map g from D to Q. Consider first the map f. The Schwarz-Christoffel theorem asserts that f^{-1} has the form

$$f^{-1}(z) = w_c + C \int_0^z \prod_{k=1}^{N} (1 - \zeta/z_k)^{-\beta_k} d\zeta \tag{1}$$

for some complex constants w_c and C and some prevertices $z_k = f(w_k)$ with $|z_k| = 1$, but the values of all of these constants are a priori unknown [8]. The problem of determining them is the Schwarz-Christoffel *parameter problem*. If parameter values are chosen arbitrarily, then in general (1) will define a map of D onto a polygonal region with the same angles as P but with different side lengths. The SCPACK package deals with the parameter problem by setting up a nonlinear system of equations embodying the conditions that the side lengths come out right. The unknowns in this system are the prevertices z_k, or more simply their arguments $\theta_k = \arg z_k$.

Unknown: N prevertex arguments $\theta_k = \arg z_k, \quad 1 \le k \le N$;

Known: N side lengths $\left| C \int\limits_{z_k}^{z_{k+1}} \prod\limits_{j=1}^{N} (1 - \zeta/z_j)^{-\beta_j} d\zeta \right| = |w_{k+1} - w_k|,$

$$1 \le k \le N. \quad (2)$$

SCPACK then applies a robust nonlinear equations solver (NS01A, by M. J. D. Powell) to adjust $\{\theta_k\}$ iteratively until (2) is satisfied. Of course we are glossing over many details, in particular the process of numerical integration; see [10].

Once the parameter problem for f has been solved, the distinguished prevertices $a', \ldots d'$ on the unit circle are known. To map D onto a rectangle Q, one now applies a second Schwarz-Christoffel formula:

$$g(z) = \int\limits_0^z [(\zeta - a')(\zeta - b')(\zeta - c')(\zeta - d')]^{-1/2} d\zeta. \quad (3)$$

(The constants w_c and C have been omitted, since they affect only the scale and position of Q.) This time there is no parameter problem, because it is the prevertices rather than the image polygon that are given. The resistance can be evaluated at the cost of three integrals of type (3):

$$R(P) = \frac{L}{W} = \frac{|g(c') - g(b')|}{|g(b') - g(a')|}. \quad (4)$$

The map g is essentially a *Jacobi elliptic function.* One can take advantage of the great amount that is known about these functions to simplify the computation (4). (The benefit of doing so is more esthetic than practical, since most of the work comes in computing f, not g.) Given $a', \ldots, d'$, let their *cross ratio* $\chi \in (-\infty, 0)$ and a real parameter $k \in (0,1)$ be defined by the formulas

$$\chi = \frac{(b' - a')(d' - c')}{(c' - b')(a' - d')}, \quad (5)$$

$$k = 1 - 2(\chi + \sqrt{\chi^2 - \chi}). \quad (6)$$

Then (4) can be rewritten as a ratio of *complete elliptic integrals* [8]

$$R(P) = \frac{2K(k)}{K(\sqrt{1 - k^2})}. \quad (7)$$

The function $K(k)$ can be computed extremely rapidly through the use of a well-known iteration involving arithmetic and geometric means:

$\alpha := 1, \quad \beta := k$

repeat until $\alpha \approx \beta$:

$$\alpha' := (\alpha + \beta)/2$$

$$\beta' := \sqrt{\alpha + \beta} \tag{8}$$

$$\alpha := \alpha', \quad \beta := \beta'$$

$K := \alpha$

This completes our description of resistor analysis by means of the Schwarz-Christoffel transformation. In summary, given a polygon P, here is the procedure:

Resistor analysis algorithm

(i) *Call SCPACK to solve the parameter problem* (2) *for the prevertices* $\{z_k\}$ *of the map* f, *and in particular to obtain the distinguished prevertices* $a', \ldots, d'$;
(ii) *Insert* $a', \ldots, d'$ *in* (3)–(4) *or* (5)–(8) *to determine* $R(P)$;
(iii) *As desired, call SCPACK to evaluate* h *or* h^{-1} *at various points from* (3) *and* (1) *for plotting or other purposes.*

Figure 2 shows four examples of successful resistance calculations with SCPACK. (All of our calculations were performed on a VAX 780 running Unix Fortran 77 in double precision.) Each part of the figure lists the resistance R obtained numerically, probably accurate to all 12 places given, and plots 11 equipotential and stream lines obtained by mapping a rectilinear grid in Q back to P. Note that these families of curves divide P into subregions that approximate rectangles of length-to-width ratio R. The polygons treated can be described as follows.

(*a*) *Regular pentagon.* Here the parameter problem for f is trivial – one can take equally spaced prevertices. The computation did not take advantage of symmetry.

(*b*) *L-shaped hexagon.* Each leg has width 1, outside length 3. Again the computation did not take advantage of symmetry. Note that the influence of the corner is obviously very weak near the ends. In fact by considering the conformal map of a straight infinite strip onto an infinite strip with a 90° corner in it, one can show that for L-shaped regions whose side length is an integer l, the fractional part of the resistance approaches $1 - 2\ln 2/\pi \approx 0.55872880$ exponentially as $l \to \infty$. The value for $l = 3$ in Fig. 2b already matches this to six places.

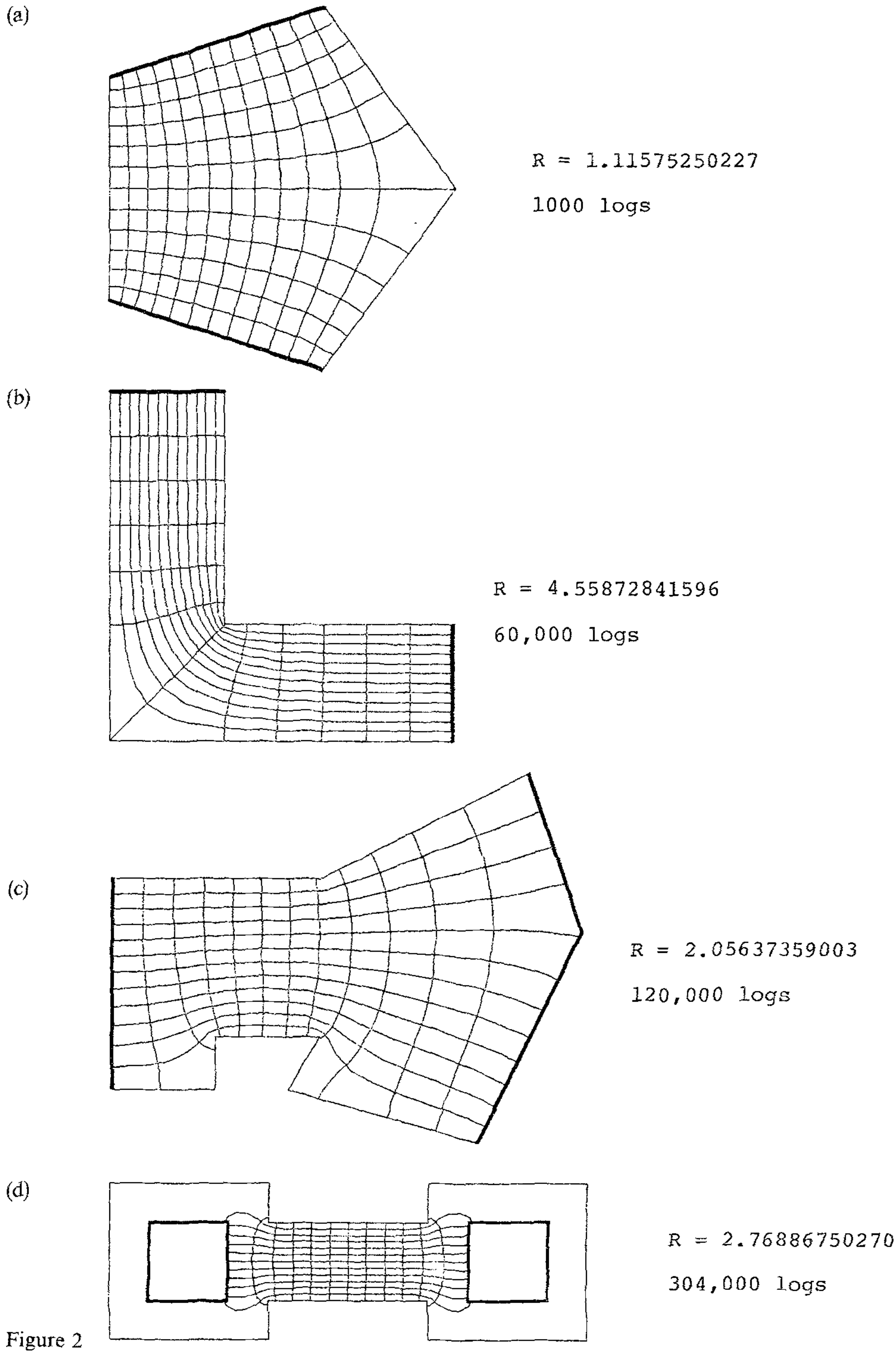

Figure 2
Four resistors.

(*c*) *Irregular 10-gon.* The vertices are located at $-4+i$, $-4-3i$, $-2-3i$, $-2-2i$, $-2i$, $-0.6-3i$, $3-4i$, 5, $4+3i$, and i.

(*d*) *Symmetrical domain with internal contacts.* This region is multiply connected, with all side lengths equal to 1, 1/2, or 1/4. We treat it by considering the quarter domain, an asymmetrical but simply connected 10-gon.

To estimate efficiency, it is convenient to take advantage of the fact that most of the computer time in all SCPACK computations goes to the repeated calculation of logarithms of complex numbers needed for evaluating the Schwarz-Christoffel integrand (1). The approximate number of logarithms required for each computation is listed in Fig. 2. (These figures do not include the somewhat larger numbers needed for plotting.) On typical machines these logarithm counts correspond to cpu times of a few seconds. Roughly speaking, they increase in proportion to the cube of the number of vertices [10].

3. Design

Again let P be a polygon as in the last section, with resistance $R(0)$. Let e denote some point on the insulated boundary of P beginning at which a straight slit of length $s>0$ has been cut in some fixed direction into P. (See Fig. 3.) Physically, the slit is a perfect insulator. Mathematically, it consists of two extra sides to the polygon (which happen to coincide) added to the part of the boundary where the Neumann condition is applied. Let P_s denote the slit polygon, with resistance $R(s)$. Obviously $R(s)$ is a monotonically nondecreasing function of s, and if the slit is oriented in such a way that for $s \geq s_\infty$ it cuts P into two halves, each containing one of the boundary arcs Γ_k, then $R(s)=\infty$ for all $s \geq s_\infty$.

Here is our inverse problem.

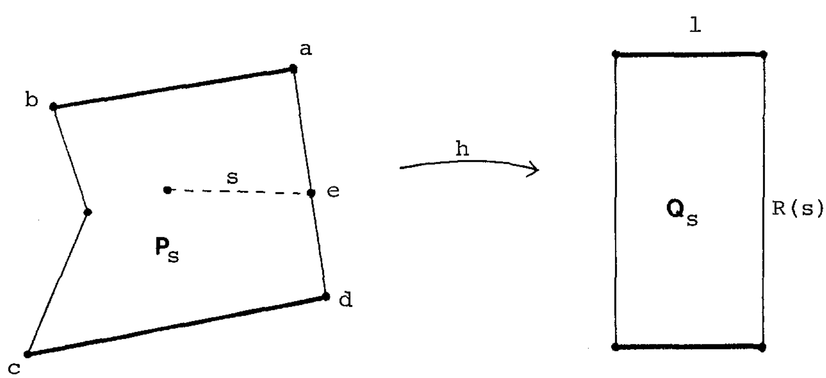

Figure 3
Polygon with a slit in it.

Resistor trimming problem

Given $R_0 > R(0)$, *find* s_0 *such that*

$$R(s_0) = R_0. \tag{9}$$

For any fixed s, P_s is a polygon whose resistance can be evaluated by solving a Schwarz-Christoffel parameter problem as in Sec. 2. In other words, we know how to evaluate the nonlinear function $R(s)$ numerically, and this ability could be used as the basis of an iterative solution of (9) [12]. However, the point of this paper is that one can do better. Let P_s have N vertices $w_1, \ldots, w_N$, including the following three which define the slit:

$$w_{J-1} = e, \qquad w_J = \text{end of slit}, \qquad w_{J+1} = e.$$

The resistor trimming problem contains the following N known quantities: $N-1$ vertices w_k $(k \neq J)$; 1 resistance R_0. Clearly it is most natural to combine precisely these conditions into a single nonlinear system of equations, rather than artificially treat w_J instead of R_0 as known and then be forced to iterate. Reducing to real variables as before, we obtain the following *generalized parameter problem* for resistor trimming. Compare with the standard parameter problem (2).

Unknown: N prevertex arguments $\theta_k = \arg z_k$, $\quad 1 \leq k \leq N$;

Known: $N-2$ side lengths $\left| C \int\limits_{z_k}^{z_{k+1}} \cdots \right| = |w_{k+1} - w_k|$, $\quad k \neq J-1, J$, (10a)

1 additional side length condition $\left| C \int\limits_{z_{J-1}}^{z_J} \cdots \right| = \left| C \int\limits_{z_J}^{z_{J+1}} \cdots \right|$, (10b)

1 resistance condition $R = R_0$. (10c)

Once the generalized parameter problem has been formulated, all that remains in principle is to solve it numerically. We will again skip over most of the details, except for two important ones that are worth mentioning.

The first concerns implementation of (10c). At any step in the iterative solution of the parameter problem, a set of trial values $\{\theta_k\}$ and in particular $a', \ldots, d'$ is in hand, and one must determine how nearly (10c) is satisfied. The direct approach would be to calculate the resistance $R(a', \ldots, d')$ by (3)–(4) or by (5)–(8). However, a more elegant method consists of computing the elliptic integrals once and for all in advance. Given R_0, begin by determining the corresponding *prescribed cross ratio* χ_0. This can be done by calling SCPACK to solve a (standard) parameter problem for a rectangle with $L/W = R$, or by performing a secant iteration based on (3)–(4) or (5)–(8), or by applying the following formulas inverse to (6)–(8) [8]:

$$k = 4\sqrt{q}\prod_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^4, \qquad q = e^{-2\pi/R}, \tag{11}$$

$$\chi = \frac{1}{4}(1-k)\left(\frac{1}{k}-1\right). \tag{12}$$

Once χ_0 is known, one can discard R_0 and use χ_0 in the main iteration. Actually it is better to work with $\log|\chi_0|$, since $|\chi|$ decreases exponentially as $R \to \infty$. Thus (10c) is replaced by

$$1 \text{ cross ratio condition } \log|\chi| = \log|\chi_0|. \tag{10c$'$}$$

The second important detail is that in both (2) and (10), we have ignored the fact that a conformal map contains three arbitrary parameters. This means that N equations as in (2) or (10) cannot fully determine all of the arguments $\{\theta_k\}$, but only $N-3$ relations among them; conversely, there must be three degrees of redundancy in these equations. We have avoided discussing this issue because there are many different ways to resolve it, and because the details are straightforward but tedious. However, the detailed formulation of the parameter problem will of course determine exactly how it changes when generalized for resistor trimming. In the particular formulation chosen in SCPACK, it happens that not N but $N-2$ side length conditions (2) are enforced, namely those with $k \neq N-2, N-1$ [10]. The remaining two side lengths come out right automatically because of elementary geometry. To adapt this formulation to the slit resistor problem, one can first number the vertices so that $J = N-1$, then simply enforce only $N-3$ side length conditions $k \neq N-3, N-2, N-1$. The geometry forces (10b) to be satisfied automatically, so this condition never needs to be imposed explicitly.

This completes our description of resistor design by means of the Schwarz-Christoffel transformation. In summary, given a class of slit polygons $\{P_s\}$ and a desired resistance R_0, here is the procedure:

Resistor design algorithm

(i) *Compute the prescribed cross ratio* χ_0 *by a secant iteration or by* (11)–(12)*;*
(ii) *Call an appropriately modified version of SCPACK to solve the generalized parameter problem* (10)*;*
(iii) *Call SCPACK to evaluate* h *or* h^{-1} *as desired.*

Figure 4 shows four examples of resistor design calculations, as follows:

(*a*) *Diamond with transverse slit.* The diamond has width 2, height 1.

(*b*) *Square with slit down the middle.* Each side has length 1.

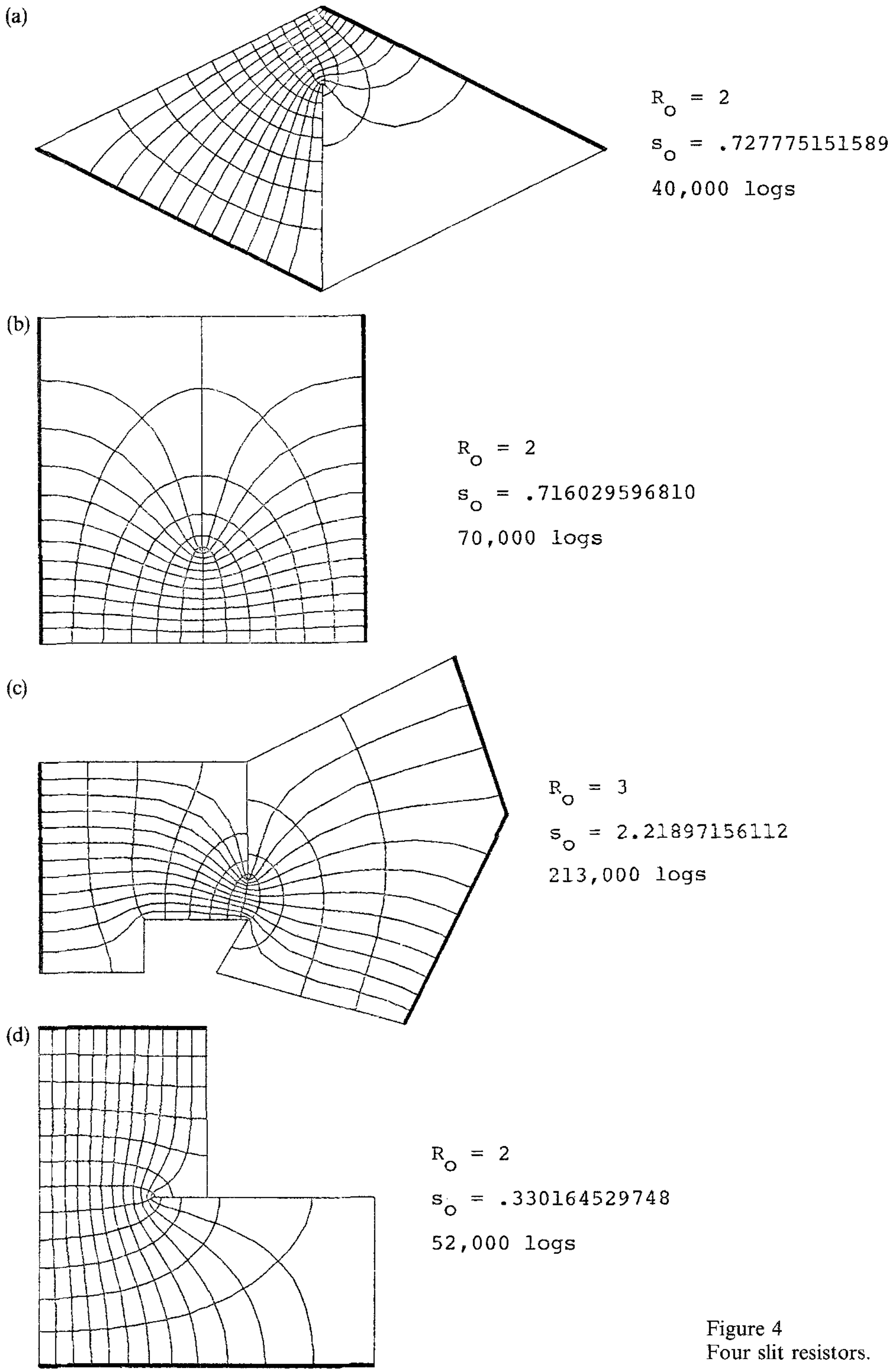

Figure 4
Four slit resistors.

(*c*) *Region of Fig. 2c with vertical slit added.* Note how long a slit must be cut to bring the resistance up from 2.06 to 3.

(*d*) *L shape.* Each leg has width 1 and outside length 2. The slit begins at the reentrant corner. With no slit, the resistance would be $\sqrt{3}$ [4].

In these resistor trimming computations, SCPACK again reliably obtains answers accurate to high precision, as indicated. The costs measured in logarithms are comparable to those listed in Fig. 2 for resistor analysis.

4. Further applications of generalized parameter problems

In many possible applications of Schwarz-Christoffel maps – perhaps most – one is not simply given a polygon whose properties require investigation. Rather, one is given a problem whose solution involves a polygon whose geometry must be determined as part of the computation. The connection of the problem with the polygon may be only implicit, and well disguised.

To find simple examples we can vary the theme of a slit resistor. One might have an L shape as in Fig. 2b of known length but unknown width, and wish to determine what width is needed to bring the resistance down to a specified level. (This is an idealization of a timing question for integrated circuits.) One might want to vary the angle of the L to achieve a desired property. And so on.

More generally, a wide variety of Laplace equation problems involving piecewise-constant boundary conditions can be solved by viewing the unknown solution φ as the real part of an analytic function f. In this process a Dirichlet condition $\varphi = const.$, for example, becomes a condition $\operatorname{Re} f = const.$, which amounts to specifying the horizontal position of a side of a polygon, but not its length. By proceeding in this way one obtains generalized parameter problems involving a mixture of known coordinates and unknown slit lengths. Details for one problem of this type, in which the generalized parameter problem turns out to be linear, can be found in the section on the Hall Effect in [15].

A particularly interesting situation appears in applications of conformal maps to two-dimensional free-streamline problems for jets, wakes, and cavities by the classical *hodograph* method [2, 16]. It is well known that this method reduces an idealized free-streamline problem to a conformal mapping problem involving a polygon in the log-hodograph plane. But it is rarely emphasized in the literature that only in special cases are the dimensions of this polygon specified a priori. Much more often, some of them are unknown, but must be chosen properly in order that certain dimensions in the physical plane come out right. Again one gets a mixture of geometric conditions in several different planes.

In these and analogous examples, Schwarz-Christoffel mapping is revealed as a broader problem than just the treatment of prescribed polygons. In each case the procedure for obtaining a solution is to first give careful thought to specifying exactly what unknown parameters exist and what mathematical facts are known that determine them uniquely. This collection of knowns and unknowns constitutes a generalized parameter problem. If this is formulated in a natural way, there is a good chance it can be solved by a general-purpose program for nonlinear systems of equations.

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Abstract

To compute the electrical resistance ($\approx$ conformal modulus) of a polygonally shaped resistor cut from a sheet of uniform resistivity, it suffices to find a conformal map of the polygon onto a rectangle. Constructing such a map requires the solution of a Schwarz-Christoffel parameter problem. First we show by examples that this is practical numerically. Then we consider an inverse

"resistor trimming" problem in which the aim is to cut a slit in a given polygon just long enough to increase its resistance to a prescribed value. We show that here the solution can be obtained by solving a "generalized parameter problem." The idea of a generalized parameter problem is applicable also in many other Schwarz-Christoffel computations.

Zusammenfassung

Um den elektrischen Widerstand eines polygonalen Resistors aus einem Material homogener Leitfähigkeit zu berechnen, genügt es, eine konforme Abbildung des Polygons auf ein Rechteck zu finden. Die Konstruktion einer solchen Abbildung erfordert die Lösung eines Schwarz-Christoffelschen Parameterproblems. Wir zeigen zunächst anhand von Beispielen, daß dies numerisch durchführbar ist. Dann betrachten wir ein inverses Problem: Die Aufgabe besteht hier darin, einen Schlitz in ein gegebenes Polygon zu schneiden, dessen Länge gerade so gewählt ist, daß der Widerstand auf einen vorgegebenen Wert erhöht wird. Wir zeigen, daß dieses Problem auf ein „verallgemeinertes Parameterproblem" zurückgeführt werden kann. Die Idee des verallgemeinerten Parameterproblems ist auch auf viele weitere Schwarz-Christoffel-Probleme anwendbar.

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