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Padé, Stable Padé, and Chebyshev-Padé Approximation

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Abstract

Chebyshev-Padé (CP) approximation is the approximation of a function $F(x)$ on $[-1,1]$ by a rational function $R^{cp}(x)$ of type (m,n) chosen to match the Chebyshev expansion of F as far as possible. We propose a new construction of R^{cp} based on a reduction of the CP problem to a problem of "stable Padé approximation of a Laurent series." This approach clarifies the troublesome issues of approximants with $m<n$ and of poles in the associated complex unit disk. To develop it we present in sequence, from a novel point of view, the elements of Padé approximation, Padé approximation of a Laurent series, formal CP approximation, stable Padé approximation, and finally true CP approximation. Among other things, we prove that the following are equivalent: (a) $F-R^{cp}=O(T_{m+n+1-\delta})$, where δ is the "defect"; (b) R^{cp} is unique; and (c) the associated Padé approximant has no poles in the disk. New results are also outlined concerning the recursive computation of CP approximants and their convergence as $m+n\to\infty$.

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Contents

Notation

I	interval $[-1,1]$
S	circle $\|z\| = 1$
D	disk $\|z\| \leq 1$
P	set of formal power series
L	set of formal Laurent series
T	set of formal Chebyshev series
R_{mn}^*	set of all rational functions of type (m,n)
R_{mn}	set of rational functions of type (m,n), no poles at 0
R_{mn}^D, R_{mn}^I	subsets of R_{mn} of functions with no poles on D, I
$\tilde{R}_{mn}$, $\tilde{R}_{mn}^D$	analogous sets of "extended rational functions"
$\hat{R}_{mn}^D$	subset of $\tilde{R}_{mn}^D$ of functions with symmetric Laurent series
(m,n)	type of a rational function
(μ,ν)	exact type ($\mu \leq m$, $\nu \leq n$)
δ	defect $\min\{m-\mu, n-\nu\}$
r^{p}	Padé approximant (in R_{mn})
$\tilde{r}^{\mathrm{p}}$	Padé approximant to a Laurent series (in $\tilde{R}_{mn}$)
r^{sp}, $\tilde{r}^{\mathrm{sp}}$	stable Padé approximants (in R_{mn}^D, $\tilde{R}_{mn}^D$)
$\hat{r}^{\mathrm{sp}}$	symmetric stable Padé approximant (in $\hat{R}_{mn}^D$)
R^{fcp}	formal Chebyshev-Padé approximant (in R_{mn}^*)
R^{cp}	Chebyshev-Padé approximant (in R_{mn}^I)
$f(z) = O(z^K)$	all Laurent coefficients of f of degree $-\infty < k < K$ are zero
$f(z) = \hat{O}(z^K)$	all Laurent coefficients of f of degree $-K < k < K$ are zero
$f(z) \equiv O(z^K)$	$f(z) = O(z^K)$, $\neq O(z^{K-1})$

0. INTRODUCTION

If $F(x)$ is continuous on $I = [-1,1]$, it has a Chebyshev expansion

$$F(x) = 2\sum_{k=0}^{\infty}{}' c_k T_k(x), \qquad c_k = \frac{1}{\pi}\int_{-1}^{1} \frac{F(x)T_k(x)}{\sqrt{1-x^2}}dx, \tag{0.1}$$

which converges uniformly if F is, say, Lipschitz continuous [5]. Here T_k is the kth Chebyshev polynomial of the first kind, and $\sum'$ denotes a sum in which the term with $k=0$ is multiplied by $\frac{1}{2}$. Let R_{mn} be the set of rational functions of type (m,n) (precise definitions begin in Section 1). A type (m,n) *Chebyshev-Padé (CP) approximant* to F on I is any function $R^{cp} \in R_{mn}$, with no poles on I, such that the Chebyshev expansions of F and R^{cp} agree to as many terms as possible:

$$(F-R^{cp})(x) = O(T_{\max}(x)). \tag{0.2}$$

The motivation behind CP approximation is that it is a natural generalization to $x \in I$ of Padé approximation at $z=0$. If $f(z)$ is a formal power series, then a type (m,n) *Padé approximant* to f is a function $r^p \in R_{mn}$ that satisfies

$$(f-r^p)(z) = O(z^{\max}). \tag{0.3}$$

To perfect the analogy we will actually take F to be a formal Chebyshev series rather than a function.

For example, Table 1 lists the L^∞ errors in type (2,2) rational approximation of various kinds to the functions $F(x) = e^x$ and $F(x) = |x|$. Notice that in both cases the CP approximant, unlike the Padé approximant, comes within a small factor of optimal. This behavior is typical. The so-called Carathéodory-Fejér (CF) approximant [20], based on the eigenvalue analysis of a Hankel matrix of Chebyshev coefficients, is often far closer to best than R^{cp}, especially when F is smooth. But for some purposes the reduction in error from CP to CF will be unimportant, while the greater simplicity of CP approximation is an attraction.

	e^x	$\lvert x\rvert$
Padé	40.0 (-4)	-
CP	1.45 (-4)	.106
CF	0.8689911 (-4)	.047
Best	0.8689910 (-4)	.044

Table 1. L^∞ errors for various type (2,2) approximants to e^x and $|x|$ on $I = [-1,1]$.

Despite the naturalness of the idea, Chebyshev-Padé approximation was introduced only in the early 1970's. (A related approximation method was devised by Maehly a decade earlier and is described in [5].) First, Frankel and Gragg published a short note in 1973 showing how to derive CP approximants by means of a transplantation to the complex unit disk $D: |z| \leq 1$, but only for $m \geq n$ [9]. Independently, Clenshaw and Lord in 1974 derived CP approximants for all m and n by a different method, and gave many practical examples [6]. Gragg and Johnson then extended their method to $m < n$ in 1974 [12,13]. Both the Clenshaw-Lord and Gragg derivations, however, are conceptually somewhat unsatisfactory, for they make use of nonlinear algebraic calculations that reduce rather inexplicably to a linear result in the end. We believe that the question of how to treat $m < n$ has been one of two main points of confusion in the CP approximation literature to date.

A new difficulty was pointed out in 1981 by Geddes in a paper on block structure and related matters [10]. Both Clenshaw and Lord and Gragg assert that their constructions are guaranteed to succeed except in cases where a pole appears on I, which corresponds, as we will explain, to the circle $S: |z| = 1$ in the complex plane. But Geddes realized that in fact, the construction breaks down for a much larger set of cases with poles in the disk D. In such a situation, he pointed out, the CP approximant defined by (0.2) may fail to be unique. A similar problem has also been mentioned by Ellacott in the context of Faber-Padé approximation on a more general domain than an interval [7]. This question of poles in D and their relationship to nonuniqueness has been the second main point of confusion in CP approximation.

We must also mention the recent work of Bultheel [3,4], who investigates *formal CP approximation,* which amounts to saying that he ignores the question of troublesome poles. (Gragg does the same, implicitly.) Bultheel derives many results that are related to ours and go well beyond them in certain directions. In particular, following Gragg, he treats the generalization to ***Laurent-Padé approximation,*** in which one approximates an arbitrary Laurent series rather than the special Chebyshev case of a series real and symmetric. An intermediate case between Chebyshev-Padé and Laurent-Padé approximation is ***Fourier-Padé approximation,*** in which the Laurent series is conjugate-symmetric but not necessarily real [12,13], but it appears that the difference between Chebyshev-Padé and Fourier-Padé approximation is not substantial.

In this paper we propose a new reduction of the CP problem on I to a Padé problem on D. Ours is the first such reduction that is a complete equivalence, transplanting the interval to the disk with no exceptional cases. We believe that this new reduction settles the two difficulties mentioned above in a natural way.

We will proceed by developing five related approximation problems in sequence, emphasizing their interconnections as far as possible. For each problem, the ideal is to set forth the essentials in the following order:

- definition of function spaces,
- definition of approximant r,
- method of construction of r,
- characterization of r,
- block structure corollary,
- examples.

Here are the five problems, treated in Sections 1-5. The various block structure results are summarized in Figures 1 and 2.

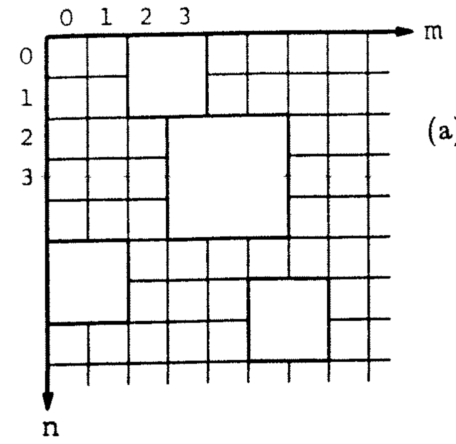
1. Padé approximation (r^{p}). f is a power series, and the Padé table fills a quarter-plane tiled by square blocks, except for one possible anomaly (Figure 1a). This problem is standard, but our "equioscillation" characterization theorem may be unfamiliar to many readers.

2. Padé approximation of a Laurent series ($\tilde{r}^{\mathrm{p}}$). Here we generalize standard Padé approximation by permitting f to be a Laurent series, and the Padé table expands to the half-plane $m \in \mathbf{Z}$, $n \geq 0$, now tiled by square blocks with no anomalies (Figure 1b). If $f(z) = f(z^{-1})$, the block pattern is symmetric about the line $m = -\frac{1}{2}$ (Figure 1c).

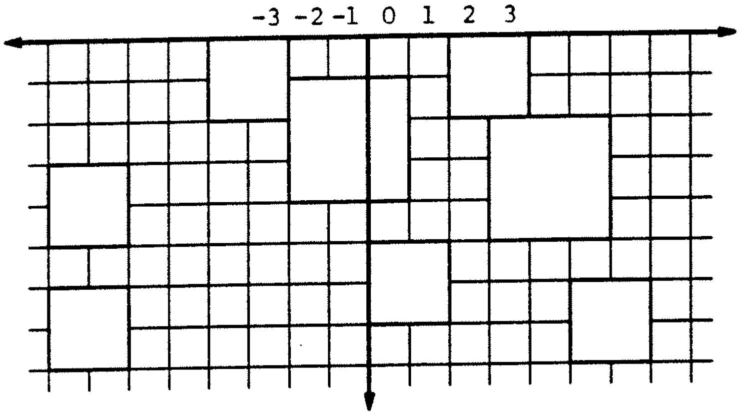
3. Formal CP approximation (R^{fcp}). This approximant is identical to the CP approximants of Gragg and Bultheel, but we define it by a new algorithm based on reduction to problem 2, which clarifies the case $m < n$. This is the only one of our five approximants that is defined by an algorithm rather than an optimality condition, and R^{fcp} is not always guaranteed to satisfy (0.2). The formal CP table occupies a quarter-plane, with square blocks throughout except that the approximant $R^{\mathrm{fcp}} \equiv 0$ may occupy one or more half-square blocks along the edge $m = 0$ (Figure 2a).

4. Stable Padé approximation (r^{sp}, $\tilde{r}^{\mathrm{sp}}$). Here problems 1 and 2 are modifed by requiring an approximant whose denominator has no zeros in D. Uniqueness fails in general, and the Padé table is tiled by square blocks containing unique entries interspersed with arbitrary regions of nonuniqueness (Figure 2b).

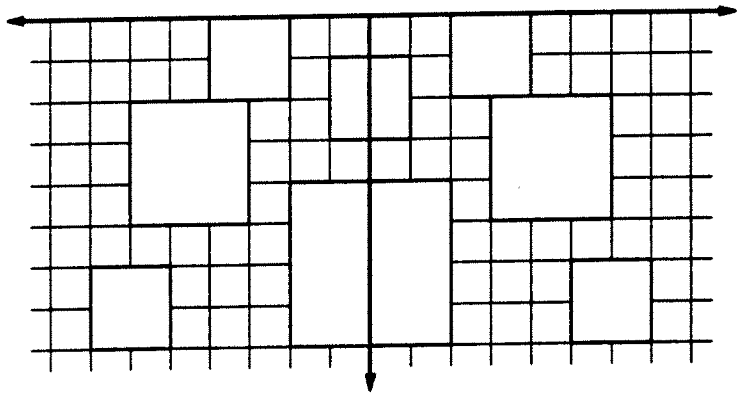
5. CP approximation (R^{cp}). R^{cp} is defined by the optimality condition (0.2). We show that in most cases, including all those with $m \geq \frac{1}{2}(n-1)$, R^{cp} can be constructed from $\tilde{r}^{\mathrm{sp}}$ exactly as R^{fcp} was constructed from $\tilde{r}^{\mathrm{p}}$. But for a universally applicable construction, one has to introduce a "symmetrized" version of $\tilde{r}^{\mathrm{sp}}$ denoted by $\hat{r}^{\mathrm{sp}}$. Like the stable Padé table, the CP table contains square blocks of unique entries surrounded by arbitrary regions of nonuniqueness (Figure 2c).



(a) Padé approximation
($r^{\mathrm{P}} \approx f$)

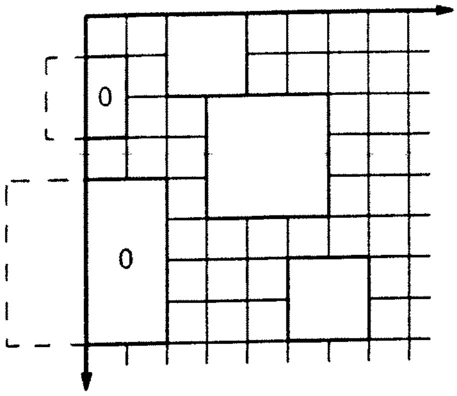


(b) Padé approximation of a
Laurent series ($\tilde{r}^{\mathrm{P}} \approx f$)

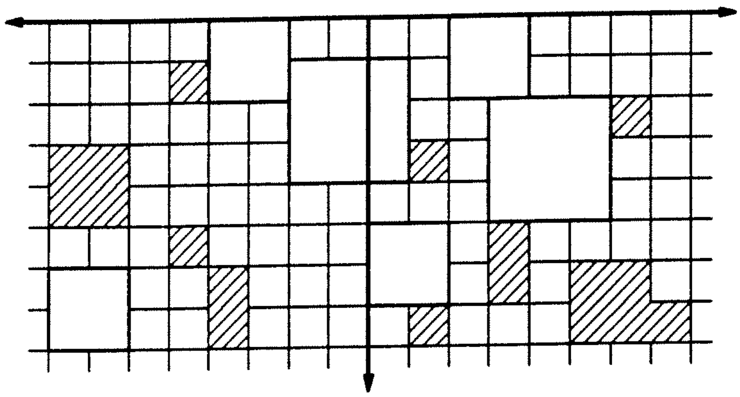


(c) Padé approximation of a
symmetric Laurent series
($\tilde{r}^{\mathrm{P}} \approx f,\ \ f(z)=f(z^{-1})$)

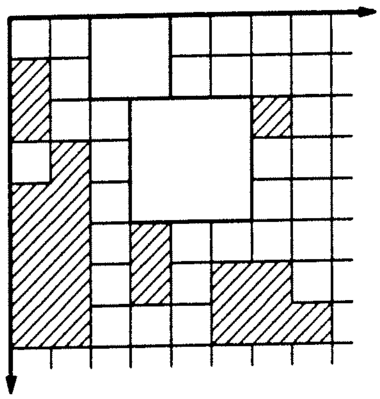
Figure 1. Examples of possible block structure patterns in Padé approximation of power series and Laurent series.



(a) Formal Chebyshev-Padé approximation ($R^{\mathrm{fcp}} \approx F$)



(b) Stable Padé approximation of a Laurent series ($\tilde{r}^{\mathrm{sp}} \approx f$)



(c) Chebyshev-Padé approximation ($R^{\mathrm{cp}} \approx F$)

Figure 2. Examples of possible block structure patterns in Chebyshev-Padé and stable Padé approximation. Shaded regions indicate nonunique approximants, for which the usual characterization and block structure results fail (see Theorems 4.1, 5.3).

The reader will notice that in contrast with the usual practice in the literature of Padé approximation, matrices and determinants rarely appear in this paper. Instead the development is based mainly on optimality criteria and the equioscillation characterization, Theorem 1.1. We believe that this is conceptually more straightforward than to rely principally on linear algebra, and that it relates Padé approximation more naturally to other areas of approximation theory. But of course linear algebra becomes the central issue as soon as one turns to computational algorithms.

Our results suggest that true CP approximation, like stable Padé approximation, has inherent difficulties that may be unavoidable. Indeed, we doubt whether any useful characterization can be found for $\tilde{r}^{sp}$ and R^{cp} in the nonunique cases. In practice, it appears that these cases do not occur very often. But for applications where they do occur it may be a good idea to abandon (0.2) and devise new classes of approximants that are more tractable, such as the pole-deletion procedure of Foster [8]. Or one can have recourse to CF approximation, which has the advantage that the approximants it delivers are always unique and stable.

1. PADE APPROXIMATION

Let R_{mn} denote the set of formal rational functions

$$r(z) = \frac{p(z)}{q(z)} = \sum_{k=0}^{m} a_k z^k \Big/ \sum_{k=0}^{n} b_k z^k, \qquad b_0 = 1, \tag{1.1}$$

where p and q are relatively prime. Given $r \in R_{mn}$, let $\mu \leq m$ and $\nu \leq n$ be the exact degrees of p and q, with $\mu = -\infty$ if $p \equiv 0$. We say that r has *exact type* (μ, ν). Throughout this paper, for simplicity, all coefficients are real. The *defect* of r with respect to R_{mn} is defined by

$$\delta = \min\{m-\mu, n-\nu\} \geq 0. \tag{1.2}$$

Let P be the set of formal power series $\sum_{k=0}^{\infty} c_k z^k$. Each $r \in R_{mn}$ will also be associated with a series $r \in P$ by a Taylor expansion at $z = 0$.

Definition of r^p. Given $f \in P$ and $m, n \geq 0$, a type (m, n) *Padé approximant* to f is any $r^p \in R_{mn}$ such that

$$(f - r^p)(z) = O(z^K), \tag{1.3}$$

where $K \geq 0$ is as large as possible.

Construction of r^p. It is well known that Padé approximants can be constructed by solving a linear system of equations in Hankel form [1,11]. We will not give the details.

The following "equioscillation characterization" of r^{p} comes from [19]. The uniqueness and "if" assertions follow easily from a zero-counting argument, but "only if" is more complicated.

THEOREM 1.1 - CHARACTERIZATION OF r^{p}. *For any $f \in P$ and $m,n \geq 0$, there exists a unique type (m,n) Padé approximant r^{p} to f. A function $r \in R_{mn}$ with defect δ is equal to r^{p} if and only if*

$$(f-r)(z) \;=\; O(z^{m+n+1-\delta}). \tag{1.4}$$

This characterization leads to the following standard result on square blocks in the Padé table. By the Padé table, we mean the set of Padé approximants r_{mn}^{p} for all $m,n \geq 0$ to a fixed function f, arranged in a quadrant in the (m,n) plane. From now on, $f(z) \equiv O(z^K)$ is an abbreviation for $f(z) = O(z^K)$, $f(z) \neq O(z^{K+1})$.

COROLLARY 1.1 - SQUARE BLOCKS *(see Figure 1a). The Padé table breaks into precisely square blocks containing identical entries. (One of these is infinite in extent if $f \in R_{mn}$ for large enough m and n.) There is one possible exception: if $f(z) \equiv O(z^K)$ for some $K > 0$, then the approximant $r^{\mathrm{p}} \equiv 0$ occupies the infinite rectangular strip $0 \leq m \leq K-1$, $0 \leq n < \infty$ at the left edge of the table.*

Proof. Suppose $r \not\equiv 0$ is a rational function of exact type (μ,ν), $\mu,\nu \geq 0$, with $(f-r)(z) \equiv O(z^{\mu+\nu+1+\Delta})$ for some $\Delta \in \mathbf{Z}$. (If this is not true for any finite Δ, then $r = f$, and we have the case of an infinite square block.) We now ask: for which (m,n) with $m \geq \mu$, $n \geq \nu$ is r the Padé approximant to f? By Theorem 1.1, the condition for this is $\mu+\nu+1+\Delta \;\geq\; m+n+1-\delta$, i.e.

$$\mu+\nu+\Delta \;\geq\; m+n - \min\{m-\mu, n-\nu\}.$$

Expanding the minimum into its two cases reduces this to

$$m \leq \mu+\Delta \qquad \text{and} \qquad n \leq \nu+\Delta.$$

Thus if $\Delta \geq 0$, $r = r^{\mathrm{p}}$ precisely for (m,n) in the square block of size $(\Delta+1)\times(\Delta+1)$ with upper-left corner (μ,ν).

If $r \equiv 0$, then by definition $\mu = -\infty$ and $\delta = n$. If $f(z) \equiv O(z^K)$ for some $K > 0$, then condition (1.4) becomes $m \leq K-1$, as asserted. ∎

Example 1.1. For which (m,n), if any, is $r(z) = 1/(1+\frac{1}{2}z^2)$ the Padé approximant to $f(z) = \cos z = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \cdots$? Here $\mu = 0$ and $\nu = 2$, and since $1/(1+\frac{1}{2}z^2) \;=\; 1 - \frac{1}{2}z^2 + \frac{1}{4}z^4 - \cdots$, $(f-r)(z) \equiv O(z^4)$. For $(m,n) = (0,2)$ we have $\delta = 0$ and therefore $m+n+1-\delta = 3 \leq 4$, so $r = r^{\mathrm{p}}$. For $(m,n) = (0,3)$ or $(1,2)$ we have $\delta = 0$ and $m+n+1-\delta = 4 \leq 4$, so $r = r^{\mathrm{p}}$. For $(m,n) = (1,3)$ we have $\delta = 1$ and $m+n+1-\delta = 4 \leq 4$ again, so $r = r^{\mathrm{p}}$ once more. All other choices $m \geq \mu$, $n \geq \nu$ lead to $m+n+1-\delta > 4$, so $r \neq r^{\mathrm{p}}$. Thus $r = r^{\mathrm{p}}$ precisely in the 2×2 block with upper-left corner (0,2). In fact the Padé table for $\cos z$ breaks into 2×2 blocks throughout. For $\sin z$ it is the same, except that the blocks are shifted one position to the right and the entire column $m=0$, $n\geq 0$ is a rectangular strip with $r^{p}\equiv 0$. //

2. PADE APPROXIMATION OF A LAURENT SERIES

Now let L be the set of formal Laurent series $\sum_{k=-\infty}^{\infty} c_k z^k$, and let $P^{\perp}$ be the subset of "coanalytic" formal series $\sum_{k=-\infty}^{-1} c_k z^k$. For $n\geq 0$ and *arbitrary* $m\in \mathbf{Z}$, we define

$$\tilde{R}_{mn} = z^{m-n+1}(P^{\perp}+R_{n-1,n}).$$

(This definition is analogous to the definition of $\tilde{R}_{mn}$ in [18], but not the same.) In other words, $\tilde{R}_{mn}$ is the set of formal expressions

$$\begin{aligned}\tilde{r}(z) &= r^-(z)+r^+(z) = r^-(z)+\frac{p(z)}{q(z)}\\ &= \sum_{k=-\infty}^{m-n} d_k z^k + \sum_{k=m-n+1}^{m} e_k z^k \Big/ \sum_{k=0}^{n} b_k z^k, \qquad b_0\neq 1,\end{aligned} \tag{2.1}$$

where p and q are relatively prime. Each $\tilde{r}\in \tilde{R}_{mn}$ will be associated with a series $\tilde{r}\in L$ by a Taylor expansion of $z^{-m+n-1}r^+(z)$ at $z=0$. A statement $f(z)=O(z^K)$, $K\in\mathbf{Z}$, will now indicate that f has zero coefficients for all k in the range $-\infty<k<K$. Again, $f(z)\equiv O(z^K)$ is an abbreviation for $f(z)=O(z^K)$, $f(z)\neq O(z^{K+1})$.

Other equivalent definitions of $\tilde{R}_{mn}$ are also possible. In particular, we might have split the Laurent series one term earlier, setting $\tilde{R}_{mn}=z^{m-n}(P^{\perp}+R_{nn})$, but this turns out to be less convenient.

We need to be able to compare functions in different spaces $\tilde{R}_{mn}$. Let us say that $\tilde{r}_1=r_1^-+r_1^+\in\tilde{R}_{m_1n_1}$ and $\tilde{r}_2=r_2^-+r_2^+\in\tilde{R}_{m_2n_2}$ are equal if they have the same formal Laurent series. This can be tested in a finite number of steps, since evidently all the coefficients in a representation (2.1) are determined once we choose the position $m-n$ at which the Laurent series is split. Set $l_1=m_1-n_1$ and $l_2=m_2-n_2$, and assume without loss of generality $l_1\leq l_2$. For $\tilde{r}_1$ to be equal to $\tilde{r}_2$, r_1^- must be equal to the degree $\leq l_1$ part of r_2^-, and the remainder of $\tilde{r}_2$ must be equivalent to the rational function r_1^+. This can be verified by cross-multiplying so that the terms $d_{l_1+1}z^{l_1+1},\ldots,d_{l_2}z^{l_2}$ of r_2^- are brought into the numerator.

The denominator q will be the same in any representation of $\tilde{r}$. Let $\nu\geq 0$ be its exact degree, and let μ be the minimal value m for which $\tilde{r}\in\tilde{R}_{m\nu}$, or $-\infty$ if there is no minimum. To determine μ, start with any representation $\tilde{r}=r^-+p/q\in\tilde{R}_{m\nu}$, and then cross-multiply to bring the term $d_{m-\nu} z^{m-\nu}$ into the numerator. If the resulting numerator has a nonzero coefficient of degree m, then $\mu = m$, but otherwise $\tilde{r} \in \tilde{R}_{m-1,\nu}$, and one can decrement m and repeat the process.

We say that $\tilde{r}$ has *exact type* (μ,ν). It is readily verified that $\tilde{r} \in \tilde{R}_{mn}$ if and only if $m \geq \mu$ and $n \geq \nu$. The defect δ is again defined by (1.2).

One can also compute μ in one step by cross-multiplying to bring $\tilde{r}$ into the form of an "extended rational function":

$$\tilde{r}(z) = \frac{\tilde{p}(z)}{q(z)} = \sum_{k=-\infty}^{m} a_k z^k \Big/ \sum_{k=0}^{n} b_k z^k . \tag{2.2}$$

Now μ, ν are simply the exact degrees of $\tilde{p}$ and q. The trouble with this representation is that it is not possible to go backwards from (2.2) to (2.1) or to a Laurent series.

Example 2.1. The function $\tilde{r}(z) = (\cdots + z^{-3} + z^{-2} + z^{-1}) + 2/(1-2z) \in \tilde{R}_{01}$ has exact type $(-1,1)$. It can be rewritten as $(\cdots + z^{-3} + z^{-2}) + z^{-1}/(1-2z) \in \tilde{R}_{-1,1}$, but not as a function in $\tilde{R}_{-2,1}$. //

Example 2.2. The function $\tilde{r}(z) = (\cdots + z^{-3} + z^{-2} + z^{-1}) + 1/(1-z) \in \tilde{R}_{01}$ has exact type $(-\infty,1)$, and can be represented in every space $\tilde{R}_{mn}$ with $n \geq 1$. This example shows that $\mu = -\infty$ does not imply $\tilde{r} \equiv 0$ unless, as in standard Padé approximation, $f(z) = O(z^K)$ for some $K > -\infty$. Of course in a sense $\tilde{r}$ is the zero function, having been obtained by subtracting one representation of $1/(1-z)$ from another, and the corresponding extended rational function (2.2) has numerator $\tilde{p} \equiv 0$. But the formal Laurent series of $\tilde{r}$ is nonzero, and therefore $\tilde{r}$ is nonzero as a member of any space $\tilde{R}_{mn}$. //

Now we are ready for Padé approximation of a Laurent series.

Definition of $\tilde{r}^{\mathrm{P}}$. Given $f \in L$ and $m \in \mathbf{Z}$, $n \geq 0$, a type (m,n) *Padé approximant* to f is any $\tilde{r}^{\mathrm{P}} \in \tilde{R}_{mn}$ such that

$$(f - \tilde{r}^{\mathrm{P}})(z) = O(z^K), \tag{2.3}$$

where $K \in \mathbf{Z}$ is as large as possible.

Construction of $\tilde{r}^{\mathrm{P}}$. It is an easy matter to construct $\tilde{r}$ from a Padé approximant of the usual sort. Divide f into two pieces,

$$f(z) = f^{-}(z) + f^{+}(z) = \sum_{k=-\infty}^{m-n} c_k z^k + \sum_{k=m-n+1}^{\infty} c_k z^k , \tag{2.4}$$

set $r^{-} = f^{-}$, and take r^{+} equal to $z^{m-n+1} r^{p}_{n-1,n}(z^{n-m-1} f^{+})$, that is, z^{m-n+1} times the type $(n-1,n)$ Padé approximant of the usual sort to $z^{n-m-1} f^{+}(z)$. (If $n = 0$, then $r^{+} \equiv 0$.) The justification of this construction comes directly from the definition of $\tilde{r}^{\mathrm{P}}$ and a comparison of (2.1) and (2.4). It leads to the following characterization theorem, a translation of Theorem 1.1 to the present context:

THEOREM 2.1 - CHARACTERIZATION OF $\tilde{r}^P$. *For any $f \in L$ and $m \in \mathbf{Z}$, $n \geq 0$, there exists a unique type (m,n) Padé approximant $\tilde{r}^P$ to f. A function $\tilde{r} \in \tilde{R}_{mn}$ with defect δ is equal to $\tilde{r}^P$ if and only if*

$$(f - \tilde{r})(z) = O(z^{m+n+1-\delta}). \tag{2.5}$$

Proof. The construction above establishes a one-to-one correspondence between approximants $r \in R_{n-1,n}$ to $z^{n-m-1}f^+(z)$ and $\tilde{r}(z) = f^-(z) + z^{m-n+1}r(z) \in \tilde{R}_{mn}$ to $f(z)$. Given $r \in R_{n-1,n}$, let its exact degrees and defect be $\mu_0+n-m-1$, ν (the same as for $\tilde{r}$), and δ_0. By Theorem 1.1, $z^{n-m-1}f^+(z) - r(z) = O(z^{2n-\delta_0})$ is a necessary and sufficient condition for $r = r^P_{n-1,n}(z^{n-m-1}f^+)$. Equivalently, $(f - \tilde{r})(z) = O(z^{m+n+1-\delta_0})$ is a necessary and sufficient condition for $\tilde{r} = \tilde{r}^P_{mn}(f)$. Therefore we are done if we can show $\delta = \delta_0$. Suppose first $\delta_0 = n - \nu \leq (n-1) - (\mu_0+n-m-1) = m - \mu_0$. Then cross-multiplication gives $\tilde{r} \in \tilde{R}_{m-n+\nu,\nu} = R_{m-\delta_0,\nu}$, so $\mu \leq m - \delta_0$, and this implies $\delta = \delta_0$. On the other hand suppose $\delta_0 = (n-1) - (\mu_0+n-m-1) = m - \mu_0 < n - \nu$. Then cross-multiplication gives $\tilde{r} \in \tilde{R}_{\mu_0,\nu}$, $\notin \tilde{R}_{\mu_0-1,\nu}$, so in fact $\mu = \mu_0$, and again $\delta = \delta_0$. ∎

Since m is an arbitrary integer, the Padé table now inhabits a half-plane instead of a quarter-plane. Here is the corresponding block structure result.

COROLLARY 2.1 - SQUARE BLOCKS *(see Figure 1b). The Padé table for a Laurent series breaks into precisely square blocks containing identical entries. (One of these is infinite in extent if $f \in \tilde{R}_{mn}$ for large enough m and n, and another if $f(z) \equiv O(z^K)$ for some $K > -\infty$.)*

Proof. Essentially the same as for Corollary 1.1. ∎

Note that in Corollary 2.1, the anomaly of an infinite rectangular strip of Corollary 1.1 has vanished. If $f(z) \equiv O(z^K)$ for some $K > -\infty$, the Padé table now has an infinite square block occupying all the positions $m \leq K-1$, $n \geq 0$. The rectangular strip was the intersection of this block with the quarter-plane $m, n \geq 0$.

Example 2.3. For $f(z) = z^{-1}+1+z$ and $(m,n) = (-1,1)$, we get $\tilde{r}^P = r^- + r^+ = 0 + z^{-1}/(1-z)$. Since $(f - \tilde{r}^P)(z) = O(z^2)$, Theorem 2.1 implies that $\tilde{r}^P$ must also be the Padé approximant of types $(0,1)$, $(-1,2)$, and $(0,2)$. If we construct the type $(-1,2)$ or $(0,2)$ approximant directly from (2.4), we get $0 + z^{-1}/(1-z)$ again, but the $(0,1)$ approximant comes out as $\tilde{r}^P = r^- + r^+ = z^{-1} + 1/(1-z)$, another representation of the same function. //

The following specialized result will be needed for applications to CP approximation (cf. [3,4]).

THEOREM 2.2 - SYMMETRIC LAURENT SERIES *(see Figure 1c). Let $f \in L$ satisfy $f(z) = f(z^{-1})$. Then the block pattern of the Padé table for f is symmetric with respect to the line $m = -\frac{1}{2}$.*

Proof. It is well known that a Padé approximant has defect $\delta > 0$ if and only if the associated Hankel matrix is singular [21,22]. If $f(z) = f(z^{-1})$, then the Hankel matrices associated with type (m,n) and type $(-m,n)$ approximation are the same except for a reflection in the anti-diagonal, so either both are nonsingular or both are singular. Therefore the pattern of singular matrices in the Padé table is symmetric about $m = 0$. Since $\delta > 0$ occurs precisely in those positions of any square block that lie outside of its top row and left column, this implies that the square block pattern is symmetric about $m = -\frac{1}{2}$. ∎

The elements of Padé approximation of power series can be recovered from the results presented in this section for Laurent series by restricting attention to functions whose coefficients of negative degree are all zero.

3. FORMAL CHEBYSHEV-PADE APPROXIMATION

In any kind of Chebyshev-Padé approximation, the motivation is the condition (0.2). However, insisting on precisely this condition leads to the problems of pole location discovered by Geddes. Therefore we will first consider instead the *formal Chebyshev-Padé approximant,* R^{fcp}, defined to be the function produced by a certain algorithm, which will not always satisfy (0.2). Our R^{fcp} is the same as the approximant investigated by Gragg and Bultheel, but we believe the present construction is more transparent for $m < n$.

R^{fcp} will sometimes have poles on I, even at $x = 0$. Let R_{mn}^{I} be the subset of R_{mn} of all rational functions of type (m,n) with no poles on I. On the other hand let $R_{mn}^{*} \supseteq R_{mn}$ be the space of all rational functions of type (m,n), regardless of pole location. We can represent these in the form (1.1), but with the normalization $b_\nu = 1$ instead of $b_0 = 1$.

Let T be the set of formal Chebyshev series $2\sum'^{\infty}_{k=0} c_k T_k(x)$, and let the variables x and z be related by

$$x = \tfrac{1}{2}(z + z^{-1}), \qquad \text{hence} \qquad T_k(x) = \tfrac{1}{2}(z^k + z^{-k}). \tag{3.1}$$

The circle $z \in S$ corresponds to the interval $x \in I$ (covered twice). If $f \in L$ and $F \in T$ are defined by

$$F(x) = 2\sum_{k=0}^{\infty}{}' c_k T_k(x), \qquad f(z) = \sum_{k=-\infty}^{\infty} c_k z^k, \qquad c_k = c_{|k|}, \tag{3.2}$$

then $F(x) = f(z)$.

Definition of R^{fcp}. Given $F \in T$ and $m, n \geq 0$, the type (m,n) *formal Chebyshev-Padé (CP) approximant* to F is the function $R^{fcp} \in R^{*}_{mn}$ produced by the following construction.

Construction of R^{fcp}. Given $F \in T$, define $f \in L$ by (3.2), and let $\tilde{r}^{p} \in \tilde{R}_{mn}$ be the type (m,n) Padé approximant to f. We simply set

$$R^{fcp}(x) = \frac{P(x)}{Q(x)} = r(z) = \tilde{r}^{p}(z) + \tilde{r}^{p}(z^{-1}) - f(z). \tag{3.3}$$

Despite appearances, this formula does not contain infinitely many terms, for the two infinite tails of $\tilde{r}^{p}(z)$ and $\tilde{r}^{p}(z^{-1})$ are cancelled by those in f. In fact by (2.1), we have

$$r(z) = \frac{p(z)}{q(z)} + \frac{p(z^{-1})}{q(z^{-1})} - f^{0}(z), \tag{3.4}$$

where

$$f^{0}(z) = \begin{cases} \displaystyle\sum_{k=m-n+1}^{n-m-1} c_k z^k & \text{if } m < n, \\ \\ \displaystyle -\sum_{k=n-m}^{m-n} c_k z^k & \text{if } m \geq n. \end{cases} \tag{3.5}$$

Cross-multiplication gives

$$r(z) = \frac{p(z)q(z^{-1}) + p(z^{-1})q(z) - f^{0}(z)q(z)q(z^{-1})}{q(z)q(z^{-1})}, \tag{3.6}$$

and since both the numerator and the denominator are symmetric with respect to the inversion $z \rightarrow z^{-1}$, they correspond under (3.1) to polynomials $P(x)$ and $Q(x)$.

Thus R^{fcp} is a rational function, but it is not obvious from (3.6) that it has the appropriate type. This is one of the matters settled by the following theorem. A function r or $\tilde{r}$ is said to be ***stable*** if its denominator has no zeros in the closed disk D, and R^{D}_{mn} and $\tilde{R}^{D}_{mn}$ denote the corresponding subsets of R_{mn} and $\tilde{R}_{mn}$.

THEOREM 3.1 - PROPERTIES OF R^{fcp}. *For any $F \in T$ and $m, n \geq 0$, the construction above yields a function $R^{fcp} \in R^{*}_{mn}$. If $\tilde{r}^{p}$ has exact type (μ, ν) with $\mu \geq 0$, then R^{fcp} has exact type (μ, ν) also, while if $\mu < 0$, then $R^{fcp} \equiv 0$. If $\tilde{r}^{p}$ is stable with $(f - \tilde{r}^{p})(z) \equiv O(z^{K})$, then $R^{fcp} \in R^{I}_{mn}$ with $(F - R^{fcp})(x) \equiv O(T_K(x))$, and hence in this case*

$$(F - R^{fcp})(x) = O(T_{m+n+1-\delta}(x)). \tag{3.7}$$

If $\tilde{r}^{p}$ is stable with $\mu < 0$, then $\nu = 0$.

The Chebyshev series for R^{fcp} implicit on the left of (3.7) is the one obtained by the standard integral (0.1).

Proof. Assume first that $\tilde{r}^{\text{p}}$ has exact type (μ,ν) with $\mu \geq 0$, and let it be represented as a function (2.1) in $\tilde{R}_{\mu\nu}$. From (3.3) and (3.6) we see that since $b_0, b_\nu \neq 0$ in $q(z)$, $Q(x)$ has exact degree ν. It will take the value 0 at $x \in I$ whenever $q(z)$ takes the value 0 at the corresponding point $z \in S$.

To determine the degree of P, we multiply (3.3) by $q(z)q(z^{-1})$ to get

$$\begin{aligned} P(x) &= r(z)q(z)q(z^{-1}) = [\tilde{r}^{\text{p}}(z^{-1}) + (\tilde{r}^{\text{p}}(z) - f(z))]\,q(z)q(z^{-1}) \\ &= \tilde{p}(z^{-1})q(z) + [\tilde{r}^{\text{p}}(z) - f(z)]\,q(z)q(z^{-1}) \qquad (3.8) \end{aligned}$$

in the notation of (2.2). The first term has exact order $O(z^{-\mu})$, while since $(f-\tilde{r}^{\text{p}})(z) = O(z^{\mu+\nu+1})$ by Theorem 2.1, the second has order $O(z^{\mu+1})$. Thus $r(z)q(z)q(z^{-1})$ has exact order $O(z^{-\mu})$, and by symmetry, it contains terms of degrees $-\mu$ through μ. Therefore $P(x)$ has exact degree μ, as claimed.

If $\mu < 0$, on the other hand, the first term in (3.8) still has exact order $O(z^{-\mu}) = O(z^{|\mu|})$, while by Theorems 2.1 and 2.2, the second has order at least $O(z^{|\mu|})$ also. Since $|\mu| \geq 1$, symmetry implies that $P(x)$ and hence R^{fcp} must be identically zero.

Now, note that by (3.3),

$$(F - R^{\text{fcp}})(x) = (f - \tilde{r}^{\text{p}})(z) + (f - \tilde{r}^{\text{p}})(z^{-1}).$$

Therefore if $(f-\tilde{r}^{\text{p}})(z) \equiv O(z^K)$, which will occur only with $K \geq 1$, it follows that $(F-R^{\text{fcp}})(x) \equiv O(T_K(x))$ always holds (regardless of stability of $\tilde{r}^{\text{p}}$) if the Chebyshev series for R^{fcp} on the left is taken to be the formal Chebyshev series derived by (3.4) from the formal Laurent series for $p(z)/q(z)$ and $p(z^{-1})/q(z^{-1})$. If $\tilde{r}^{\text{p}}$ is stable, then these latter series actually converge on S, so they coincide with the Laurent series defined by Cauchy integrals on S. A transplantation to I gives the integral (0.1) that defines Chebyshev coefficients. Since $K \geq m+n+1-\delta$ by Theorem 2.1, this establishes (3.7).

Finally, if $\tilde{r}^{\text{p}}$ is stable and $\mu < 0$, both arguments above apply and we have $R^{\text{fcp}} \equiv 0$ and $(F-R^{\text{fcp}})(x) = O(T_{m+n+1-\delta}(x))$, where δ is the defect of $\tilde{r}^{\text{p}}$. Therefore $F(x) = O(T_{m+n+1-\delta}(x))$, i.e. $c_k = 0$ for $|k| < m+n+1-\delta$ $(\geq \mu+\nu+\delta+1)$. If we construct $\tilde{r}^{\text{p}}$ via (2.4) with $(m,n) = (\mu,\nu)$, then

$$z^{\nu-\mu-1} f^{+}(z) = z^{\nu-\mu-1} O(z^{\mu+\nu+\delta+1}) = O(z^{2\nu+\delta}),$$

hence $r^{+} = z^{\mu-\nu+1} r^{p}_{\nu-1,\nu}(z^{\nu-\mu-1} f^{+}) \equiv 0$, i.e. $\tilde{r}^{\text{p}} = f^{-}$ and $\nu = 0$. ∎

Example 3.1. The function $F(x) = a + 2x = z^{-1} + a + z$, $a \in \mathbf{R}$, has the type (0,1) Padé approximant

$$\tilde{r}^{p}(z) = z^{-1} + \frac{a^2}{a-z},$$

and substituting this in (3.3) gives

$$R^{fcp}(x) = (z^{-1} + \frac{a^2}{a-z}) + (z + \frac{a^2}{a-z^{-1}}) - (z^{-1} + a + z)$$

$$= \frac{a^3 - a}{(a^2+1) - a(z+z^{-1})} = \frac{a^3 - a}{(a^2+1) - 2ax}.$$

If $|a| > 1$, then $\tilde{r}^p$ and R^{fcp} have exact type (0,1), $\tilde{r}^p$ is stable, $R^{fcp} \in R^I_{mn}$, and $(F - R^{fcp})(x) \equiv O(T_2(x))$. If $|a| < 1$, then $\tilde{r}^p$ and R^{fcp} have exact type (0,1) and $R^{fcp} \in R^I_{mn}$, but $\tilde{r}^p$ is unstable and $(F - R^{fcp})(x) \equiv O(T_0(x))$. If $a = \pm 1$, then $\tilde{r}^p$ has exact type (-1,1) and is unstable, $R^{fcp} \equiv 0$, and $(F - R^{fcp})(x) \equiv O(T_0(x))$ again. //

The formal CP table occupies the quarter-plane $m, n \geq 0$. Here is the block structure result that follows from Theorem 3.1.

COROLLARY 3.1 - SQUARE BLOCKS *(see Figure 2a). The formal CP table for a Chebyshev series $F \in T$ breaks into precisely square blocks containing identical entries, except that the entry $R^{fcp} \equiv 0$ appears not in a square block but in some number (zero, finite, or infinite) of half-square blocks along the left edge of the table of the form $0 \leq m \leq J$, $n_0 \leq n \leq n_0 + 2J + 1$ for some $J \geq 0$.*

Proof. By Corollary 2.1, the Padé table for f is a half-plane divided into square blocks. In any such block that lies entirely in the quarter-plane $m, n \geq 0$, $\tilde{r}^p$ is a function of exact type (μ, ν) for some $\mu \geq 0$, and by Theorem 3.1, there will then be a corresponding block in the formal CP table containing an entry R^{fcp} of exact type (μ, ν). (Since different blocks of this sort contain functions of differing exact type, the functions themselves are of course distinct.) On the other hand in the Padé table for f there may be some blocks of even dimension that straddle the line of symmetry $m = -\frac{1}{2}$, for which one has $\mu < 0$, and by Theorem 3.1, all of these map to the entry $R^{fcp} \equiv 0$. ∎

Example 3.1, continued. For the function $F(x) = a + 2x$, the Padé table for f and the formal CP table for F both contain an infinite square block $m \geq 1$, $n \geq 0$, so the question is what happens for $m < 1$. We have already seen that for $a = 1$, the Padé table for f contains a 2×2 block with upper-left corner (-1,1). By considering Hankel determinants, one can show that the strip $-1 \leq m \leq 0$ in the Padé table for f is actually tiled by an infinite sequence of a pair of 1×1 blocks $(-1, 3J)$, $(0, 3J)$ alternating with a 2×2 block with upper-left corner $(-1, 3J+1)$, for all $J \geq 0$. Therefore in this case the formal CP table for F has a nonzero entry in every third position along the left edge $m = 0$, but $R^{fcp} \equiv 0$ in each 2×1 area in-between.

For $a=0$, on the other hand, the Padé table for f is tiled by 2×2 blocks throughout the strip $-1\le m\le 0$, and in the formal CP table for F, these fuse into an infinite rectangular strip $m=0$, $n\ge 0$ of entries $R^{fcp}\equiv 0$. //

Example 3.2. The function $F(x)=|x|$ has the Chebyshev expansion [5, p. 132]

$$F(x)=\frac{2}{\pi}+\frac{4}{\pi}\left(\tfrac{1}{3}T_2(x)-\tfrac{1}{15}T_4(x)+\tfrac{1}{35}T_6(x)-\cdots\right),$$

which corresponds to

$$f(z)=\frac{2}{\pi}\left(\cdots-\tfrac{1}{15}z^{-4}+\tfrac{1}{3}z^{-2}+1+\tfrac{1}{3}z^{2}-\tfrac{1}{15}z^{4}+\cdots\right).$$

To compute R^{fcp} for $(m,n)=(0,2)$, we first compute the type (0,2) Padé approximant to f,

$$\tilde{r}^{P}(z)=\frac{2}{\pi}\left[\left(\cdots-\tfrac{1}{15}z^{-4}+\tfrac{1}{3}z^{-2}\right)+\frac{3}{3-z^2}\right],$$

and now (3.3) gives

$$R^{fcp}(x)=\frac{2}{\pi}\left[\frac{3}{3-z^2}+\frac{3}{3-z^{-2}}-1\right]$$

$$=\frac{2}{\pi}\left(\frac{8}{10-3(z^2+z^{-2})}\right)=\frac{4}{\pi(4-3x^2)}.$$

Since $\tilde{r}^{P}$ is stable, $|x|-R^{fcp}(x)=O(T_3(x))$ in this case, actually $O(T_4(x))$ since the functions are even. The L^∞ error on I is $1/\pi\approx .318$, as compared to .268 for the optimal Chebyshev approximant.

In approximation to $|x|$ of type (2,2), the results are

$$\tilde{r}^{P}(z)=\frac{2}{\pi}\left[\left(\cdots+\tfrac{1}{3}z^{-2}+1\right)+\frac{z^2/3}{1+z^2/5}\right],\qquad R^{fcp}(x)=\frac{4}{3\pi}\left(\frac{1+20x^2}{4+5x^2}\right).$$

Now the L^∞ error is $1/3\pi\approx .106$, as reported in Table 1. Again $\tilde{r}^{P}$ is stable, and we have $|x|-R^{fcp}(x)\equiv O(T_6(x))$. //

4. STABLE PADE APPROXIMATION

In stable Padé approximation, we are going to add the new condition that there are no poles in the disk D. This condition will be troublesome, but it reflects the difficulty inherent in CP approximation. Recall that $\tilde{r}\in\tilde{R}_{mn}$ is *stable* if its denominator has no zeros in D, and that $\tilde{R}^{D}_{mn}$ is the corresponding subset of $\tilde{R}_{mn}$. The definitions and theorems below treat approximation of Laurent series only, but the analogous developments for power series can be obtained by setting all Laurent coefficients of negative degree equal to zero. On the other hand for simplicity, the examples we give involve power series rather than Laurent series.

Definition of $\tilde{r}^{sp}$. Given $f \in L$ and $m \in \mathbf{Z}$, $n \geq 0$, a type (m,n) *stable Padé approximant* to f is any $\tilde{r}^{sp} \in \tilde{R}^D_{mn}$ such that

$$(f - \tilde{r}^{sp})(z) = O(z^K), \tag{4.1}$$

where $K \in \mathbf{Z}$ is as large as possible.

If $\tilde{r}^p$ is stable, the definition implies that $\tilde{r}^{sp} = \tilde{r}^p$, and therefore $\tilde{r}^{sp}$ is unique. The following example shows that uniqueness may be lost if $\tilde{r}^p$ is unstable.

Example 4.1. For $f(z) = 1 + z/2 + z^2/2$, the first four entries in the Padé table are

$$r^p_{00}(z) = 1, \quad r^p_{10}(z) = 1 + z/2, \quad r^p_{01}(z) = \frac{1}{1-z/2}, \quad r^p_{11}(z) = \frac{1-z/2}{1-z}.$$

The first three of these are stable, but r^p_{11} is not, and since r^p_{11} is the only function in R_{11} with $(f-r)(z) = O(z^3)$, it follows that any $r \in R^D_{11}$ with $(f-r)(z) = O(z^2)$ will suffice for r^{sp}_{11}. In particular, $r^{sp}_{11} = r^p_{01}$ and $r^{sp}_{11} = r^p_{10}$ are suitable candidates, and thus r^{sp}_{11} is nonunique. //

We generalize these observations in the following theorem:

THEOREM 4.1 - PROPERTIES OF $\tilde{r}^{sp}$. *For any $f \in L$ and $m \in \mathbf{Z}$, $n \geq 0$, there exists a type (m,n) stable Padé approximant $\tilde{r}^{sp}$ to f. If a function $\tilde{r} \in R^D_{mn}$ with defect δ satisfies $(f - \tilde{r})(z) = O(z^{m+n+1-\delta})$, then $\tilde{r} = \tilde{r}^{sp}$, but the converse does not hold except when $\tilde{r} \equiv 0$. The following conditions are equivalent:*

(i) $(f - \tilde{r}^{sp})(z) = O(z^{m+n+1-\delta})$;

(ii) $\tilde{r}^{sp} = \tilde{r}^p$;

(iii) $\tilde{r}^p$ *is stable;*

(iv) $\tilde{r}^{sp}$ ***is unique.***

Proof. Theorem 2.1 and the definitions of $\tilde{r}^p$ and $\tilde{r}^{sp}$ imply the first two claims and the equivalence of (i), (ii), and (iii), and also that these conditions imply (iv). To complete the proof, it will suffice to show that (iv) implies (i).

Suppose that $\tilde{r} \in \tilde{R}^D_{mn}$ is a type (m,n) stable Padé approximant to f and that $(f - \tilde{r})(z) \neq O(z^{m+n+1-\delta})$. Set $(m_0, n_0) = (\mu, n)$ if $\mu = m - \delta$, otherwise $(m_0, n_0) = (m, \nu)$ (in which case $\nu = n - \delta$). Then $\tilde{r} \in \tilde{R}^D_{m_0 n_0}$, the corresponding defect is $\delta_0 = 0$, and $m + n + 1 - \delta = m_0 + n_0 + 1$. Now of course, $\tilde{r}$ is the type (m_0, n_0) ordinary Padé approximant to the Laurent series that represents itself – call it $f_{\tilde{r}}$. Moreover, since $\delta_0 = 0$, the coefficients of this Padé approximant depend continuously on the Laurent coefficients of

$f_{\tilde{r}}$ if these are perturbed slightly [21,22]. Consider a new Laurent series $f_{\tilde{r}}'$ in which the coefficient of degree m_0+n_0 has been changed by a sufficiently small amount. Its (m_0,n_0) Padé approximant will be a new function $\tilde{r}' \in \tilde{R}_{m_0 n_0}$ with coefficients close to those of $\tilde{r}$ and satisfying $(f_{\tilde{r}}' - \tilde{r}')(z) = O(z^{m_0+n_0+1}) = O(z^{m+n+1-\delta})$.

We claim that $\tilde{r}'$ is a new type (m,n) stable Padé approximant to f, which establishes nonuniqueness. For by construction, $(\tilde{r} - \tilde{r}')(z) = O(z^{m_0+n_0}) = O(z^{m+n-\delta})$. Since $(f - \tilde{r})(z) \neq O(z^{m+n+1-\delta})$, this means that $\tilde{r}'$ matches the Laurent series f as far as $\tilde{r}$ does. At the same time, if the perturbation was sufficiently small, $\tilde{r}'$ must also be stable, for the coefficients of its denominator are close to those of the denominator of $\tilde{r}$, which means that the two denominators have nearly equal zeros on the Riemann sphere $\mathbb{C}\cup\{\infty\}$.

We have actually shown not only that $\tilde{r}^{sp}$ is nonunique, but that there is a continuum of such functions. ∎

What can be said about $\tilde{r}^{sp}$ in the case where $\tilde{r}^{p}$ is unstable? We are afraid that the answer may be: very little. For example, given f and (m,n), one might hope that as in Example 4.1, $\tilde{r}^{sp}$ could always be taken equal to the entry $\tilde{r}^{p}$ in some position (m',n') with $m' \le m$, $n' \le n$. Or one might hope that if $\tilde{r}^{p}$ is unstable in a square block, then although $\tilde{r}^{sp} \neq \tilde{r}^{p}$ there, at least one could find a single choice of $\tilde{r}^{sp}$ valid throughout that block. But the following example shows that neither hope is justified.

Example 4.2. If $f(z) = 1 + z + z^2$, then $r_{01}^{p} = r_{02}^{p} = 1/(1-z)$, which is unstable, so $(f - r_{01}^{sp})(z) \equiv O(z)$. But for $(m,n) = (0,2)$, we can do better by taking for example $r_{02}^{sp}(z) = 1/(1-z+z^2/4)$. Thus r_{02}^{sp} is necessarily different from r_{01}^{sp}, and cannot be found anywhere in the Padé table. //

The following corollary to Theorem 4.1 summarizes the situation.

COROLLARY 4.1 - SQUARE BLOCKS *(see Figure 2b). In the stable Padé table for a Laurent series, the portion of the table in which $\tilde{r}^{sp}$ is unique breaks into precisely square blocks containing identical entries. (One of these will be infinite in extent if $f \in \tilde{R}_{mn}^{D}$ for large enough m and n, and another if $f(z) \equiv O(z^K)$ for some $K > -\infty$.) The remainder of the table does not in general break into square blocks.*

5. CHEBYSHEV-PADE APPROXIMATION

At last, we are ready to solve the true CP approximation problem by reducing it to a stable Padé approximation problem. The essential idea is that this can be very nearly achieved by the same formula (3.3) that was used to define R^{fcp}, if $\tilde{r}^{p}$ is replaced by $\tilde{r}^{sp}$. However, this prescription fails in exceptional cases where $m < \frac{1}{2}(n-1)$ and $(f - \tilde{r}^{sp})(z)$

$\neq O(z^{n-m})$, and we will have to modify it slightly to get an exact reduction.

Recall that R_{mn}^{I} is the subset of R_{mn} of all rational functions with no poles on I. Any $R \in R_{mn}^{I}$ will be associated with a series in T by means of an integral analogous to (0.1).

Definition of R^{cp}. Given $F \in T$ and $m,n \geq 0$, a type (m,n) *Chebyshev-Padé approximant* to F is any $R^{cp} \in R_{mn}^{I}$ such that

$$(F-R^{cp})(x) = O(T_K(x)), \tag{5.1}$$

where $K \geq 0$ is as large as possible.

Throughout this section, we continue to make the identifications (3.1), (3.2) between $x \in I$ and $z \in S$ and between $F(x)$ and $f(z)$. Similarly, a rational function $R(x) = 2{\sum}'_{k=0}^{m} A_k T_k(x) / 2{\sum}'_{k=0}^{n} B_k T_k(x) \in R_{mn}^{I}$ with the Chebyshev series $2{\sum}'_{k=0}^{\infty} d_k T_k(x)$ will be identified with the rational function $r(z) = \sum_{k=-m}^{m} A_k z^k / \sum_{k=-n}^{n} B_k z^k$, which is pole-free on S and has Laurent series $\sum_{k=-\infty}^{\infty} d_k z^k$ there ($A_k = A_{|k|}$, $B_k = B_{|k|}$, $d_k = d_{|k|}$).

We now introduce our final space of functions. Let $\hat{R}_{mn}^{D} \subseteq \tilde{R}_{mn}^{D}$ be the set of all functions in $\tilde{R}_{mn}^{D}$ whose formal Laurent series are symmetric: $\hat{r}(z) = \hat{r}(z^{-1})$. This is a strong assumption: stability implies that the positive-degree part of the formal Laurent series $\hat{r}$ converges on S, and if $\hat{r}$ is symmetric, then this conclusion holds also for the whole series. The basis of our treatment of CP approximation is that $\hat{R}_{mn}^{D}$ and R_{mn}^{I} are equivalent. That is, the Laurent series induces a one-to-one correspondence between functions $\hat{r}(z) \in \hat{R}_{mn}^{D}$ and $R(x) \in R_{mn}^{I}$. The following two paragraphs explain this correspondence, and Figure 3 summarizes what is going on.

$\hat{R}_{mn}^{D} \rightarrow R_{mn}^{I}$. Suppose $\hat{r}(z) = \sum_{k=-\infty}^{\infty} d_k z^k \in \hat{R}_{mn}^{D}$ has exact type (μ,ν), $\mu \geq 0$. Let it be written in the form (2.1) as

$$\hat{r}(z) = \hat{r}^{-}(z) + \hat{r}^{+}(z) = \sum_{k=-\infty}^{\mu-\nu} d_k z^k + \sum_{k=\mu-\nu+1}^{\mu} e_k z^k \Big/ \sum_{k=0}^{\nu} b_k z^k, \tag{5.2}$$

and in analogy to (3.5), define

$$\tilde{r}^{0}(z) = \begin{cases} \displaystyle\sum_{k=\mu-\nu+1}^{\nu-\mu-1} d_k z^k & \text{if } \mu < \nu, \\ \\ \displaystyle -\sum_{k=\nu-\mu}^{\mu-\nu} d_k z^k & \text{if } \mu \geq \nu. \end{cases} \tag{5.3}$$

From this definition and the symmetry of $\hat{r}(z)$, it follows that if $\hat{r}$ and $\hat{r}^{+}$ are viewed as formal Laurent series in the usual way, then we have the following identity of formal Laurent series, analogous to (3.4):

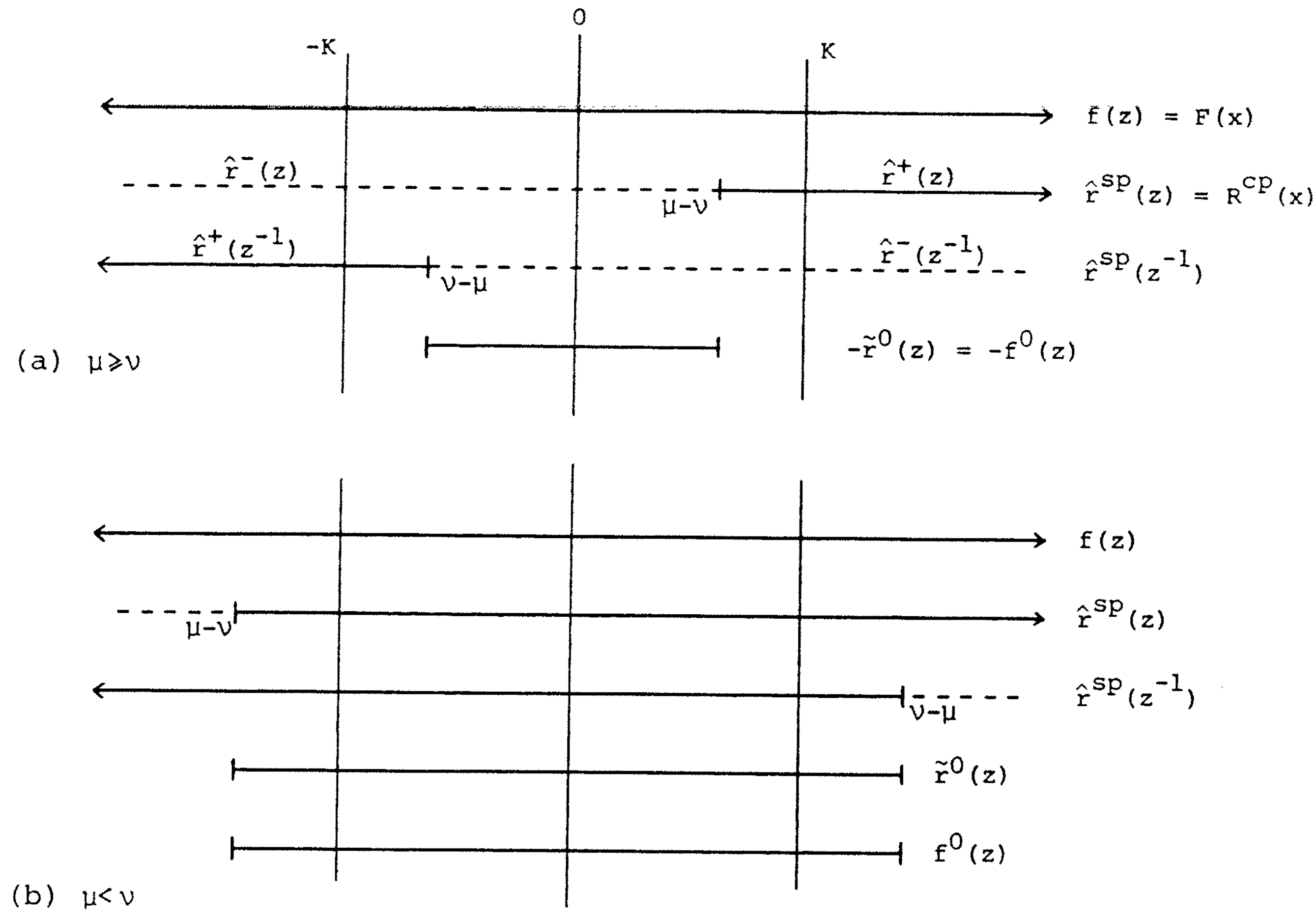


Figure 3. Decomposition of the Laurent series in CP approximation. Solid and dashed lines emphasize the splitting between $\hat{r}^-$ and $\hat{r}^+$.

$$\hat{r}(z) = \hat{r}^+(z) + \hat{r}^+(z^{-1}) - \tilde{r}^0(z). \qquad (5.4)$$

(See Figure 3.) We claim that this same series $\hat{r}(z)$ is the Laurent series on $z \in S$ corresponding to a function $R(x) \in R^I_{mn}$ of exact type (μ,ν). For since $\hat{r}^+(z)$ is stable, its formal Laurent series converges in $0 < |z| \leq 1$ and is the same as the Laurent series one gets by computing integrals on S. Since $\hat{r}^+(z)$ has exactly ν poles in $0 < |z| < \infty$, all outside S, it follows from (5.4) that $\hat{r}(z)$ is the Laurent series on S of a symmetric rational function $r(z)$ in z and z^{-1} with exactly ν poles in $1 < |z| < \infty$ and ν poles symmetrically located in $0 < |z| < 1$. All that remains is to verify that $r(z)$ has exact order $O(z^{\mu-\nu})$ as $z \to \infty$. This follows from the representation (5.2) and the assumption that it has exact type (μ,ν) as a member of $\tilde{R}^D_{mn}$, which implies that the terms $d_{\mu-\nu} z^{\mu-\nu}$

and $e_\mu z^{\mu-\nu}/b_\nu$ in (5.2) do not cancel.

$R^I_{mn} \rightarrow \hat{R}^D_{mn}$. Conversely, suppose $R \in R^I_{mn}$ has exact type (μ,ν) with $\mu \geq 0$, and as usual write $R(x) = r(z) = \sum_{k=-\infty}^{\infty} d_k z^k$ on S. We claim that $r(z)$ is the formal Laurent series corresponding to a function in $\hat{R}^D_{mn}$ of exact type (μ,ν). To see this, define $\hat{r}^+(z)$ to be the formal Laurent series

$$\hat{r}^+(z) = \sum_{k=\mu-\nu+1}^{\infty} d_k z^k$$

(see Figure 3). Since R has exact type (μ,ν), the function $z^{\nu-\mu} r(z)$ has exactly ν poles in $1 < |z| < \infty$ and is bounded at $z=\infty$. On the other hand since the rightmost term in

$$z^{\nu-\mu}\hat{r}^+(z) = z^{\nu-\mu} r(z) - z^{\nu-\mu} \sum_{k=-\infty}^{\mu-\nu} d_k z^k$$

is analytic in $1 < |z| \leq \infty$, $z^{\nu-\mu}\hat{r}^+(z)$ must also have exactly ν poles in $1 < |z| < \infty$ and must be bounded at $z=\infty$. Since $z^{\nu-\mu}\hat{r}^+(z)$ is also analytic in D, it follows that it belongs to $R^D_{\nu\nu}$. This implies that $\hat{r}^+$ is a fraction of the form given in (5.2), which proves $r(z) \in \hat{R}^D_{\mu\nu}$. And since $r(z)$ has exact order $z^{\mu-\nu}$ as $z \rightarrow \infty$, it must again have exact type (μ,ν).

The constructions we have just described amount to a generalization of the "splitting lemmas" given as Lemma 1.1 and Theorem 2.3 in [14]. We record these conclusions in the following lemma:

LEMMA 5.1 - SPLITTING OF R^{cp}. *Given any $\hat{r}(z) \in \hat{R}^D_{mn}$ of exact type (μ,ν), $\mu \geq -\infty$, there is a unique $R(x) \in R^I_{mn}$ such that the Laurent series on S of the corresponding function $r(z)$ is the same as the formal Laurent series of $\hat{r}(z)$, and R has exact type (μ,ν). Conversely, given any $R(x) \in R^I_{mn}$ of exact type (μ,ν), $\mu \geq -\infty$, with corresponding function $r(z)$ on S, there is a unique $\hat{r}(z) \in \hat{R}^D_{mn}$ whose formal Laurent series is the same as the Laurent series on S of $r(z)$, and $\hat{r}$ has exact type (μ,ν).*

Proof. We have already proved everything except uniqueness and the splitting in the case $\mu < 0$. The former is immediate, for in both $\hat{R}^D_{mn}$ and R^I_{mn}, two functions with the same Laurent series are identical. As for the latter, it is readily seen that if $\hat{r} \in \hat{R}^D_{mn}$ has exact type (μ,ν) with $\mu < 0$, then the construction above leads to the zero function in R^I_{mn}. By uniqueness, this implies that $-\infty \leq \mu < 0$ (as in Examples 2.1, 2.2) is impossible in $\hat{R}^D_{mn}$ except in the case $\mu = -\infty$, $\hat{r} \equiv 0$. So $\hat{r} \equiv 0$ corresponds to $R \equiv 0$, with $\mu = -\infty$, and all other functions in $\hat{R}^D_{mn}$ and R^I_{mn} have $\mu \geq 0$. ∎

Now we want to apply the $\hat{R}^D_{mn} \leftrightarrow R^I_{mn}$ correspondence to construct R^{cp}. To make this possible, let $\hat{O}$ be a "big-oh" symbol to indicate that a Laurent series has zero coefficients in a symmetric pattern about degree 0: for $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k \in L$ and

$K \geq 0$,

$$f(z) = \hat{O}(z^K) \iff c_k = 0, \quad -K < k < K.$$

This definition implies that for any $F(x) \in T$ and corresponding $f(z) \in L$,

$$F(x) = O(T_K(x)) \iff f(z) = \hat{O}(z^K). \tag{5.5}$$

As usual, $f(z) \equiv \hat{O}(z^K)$ is an abbreviation for $f(z) = \hat{O}(z^K), \quad \neq \hat{O}(z^{K+1})$.

The following approximant is of little intrinsic interest, but is the intermediate tool we need.

Definition of $\hat{r}^{sp}$. Given $f \in L$ with $f(z) = f(z^{-1})$ and $m \in \mathbf{Z}$, $n \geq 0$, a type (m,n) *symmetric stable Padé approximant* to f is any $\hat{r}^{sp} \in \hat{R}^D_{mn}$ such that

$$(f - \hat{r}^{sp})(z) = \hat{O}(z^K), \tag{5.6}$$

where $K \geq 0$ is as large as possible.

Here is the construction of R^{cp}. Again, refer to Figure 3 for clarification.

THEOREM 5.1 - CONSTRUCTION OF R^{cp}. *Let $m,n \geq 0$ and $F(x) \in T$ be given, and let $f(z) \in L$ be the corresponding function on S. Then the equivalence described in Lemma 5.1 provides a one-to-one correspondence between type (m,n) CP approximants R^{cp} to F and type (m,n) symmetric stable Padé approximants $\hat{r}^{sp}$ to f, which preserves both the exact type (μ,ν) and the exact order of agreement K.*

Proof. The candidates for $R^{cp} \approx F$ and $\hat{r}^{sp} \approx f$ come from R^I_{mn} and $\hat{R}^D_{mn}$, respectively, and by Lemma 5.1, these spaces are the same. This is half of the proof. The remainder consists of pointing out that by (5.5), any candidate R matches F in the $O(T_K(x))$ sense to the same order as the corresponding $\hat{r}$ matches f in the $\hat{O}(z^K)$ sense. ∎

Construction of R^{cp} when $(f - \tilde{r}^{sp})(z) = O(z^{n-m})$. Having presented the construction of R^{cp} in the general case, we can now describe the simpler construction that is applicable almost all of the time. The key assumption is that $(f - \tilde{r}^{sp})(z) \equiv O(z^K)$ with $K \geq n-m$. Assuming this, consider the following one-to-one correspondence between stable Padé approximants $\tilde{r}^{sp}$ to f and symmetric stable Padé approximants $\hat{r}^{sp}$, which by Theorem 5.1 are equivalent to CP approximants $R^{cp}(x)$. See Figure 4. Given $\tilde{r}^{sp}(z) = \sum_{k=-\infty}^{\infty} d_k z^k \in \tilde{R}^D_{mn}$, we obtain $\hat{r}^{sp}$ by replacing d_k by $d_{|k|}$ for $k \leq -K$. Since f has a symmetric Laurent series, so does the function $\hat{r}^{sp}$ obtained in this way, and since $-K \leq m-n$, $\hat{r}^{sp}$ belongs to $\hat{R}^D_{mn}$. Conversely, given $\hat{r}^{sp}(z) = \sum_{k=-\infty}^{\infty} d_k z^k \in \hat{R}^D_{mn}$, with $(f - \hat{r}^{sp})(z) = O(z^K)$ for some $K \geq n-m$, we obtain $\tilde{r}^{sp}$ by replacing d_k by c_k (the corresponding Laurent coefficient of f) for $k \leq -K$. Again since $-K \leq m-n$, $\tilde{r}^{sp}$ belongs to $\tilde{R}^D_{mn}$. And since the order of agreement K is obviously preserved in both directions by this correspondence, stable Padé approximants map into symmetric stable

Padé approximants and vice versa.

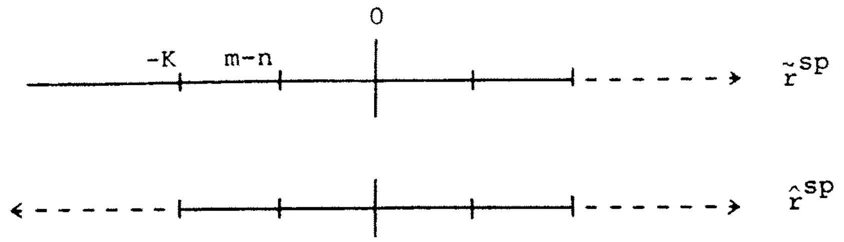


Figure 4. Laurent series for $\tilde{r}^{sp}$ and $\hat{r}^{sp}$ in the case $(f-\tilde{r}^{sp})(z) = O(z^K)$, $K \geq m-n$. Solid lines indicate Laurent coefficients that coincide with those of f.

Let f^0 be defined by (3.5), and $\tilde{r}^0$ by (5.3) with (μ,ν) replaced by (m,n). Since $(f-\tilde{r}^{sp})(z) = O(z^{n-m})$, $f^0 = \tilde{r}^0$. If $\tilde{r}^{sp}$ is written $\tilde{r}^- + \tilde{r}^+$ as a function in $\tilde{R}^D_{mn}$, then we have in fact

$$R^{cp}(x) = \hat{r}^{sp}(z) = \tilde{r}^+(z) + \tilde{r}^+(z^{-1}) - f^0(z) \tag{5.7a}$$

$$= \tilde{r}^+(z) + \tilde{r}^+(z^{-1}) - \tilde{r}^0(z) \tag{5.7b}$$

$$= \tilde{r}^{sp}(z) + \tilde{r}^{sp}(z^{-1}) - f(z). \tag{5.7c}$$

The following theorem confirms that R^{cp} can be constructed from these formulas, just as R^{fcp} was constructed from the analogous formulas (3.3) and (3.4).

THEOREM 5.2 - CONSTRUCTION OF R^{cp} WHEN $(f-\tilde{r}^{sp})(z) = O(z^{n-m})$. *Given m, n, F, f as in Theorem 5.1, suppose that the type (m,n) stable Padé approximants to f satisfy $(f-\tilde{r}^{sp})(z) \equiv O(z^K)$ with $K \geq n-m$; this will always hold if $m \geq \frac{1}{2}(n-1)$ or if $\tilde{r}^{sp} = \tilde{r}^p$. Then equations (5.7) provide a one-to-one correspondence between type (m,n) CP approximants R^{cp} to F and type (m,n) stable Padé approximants $\tilde{r}^{sp}$ to f, which preserves the exact order of agreement K. The exact type (μ,ν) is also preserved whenever $\tilde{r}^{sp}$ or $\hat{r}^{sp}$ has exact type (μ,ν) with $K \geq \nu+1-\mu$*

Proof. Theorem 5.1 established the equivalence between R^{cp} and $\hat{r}^{sp}$, and we have just shown that if $(f-\tilde{r}^{sp})(z) = O(z^{n-m})$, the latter is equivalent to $\tilde{r}^{sp}$. The claim regarding (μ,ν) follows directly from the construction above if $\tilde{r}^{sp}$ or $\hat{r}^{sp}$ is represented as a function in $R^D_{\mu\nu}$. It remains to prove the assertion following the semicolon. If $m \geq \frac{1}{2}(n-1)$, i.e. $m \geq n-m-1$, then since the type $(m,0)$ Padé approximant is trivially stable, we have $K \geq m+1 \geq n-m$. If $\tilde{r}^{sp} = \tilde{r}^p$, then by (1.2) and Theorem 2.1, we have $K \geq m+n+1-\delta \geq n$ once again. ∎

Our first example has $\tilde{r}^{sp}=\tilde{r}^{p}$, so Theorem 5.2 applies.

Example 5.1. If $F(x)=2+2x$ and $(m,n)=(0,1)$, we have $f(z)=z^{-1}+2+z$ and $\tilde{r}^{sp}(z)=\tilde{r}^{p}(z)=z^{-1}+2/(1-z/2)$. By (5.7b), this implies

$$R^{cp}(x) = \hat{r}^{sp}(z)=\frac{2}{1-\frac{1}{2}z}+\frac{2}{1-\frac{1}{2}z^{-1}}-2=\frac{\frac{3}{2}}{\frac{5}{4}-\frac{1}{2}(z+z^{-1})}=\frac{6}{5-4x}.$$

Since $\tilde{r}^{sp}=\tilde{r}^{p}$, this is the same as the formal CP approximant R^{fcp} computed in Example 3.1. //

Our second example has $\tilde{r}^{sp}\neq\tilde{r}^{p}$ but $m\geq\frac{1}{2}(n-1)$, so Theorem 5.2 applies again.

Example 5.2. If $F(x)=1+2x$ and $(m,n)=(0,1)$, then $f(z)=z^{-1}+1+z$, and $\tilde{r}^{p}$ is unstable. (This is essentially the same function f as in Example 4.2.) Therefore any $\tilde{r}\in\tilde{R}_{01}^{D}$ with $(f-\tilde{r})(z)=O(z)$ will serve as a stable Padé approximant. One such function is $\tilde{r}^{sp}(z)=z^{-1}+1$, and from (5.7b) we compute

$$R^{cp}(x) = \hat{r}^{sp}(z) = 1+1-1 = 1.$$

The general stable Padé approximant in this case is $\tilde{r}^{sp}(z)=z^{-1}+1/(1-az)$, $|a|<1$, and now (5.7b) gives

$$\hat{r}^{sp}(z) = \frac{1}{1-az}+\frac{1}{1-a/z}-1=\frac{1-a^2}{1+a^2-a(z+z^{-1})},$$

and therefore

$$R^{cp}(x) = \frac{1-a^2}{1+a^2-2ax}. \quad //$$

Our third example is one to which Theorem 5.2 does not apply.

Example 5.3. Suppose $F(x)=1+2x$, $f(z)=z^{-1}+1+z$ again, but $(m,n)=(0,2)$. Since a stable Padé approximant $\tilde{r}^{sp}$ to f will satisfy at least $(f-\tilde{r}^{sp})(z)=O(z)$, any such function can be written in the form

$$\tilde{r}^{sp}(z) = \tilde{r}^{+}(z) = \frac{z^{-1}+1-b-c}{(1-bz)(1-cz)}$$

for some $|b|,|c|<1$. If $(f-\tilde{r}^{sp})(z)=O(z^2)$, then the terms of degree 1 in the Laurent expansion of this fraction must add up to 1, i.e. $(b^2+c^2+bc)+(b+c)(1-b-c)=1$, or $(1-c)(b-1)=0$, a condition that cannot be satisfied. Therefore $(f-\tilde{r}^{sp})(z)\equiv O(z)$, and any $|b|,|c|<1$ will suffice. For example, we might take $b=1/\sqrt{2}$, $c=-1/\sqrt{2}$, $\tilde{r}^{sp}(z)=(z^{-1}+1)/(1-z^2/2)$. But now (5.7b) leads to $R^{cp}(x)=(3+2x)/(9-8x^2)$, which does not have the required type $(0,2)$.

To construct R^{cp} properly in this example, we must revert to Theorem 5.1 and replace $\tilde{r}^{sp}$ by a symmetric stable Padé approximant $\hat{r}^{sp}$. The general form for such a function with error $\hat{O}(z)$ is $\hat{r}^{sp} = \hat{r}^- + \hat{r}^+$ with

$$\hat{r}^+(z) = \frac{az^{-1} + 1-ab-ac}{(1-bz)(1-cz)}$$

for some $|b|, |c| < 1$, and $a = (b+c)/(1+bc)$. An error $\hat{O}(z^2)$ would require $a = 1$, which is impossible, as shown above. Taking $b = 1/\sqrt{2}$ and $c = -1/\sqrt{2}$ again, we get $a = 0$, and now (5.4) leads to $R^{cp}(x) = 3/(9-8x^2)$. //

Here is the theorem mentioned in the Abstract (compare Theorem 4.1).

THEOREM 5.3 - PROPERTIES OF R^{cp}. *For any $F \in T$ and $m, n \geq 0$, there exists a type (m,n) Chebyshev-Padé approximant R^{cp} to F. If a function $R \in R^I_{mn}$ with defect δ satisfies $(F-R)(x) = O(T_{m+n+1-\delta}(x))$, then $R = R^{cp}$, but the converse does not hold except when $R \equiv 0$. The following three conditions are equivalent:*

(i) $(F-R^{cp})(x) = O(T_{m+n+1-\delta}(x))$;

(ii) the associated Padé approximant $\tilde{r}^p(z)$ to $f(z) = F(x)$ is stable (or $R^{cp} \equiv 0$, which is equivalent to $F(x) = O(T_{m+1}(x))$);

(iii) R^{cp} is unique.

If these conditions hold and $R^{cp} \not\equiv 0$, then

(iv) $R^{cp} = R^{fcp}$.

Proof. The existence of R^{cp} follows from its definition. Examples 5.2 and 5.3 show that $(F-R)(x) = O(T_{m+n+1-\delta}(x))$ does not necessarily hold for $R = R^{cp}$, but if $R = R^{cp} \equiv 0$, then $T_{m+n+1-\delta}(x) = T_{m+1}(x)$ and so it does hold, since otherwise there would be a nonzero polynomial that approximated F better than R. To show that this condition is sufficient for $R = R^{cp}$, we can argue directly from the definition of R^{cp}, ignoring the developments of the past few pages. For if R^{cp} were an equally good or better approximation to F than R, we would have

$$(R - R^{cp})(x) = O(T_{m+n+1-\delta}(x)),$$

or in other words, $R-R^{cp}$ would be orthogonal to all polynomials of degree at most $m+n-\delta$ with respect to the Chebyshev weight. But this implies that $R-R^{cp}$ has at least $m+n+1-\delta$ zeros in I, which can only occur if $R = R^{cp}$.

To show that (i)-(iii) are equivalent, we will now prove that (i) is equivalent to (iii) and then (ii). The last argument also establishes (ii) $\Longrightarrow$ (iv).

(i) $\Longrightarrow$ (iii). The argument just given shows also that R^{cp} is unique.

(iii) $\Longrightarrow$ (i). See the proof of Theorem 4.1. Suppose $(F-R^{cp})(x) \neq O(T_{m+n+1-\delta}(x))$, and consider the equivalent function $\hat{r}^{sp}$ provided by Theorem 5.1, with exact type (μ,ν). Define m_0 and n_0 as before, and perturb the coefficient of $\hat{r}^{sp}$ of degree m_0+n_0 by a sufficiently small amount. If the resulting perturbed Padé

approximant is extended symmetrically to coefficients of degree $\leq\mu-\nu$, it becomes a new symmetric stable Padé approximant $\hat{r}^{sp'}$ to f, hence a new CP approximant $R^{cp'}$ to R.

(i) $\Rightarrow$ (ii). If $R^{cp}\equiv 0$ there is nothing to prove, so assume $R^{cp}\not\equiv 0$, which implies that R^{cp} and $\hat{r}^{sp}$ have exact type (μ,ν) for some $\mu\geq 0$. Since $(f-\hat{r}^{sp})(z)=\hat{O}(z^{n+1})$, the correspondence described in Theorem 5.2, cf. Figure 4, leads to a function $\tilde{r}^{sp}$ of the same exact type with $(f-\tilde{r}^{sp})(z)=O(z^{m+n+1-\delta})$. By Theorem 2.1, this must be the Padé approximant $\tilde{r}^{p}$, which is accordingly stable.

(ii) $\Rightarrow$ (i), (iv). If $\tilde{r}^{p}$ is stable and has exact type (μ,ν) with $\mu\geq 0$, then by Theorem 3.1, R^{fcp} has exact type (μ,ν) also and satisfies $(F-R^{fcp})(x)=O(T_{m+n+1-\delta}(x))$. Therefore $R^{fcp}=R^{cp}$ by the sufficient condition above, and so (i) holds. If $\tilde{r}^{p}$ is stable but with $\mu<0$, then by Theorem 3.1, $R^{fcp}\equiv 0$ and $(F-R^{fcp})=O(T_{m+1}(x))$, so $R^{fcp}=R^{cp}$, and (i) holds again. Finally, if $R^{cp}\equiv 0$, we have already established (i) above. In this case (iv) does not always hold, whence the condition $R^{cp}\not\equiv 0$ in the statement of the theorem; as an example one can consider type (0,2) approximation of $T_1(x)+T_2(x)$. ∎

Note that the last sentence of Theorem 5.3 asserts that whenever the CP approximant is nonzero and unique, the formal CP procedure is guaranteed to get the right answer. We do not know whether the converse also holds, that is, whether (iv) implies (i)-(iii).

Note also one of the implications of condition (ii) – that the approximant $R^{cp}\equiv 0$ can appear in the CP table only when it is the unique CP approximant and satisfies (i).

Regarding block structure in the CP table, we know nothing more than the analog of Corollary 4.1, which is essentially the same as Theorem 5.1 of Geddes [10], although the derivation here has been completely different from his.

COROLLARY 5.1 - SQUARE BLOCKS *(see Figure 2c). In the Cheybshev-Padé table for a formal Chebyshev series, the portion of the table in which R^{cp} is unique breaks into precisely square blocks containing identical entries. (One of these is infinite in extent if $F\in R^{I}_{mn}$ for large enough m and n.) The remainder of the table does not in general break into square blocks. There is one possible exception: if $F(x)\equiv O(T_K(x))$ for some $K>0$, then the approximant $R^{cp}\equiv 0$ occupies the infinite rectangular strip $0\leq m\leq K-1$, $0\leq n\leq\infty$ at the left edge of the table. This is the only situation in which $R^{cp}\equiv 0$ can appear.*

6. REMARKS ON RECURSIVE COMPUTATION AND CONVERGENCE

In this paper we have reduced formal CP approximation to Padé approximation, even for $m < n$, and the reduction carries over to true CP approximation whenever $\tilde{r}^{\mathrm{p}}$ is stable. Here is the schematic:

$$r^{\mathrm{p}} \underset{\text{Sect. 2}}{\longleftrightarrow} \tilde{r}^{\mathrm{p}} \underset{\text{Sect. 3}}{\longleftrightarrow} R^{\mathrm{fcp}} \underset{\text{Sects. 4,5}}{\longleftrightarrow} R^{\mathrm{cp}} \qquad (6.1)$$

Before closing, we will mention the relevance of these developments to two further topics in Padé and CP approximation.

The first is the derivation of recursive formulas to compute approximants. This is an old idea that has been worked out quite fully for Padé approximation; see [2,11]. Our approach to formal CP approximation, based on (3.3), makes it an easy matter to extend any recursive procedure for computing $\tilde{r}^{\mathrm{p}}$ to a corresponding procedure for computing R^{fcp}, regardless of m and n. To complete the link with the Padé literature, we must relate recursions for r^{p} to recursions for $\tilde{r}^{\mathrm{p}}$. This is straightforward too, if one modifies the construction of Section 2 in one respect. Rather than splitting the Laurent series always at degree $m-n$, as in (2.1) and (2.4), let the whole derivation be based on a splitting at some lower degree $l \leq m-n$,

$$f(z) = f^{-}(z) + f^{+}(z) = \sum_{k=-\infty}^{l} c_k z^k + \sum_{k=l+1}^{\infty} c_k z^k . \qquad (6.2)$$

If $l \in \mathbf{Z}$ is fixed, this same splitting can now be applied to construct all approximants $\tilde{r}^{\mathrm{p}}$ in the sector $m \geq n+l$ of the Padé table. All of the recursion formulas for Padé approximation then carry over to CP approximation, so long as l is taken low enough that they remain in the sector.

Various recursion formulas for R^{fcp} have appeared in the literature [3,6,13,16]. Although we have not worked out the details, we believe that these and others can be derived and understood in a uniform way by the method just outlined.

The second topic is the question of convergence as $m+n \to \infty$. In Padé approximation, the classical result is the de Montessus theorem for convergence as $m \to \infty$ with fixed n [1]. This theorem has the great advantage of asserting that for large enough m, there are no spurious poles, so that stability can be guaranteed. As a result all three reductions in (6.1) can be carried out successfully, and we get the following theorem, which is due in various forms to Suetin [17] and Bultheel [3]. Here E denotes the open region bounded by the ellipse with foci ± 1 and principal semiaxis $\frac{1}{2}(\rho + 1/\rho)$ for some $\rho > 1$.

THEOREM 6.1 - CONVERGENCE ALONG ROWS OF THE CP TABLE. *Let $F(x)$ be analytic on I and meromorphic in E, with exactly n poles in E.*

counted with multiplicity. Then for all sufficiently large m, the CP approximant R^{cp} of type (m,n) to F is unique, and as $m \to \infty$ it converges uniformly to F on I with

$$\limsup_{m \to \infty} \| F - R^{cp} \|_\infty^{1/m} \le \frac{1}{\rho}.$$

There are also many more specialized results in Padé approximation on convergence as $m+n \to \infty$ with variable n, due to Gončar, Nuttall, Pommerenke, Stahl, Wallin, and others [1]. Most of these assert convergence in capacity, under restrictive assumptions, for limits along diagonals with $m/n \to const$. If there were no problem of stability, the reductions of the present paper would convert these to corresponding convergence results for CP approximation, even for $m < n$. Unfortunately, stability cannot be guaranteed, and we are able to say very little. Some convergence results related to CP approximation have been described by Gragg [12] and by Lubinsky and Sidi [15], but their validity is based on an explicit assumption that the Chebyshev series is matched to an appropriate degree, i.e., that $\tilde{r}^p$ is stable.

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