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Journal of Computational and Applied Mathematics 135 (2001) 23–40

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

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Eigenvalues and musical instruments ☆

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Received 6 October 1998; received in revised form 5 May 2000

Abstract

Most musical instruments are built from physical systems that oscillate at certain natural frequencies. The frequencies are the imaginary parts of the eigenvalues of a linear operator, and the decay rates are the negatives of the real parts, so it ought to be possible to give an approximate idea of the sound of a musical instrument by a single plot of points in the complex plane. Nevertheless, the authors are unaware of any such picture that has ever appeared in print. This paper attempts to fill that gap by plotting eigenvalues for simple models of a guitar string, a flute, a clarinet, a kettledrum, and a musical bell. For the drum and the bell, simple idealized models have eigenvalues that are irrationally related, but as the actual instruments have evolved over the generations, the leading five or six eigenvalues have moved around the complex plane so that their relative positions are musically pleasing.

Keywords: Musical instruments; Eigenvalues; Normal modes; Drum; Bell; Recorder

1. Introduction

Linear systems like strings and bars and drums have eigenvalues, which are numbers in the complex plane $\mathbb{C}$. If the system is governed by an equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{2\pi}Au,$$

where A is a matrix or linear operator, then an eigenvalue $\lambda \in \mathbb{C}$ of A corresponds to a solution

$$u(t) = \mathrm{e}^{\lambda t/2\pi}u(0),$$

where $u(0)$ is a corresponding eigenvector. An eigenvalue on the imaginary axis corresponds to oscillation without decay, and an eigenvalue to the left of the imaginary axis corresponds to

☆ The work of both authors was supported in part by NSF Grant DMS-9500975CS.

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PII: S0377-0427(00)00560-4

oscillation with decay. Thus $\mathrm{Im}\,\lambda$ is the (real) frequency of the oscillation and $-\mathrm{Re}\,\lambda$ is its decay rate.

Nonlinear oscillators have eigenvalues too, if one linearizes by considering infinitesimal motions. This may give a good initial picture of the behavior, which can then be improved by considering changes introduced by the nonlinearity.

In this report our goal is to bring these ideas to life by showing a sequence of more than a dozen pictures of eigenvalues in the complex plane of various oscillating systems. The systems we consider are musical instruments. Typically we ask, what do the eigenvalues look like for the simplest idealization, such as a perfect string or tube or membrane? How do they change when terms are added to the model corresponding to stiffness, or friction, or sound radiation? And how does this all match frequencies and attenuation rates that have been measured in the laboratory?

Eigenvalues do not tell us everything about the sound of a musical instrument. If we wanted to fully distinguish a guitar from a flute, for instance, even without getting into nonlinear effects, we would want to know the relative amplitudes of the excited modes. Still, the eigenvalues represent a good starting point.

There is nothing new in the study of the physics of musical instruments, which is set forth beautifully in the books in [2,3,7], nor in the portrayal of eigenvalues in the complex plane. Nevertheless, we have not found a book or paper that combines the two as we do here. Thus, this paper is solely expository — but the basis of the exposition, we think, is new, and well suited to applied mathematicians.

2. Guitar string

Let us start with the eigenvalues of an ideal guitar string. By "ideal" we mean that the string is perfectly flexible, that there are no internal losses or losses to the string supports, and that there is no damping due to the surrounding air. To be definite, let us suppose the length of our string is $L = 65$ cm, the mass density is $\rho = 1.15\ \mathrm{g/cm}^3$, and the radius is $r = 0.032$ cm. These values correspond approximately to the high E string on a nylon string guitar. Typical tension values are between 50 and 80 N [7, p. 212]. Given the values for mass density, cross-sectional area, and string length just chosen, we can "tune" our ideal guitar string to high E (329.63 Hz $\approx$ 330 Hz) by setting the tension to $T = 67.09$ N.

The motion of an ideal string is governed by the second-order wave equation, $u_{tt} = c^2 u_{xx}$, where $c^2 = T/\rho S$ is the square of the wave speed. Fig. 1 shows the eigenvalues for these choices or parameters.

In this figure, the fundamental frequency, the first eigenvalue, is a good approximation to the expected fundamental frequency, $2^{-5/12} \times 440$ Hz ≈ 330 Hz, for a high E guitar string (five half-steps below concert A, which has a frequency of 440 Hz). The remaining eigenvalues, the higher harmonics, are integer multiples of the fundamental frequency. The eigenvalues have zero real part, corresponding to our ideal string experiencing no losses.

For every positive frequency, we always have a corresponding negative frequency of the same magnitude. From now on, we will only show the upper half-plane.

If we now gently touch the guitar string in the middle, we damp out the fundamental and all of the odd harmonics. The odd eigenvalues all shift into the left half-plane. Ideally speaking, the

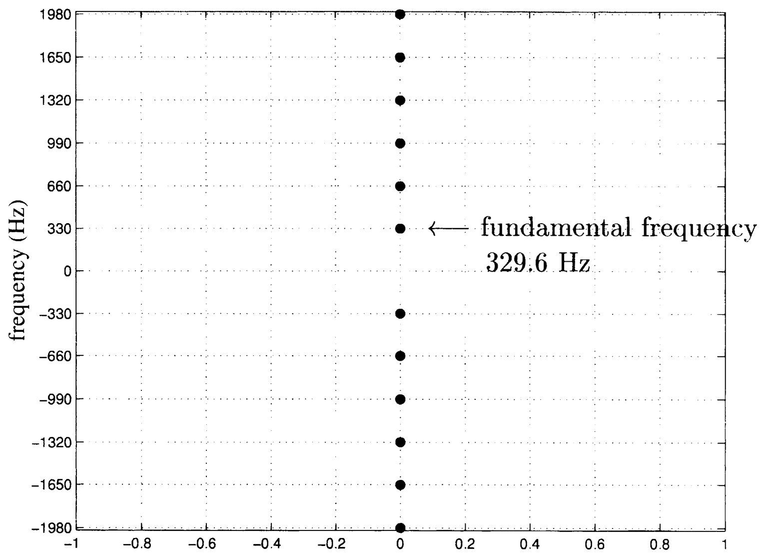

Fig. 1. Eigenvalues for an ideal (even-tempered) high E guitar string. There are no losses, so the eigenvalues are pure imaginary.

even eigenvalues are not much affected, since their corresponding eigenmodes already have a node at the point we are touching. However, in reality they are affected somewhat since a finger is not sharp and cannot damp the string exactly in the middle, but damps a small region at the middle of the string. Fig. 2 suggests schematically what happens to the eigenvalues when we gently touch the middle of a high E string with a very narrow finger.

We are still assuming in this picture that the string experiences no losses other than the damping from touching it in the middle. A real guitar is not so simple (if it were, we couldn't hear it!). Some of the physical phenomena that make it deviate from the ideal are stiffness in the strings and damping. The effect of stiffness is to stretch the frequencies slightly, making them inexact harmonics of one another. However, the stiffness of a string depends on its thickness, and guitar strings are not generally thick enough for stiffness to be an important factor. Note that stiffness is an extremely important factor in the eigenvalues of a piano string, since piano strings are much thicker than guitar strings. On the other hand, losses from damping are indeed important. Their effect is to shift the eigenvalues into the left half of the complex plane, some of them further than others.

The energy losses associated with nonrigid end supports are in one sense the most important, for it is these losses that couple the string to the soundboard and thus provide most of the sound volume that we hear. However, this type of loss is actually not very large in magnitude, and thus it is not essential to our picture of the eigenvalues.

In a real guitar, the most important phenomena affecting the eigenvalues are damping due to air viscosity and internal losses. Which is more important depends on the materials used. In a nylon string guitar, the higher modes decay mainly as a result of internal damping in the string. In a steel string guitar, on the other hand, the main damping mechanism is air viscosity. Steel strings are actually damped less by air viscosity than nylon strings, since they are thinner, but the effect is relatively more important because the internal losses are negligible.

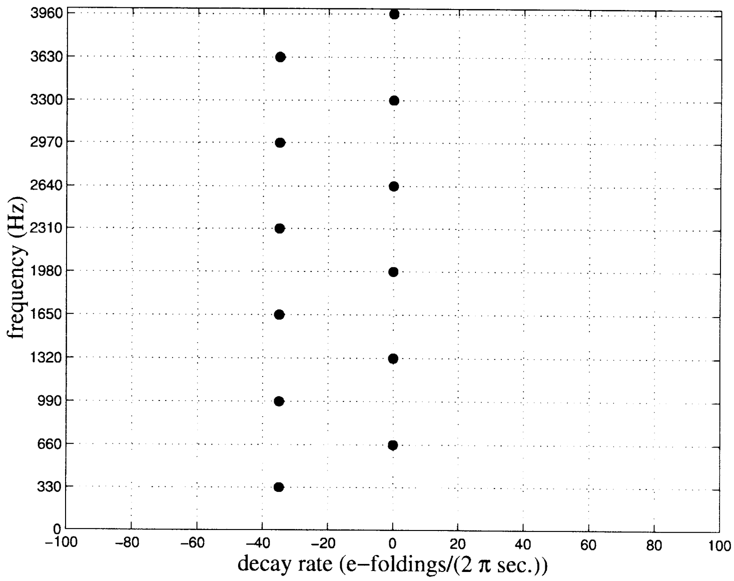

Fig. 2. The effect of gently touching the guitar string in the middle — schematic. Odd harmonics of this fundamental are greatly attenuated, while even harmonics are little affected. The unattenuated frequencies make up a harmonic series of a pitch whose fundamental is twice the original fundamental, so the pitch goes up an octave. This effect is analogous to the effect of opening the register hole in a recorder or clarinet.

Consider first damping from the viscosity of the air in which the string is vibrating. Viscous damping depends on the velocity of the string, and so affects each frequency differently. In particular, the decay rate due to air viscosity is roughly proportional to the square root of the frequency. Specifically, the decay rate is

$$\alpha_a = \frac{\pi\rho_a f(2\sqrt{2}M + 1)}{\rho M^2},$$

where ρ is the density of the string material, ρ_a is the air density, f is the frequency, S is the cross-sectional area of the string, $r = \sqrt{S/\pi}$ is the radius of cross-section of the string, η_a is the kinematic viscosity of air, and $M = (r/2)\sqrt{f/\eta_a}$ [7, p. 50].

Consider a steel string guitar. Let us assume that the string is still 65 cm long, the radius is $r = 0.017$ cm, the mass density is $\rho = 7.8$ g/cm^3, and the tension is $T = 1.228 \times 10^7$ dyn. (Typical tension values for steel strings are between 10^7 and 1.8×10^7 dyn [7, p. 212]; we have again "tuned" the tension to get a fundamental frequency of approximately 330 Hz.) Assume the air density is $\rho_a = 0.0012$ g/cm^3 and the kinematic density is $\eta_a = 0.15$ cm^2/s. In Fig. 3, the eigenvalues have shifted into the left half-plane, the magnitude of the shift being proportional to the square root of the frequency.

Going back to the nylon string guitar parameters we considered above, when we include air damping, the picture changes much more than it did for the steel string (Fig. 4). The eigenvalues shift farther into the left half-plane, since the nylon string is thicker and less dense.

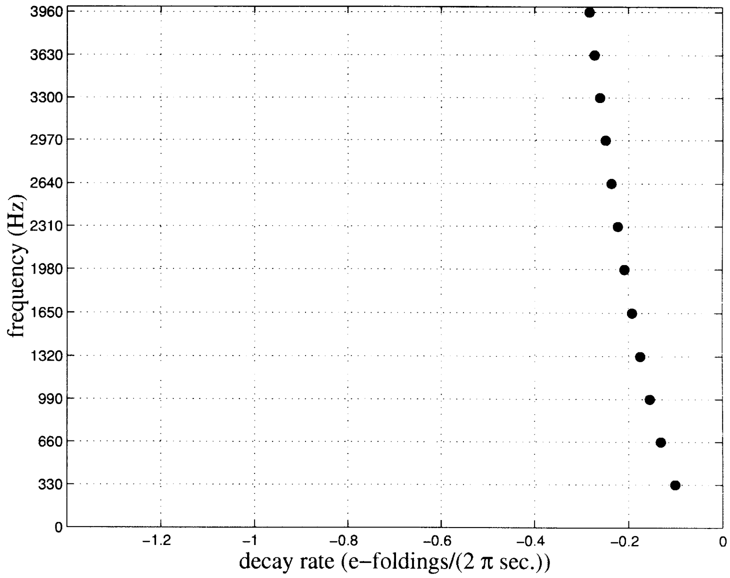

Fig. 3. Steel guitar string with air damping. Higher frequencies are attenuated more than lower ones.

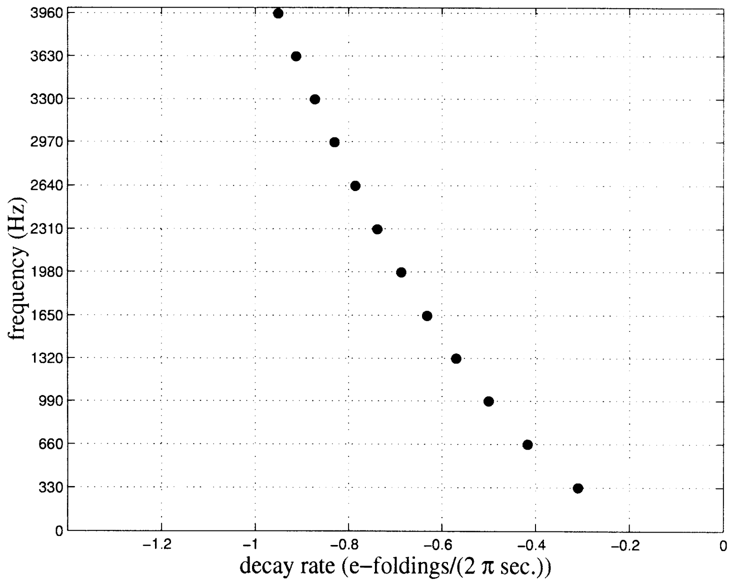

Fig. 4. Nylon guitar string with air damping. Since the nylon string is thicker and less dense than the steel one, the losses are greater.

Next, let us consider internal damping. This type of damping occurs because of energy lost in bending the string. It is generally negligible for metal strings, but it can be important for nylon strings. Internal losses can be represented by assuming a complex Young's modulus for the string material. The characteristic decay rate α_i from internal damping is roughly proportional to the frequency and

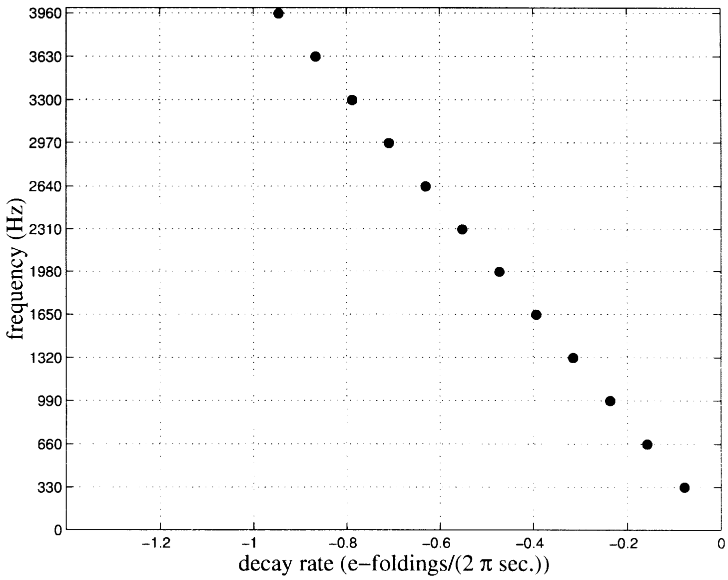

Fig. 5. Nylon guitar string with internal losses. The attenuation rates are great, representing the chief energy loss mechanism for nylon strings at high frequency.

is given by

$$\alpha_i = \pi f \, \frac{Q_2}{Q_1},$$

where f is the frequency and $Q_1 + \mathrm{i}Q_2$ is the complex Young's modulus [7, p. 51]. Q_1 relates to the elastic bond distortions and Q_2 relates to relaxation processes such as dislocation motion or the movement of kinks in the polymer chains [7, p. 51]. Figs. 5 and 6 show the pictures we get for our same representative nylon and steel high E guitar strings, if we assume somewhat arbitrarily that $Q_2/Q_1 \approx 3.75 \times 10^{-6}$ for steel and 4.78×10^{-4} for nylon. (These ratios are chosen roughly to match the data on internal damping in [12, Table 1; 8, Table 5B].) Notice that the effect is large for the nylon string, but very small for the steel string.

Putting together both types of damping, we get the pictures of the nylon and steel strings summarized in Figs. 7 and 8.

Of course, there are other instruments based upon strings than guitars, and in these, other phenomena may be important in determining the eigenvalues. In guitars, energy loss from nonrigid end supports is only important in the bridge end, since the neck end is stopped with a rigid end support, the fret. However, losses to end supports can be very important, for example, in violins, where the string is stopped at the neck end with a finger pressing the string against the fingerboard instead of against a fret (especially in a violin played pizzicato [5, p. 15]). In pianos, the increased spacing of eigenvalues caused by stiffness of the strings becomes important.

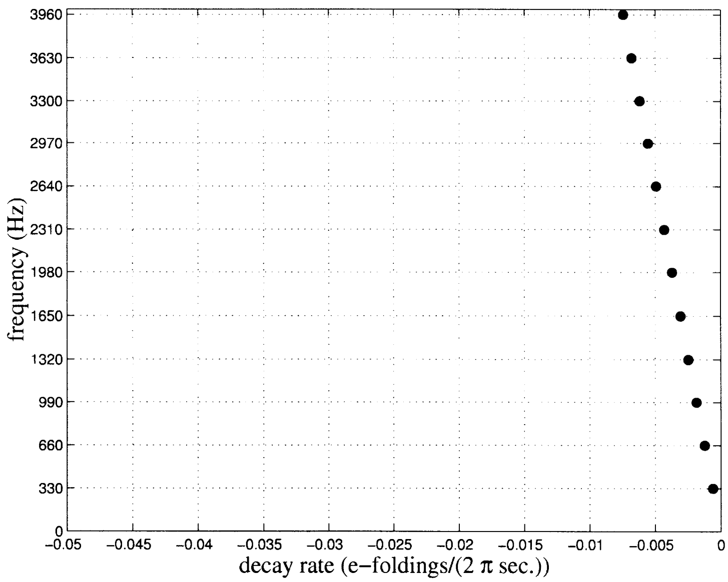

Fig. 6. Steel guitar string with internal losses. Since a steel string is thin and elastic, these attenuation rates are small, much less significant than the damping introduced by air viscosity.

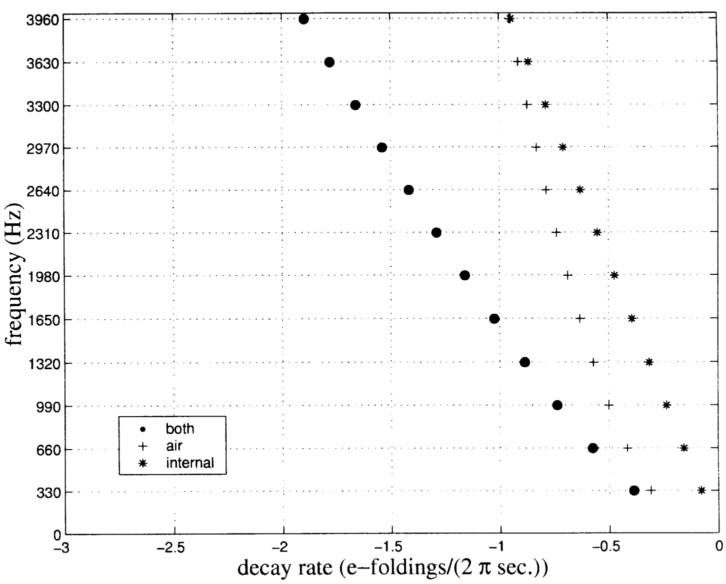

Fig. 7. Nylon guitar string with both types of damping. Internal losses dominate at higher frequency, making the overall decay rate approximately proportional to the frequency.

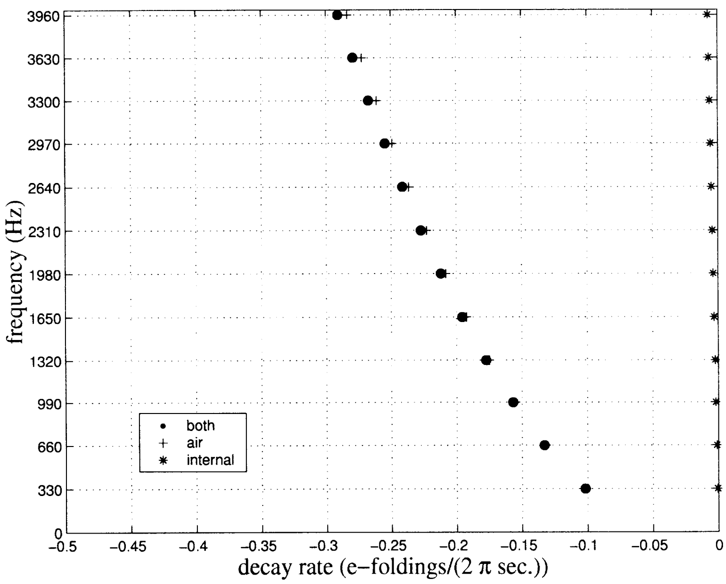

Fig. 8. Steel guitar string with both types of damping. Air damping dominates, making the overall decay rate approximately proportional to the square root of the frequency.

3. Flute

In this section, we start with the eigenvalues of an ideal flute and gradually add in the effects of some additional physical phenomena, as we did with the guitar string.

Consider a flute of length L with no tone holes. The oscillator in a flute is the column of air in the tube, which is kept oscillating by the air blowing across a sharp ridge. In the simplest model, inside the tube are plane pressure waves that obey the same equation as do transverse waves on a string,

$$u_{tt} = c^2 u_{xx},$$

where c is the speed of sound in air under normal conditions, u is the deviation from equilibrium pressure, and the x-axis is oriented along the center of the flute bore.

Since the flute is open at both ends and the openings are narrow compared with the sound wavelength, there are pressure nodes at both the head and foot of the flute (where the air column is in contact with the surrounding air) [17, p. 513]. It follows that the eigenvalues are exactly the same as those for an ideal string fixed at both ends. They are given (in Hertz) by $nc/(2L)$, where n is the mode number.

Under typical conditions, the speed of sound in air is 34400 cm/s [10, p. 222]. If we want a simple flute whose lowest note has a fundamental frequency approximately equal to the C below concert A (approximately 261.6 Hz, corresponding, for example, to a modern flute), its length should be $L = c/(2^{-9/12} \times 440\ \mathrm{s}^{-1} \times 2) \approx 65.7$ cm. The eigenvalues are all the integer multiples of this frequency, exactly as in the case of the ideal guitar string (Fig. 9). The only difference in the picture is that the fundamental frequency is 261.6 Hz.

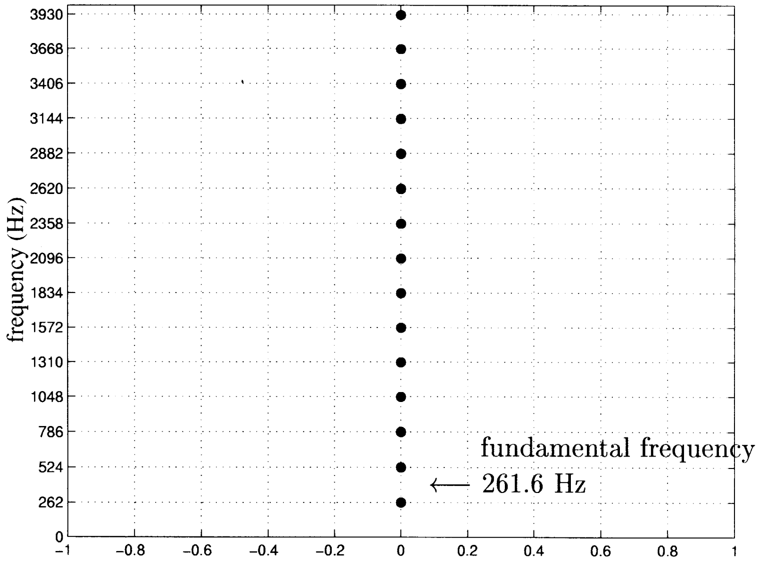

Fig. 9. Eigenvalues of an ideal C flute, of length 65.7 cm. There are no losses, and the flute has no finger holes.

Some pitches on a flute are raised an octave by opening a register hole. Opening the register hole is analogous to lightly touching a guitar string in the middle. An ideally located register hole in a flute is at half the acoustic length of the tube. Opening the hole greatly attenuates the fundamental and all of the odd harmonics of the initial pitch. The unattenuated frequencies make up a harmonic series of a pitch whose fundamental is twice the original fundamental, raising the pitch of the note by an octave as in Fig. 2.

In this model, as in our ideal guitar, we are ignoring all losses (other than those from the register hole). To begin to develop a more realistic picture of the flute, we consider length corrections, wall losses, and losses from sound radiation.

In a real flute (even one without tone holes), the air pressure does not reach equilibrium precisely at the ends of the tube. The pressure waves extend slightly past the ends before coming to equilibrium with the outside air. How far they extend depends on the cross-sectional area of the tube and the acoustic conductivity of the open ends. This open-end correction lowers the frequencies of all the harmonics, making the "acoustic length" of the tube somewhat longer than its physical length. A reasonable approximation is that the acoustic length is $L + S/c_{\mathrm{a}}$, where L is the physical length of the tube, S is its cross-sectional area, and c_{a} is the acoustic conductivity of the open end [11, p. 613]. Therefore the nth eigenvalue becomes $nc/2(L + S/c_{\mathrm{a}})$.

Given a flute with an internal radius of 0.95 cm and assuming that the acoustic conductivity is 1.9 cm (the acoustic conductivity is approximately equal in magnitude to the diameter of the flute [11, p. 613]), for the flute to have a fundamental frequency of approximately 261.6 Hz, its (physical) length would have to be $L = c/(2^{-9/12} \times 440 \times 2) - S/c_{\mathrm{a}} \approx 64.3$ cm instead of 65.7 cm. This length is a little closer to the length of a real flute (approximately 62 cm).

The two main mechanisms for energy loss in a flute are frictional and thermal energy transfer to the instrument walls and sound radiation. The wall losses have the greatest effect on the eigenvalues [1, p. 142].

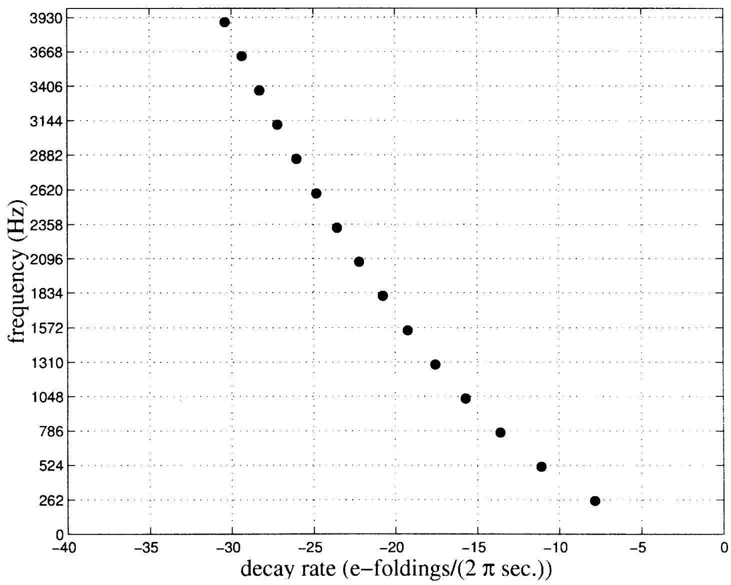

Fig. 10. Flute eigenvalues with wall losses. Note that the decay rates are large. Fortunately, the player keeps blowing, supplying the system with energy.

The damping rate due to wall losses is approximately proportional to the square root of the frequency. A reasonable approximation to the damping rate α is

$$\alpha = 2A\omega^{1/2}c/r,$$

where A is a constant depending on the wall material and surface condition (we use $A \approx 2.5 \times 10^{-5}\,\mathrm{s}^{1/2}$, which is a good approximation for some woodwinds), ω is the angular frequency, and c is the speed of sound in air [1, pp. 142–143]. Fig. 10 shows the effect of wall losses in the flute.

Next consider losses from sound radiation. It is this energy loss that lets us hear the flute. In our ideal flute, without any tone holes, all of the sound radiation is from the end. Losses from sound radiation at the end of a tube are approximately proportional to the square of the frequency [7, p. 183],

$$\alpha = \frac{\pi}{4}\left(\frac{r}{l}\right)^2 (2n-1)\omega,$$

where r is the bore radius, l is the length of the flute, n is the mode number, and ω is the angular frequency [1, p. 143]. Fig. 11 shows the effects of radiation losses on the eigenvalues of our C flute. Fig. 12 shows the combined effect of both types of losses.

A real flute has many more features that would affect our picture of the eigenvalues. However, we will stick with this highly idealized flute and compare its eigenvalues with a similarly idealized clarinet. Even at this level of idealization, there are clear differences to be seen between the eigenvalues of a flute and those of a clarinet.

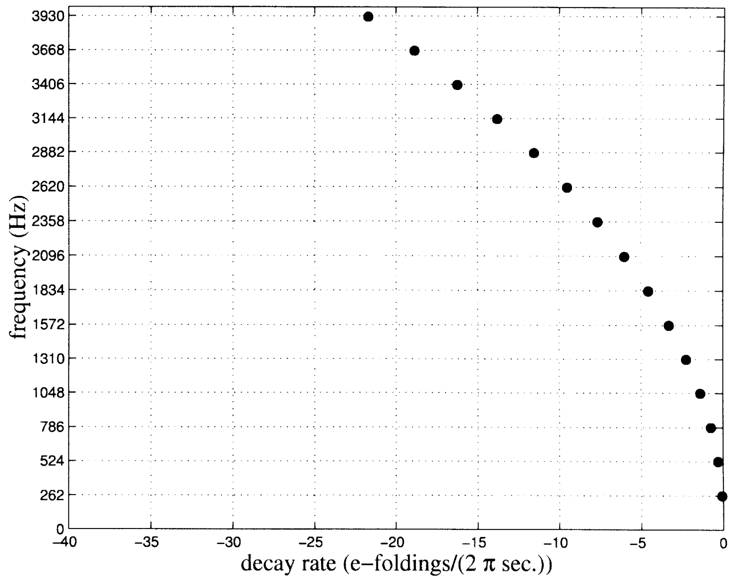

Fig. 11. Flute eigenvalues with radiation losses from the end the tube.

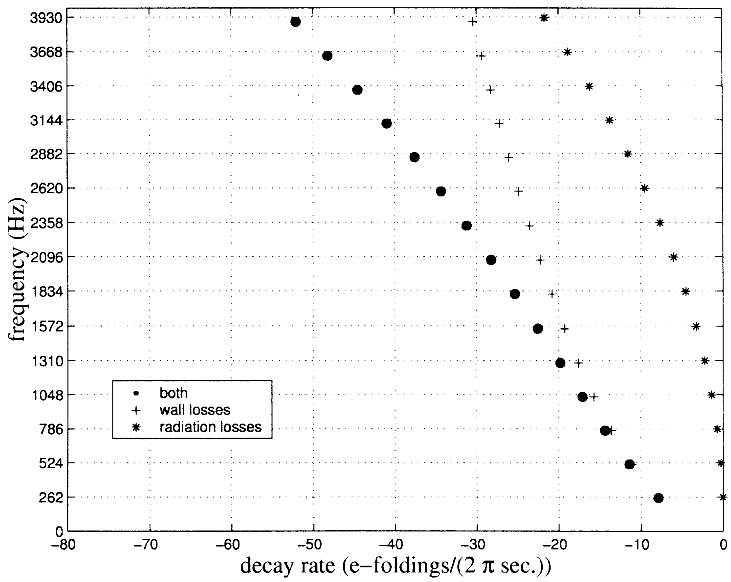

Fig. 12. Flute eigenvalues with both wall losses and radiation losses.

4. Clarinet

The most important difference between a clarinet and a flute is that the oscillations in a clarinet are driven by a reed. The main effect of the reed is to essentially close the reed end of the instrument.

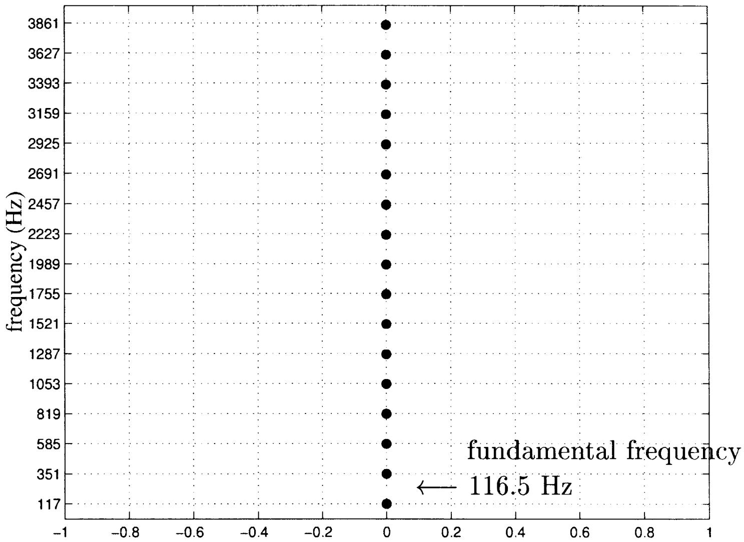

Fig. 13. Eigenvalues for an ideal clarinet (even-tempered $B^{\flat}$). The fundamental is an octave lower than that of an ideal flute of the same length, and the harmonics are all odd multiples of the fundamental.

A simple model of a clarinet (without tone holes) is therefore a cylindrical tube open at one end and closed at the other. The closed end lowers the fundamental frequency by an octave compared with that of a flute of the same length, since we now have a pressure antinode at the closed end instead of a node. The wavelength of the fundamental is thus four times the length of the tube instead of twice the length, and the fundamental frequency is halved.

The closed end also means that the overtones of the clarinet are only the odd multiples of the fundamental. The even harmonics would have a pressure node at the reed end of the instrument, which is impossible since the closed end forces an antinode at that point. Fig. 13 shows the eigenvalues for an ideal $B^{\flat}$ clarinet.

Because we only have the odd harmonics, the register hole in a clarinet is situated differently than in a flute. In a flute, the register hole is ideally located at half the acoustic length of the tube. Opening the hole damps the fundamental and all of the odd harmonics, raising the pitch by an octave. In a clarinet, the register hole is ideally located at a third of the acoustic length of the tube. Opening the hole damps the fundamental and all but every third (odd) upper harmonic of the original pitch, raising the pitch by a twelfth. See Fig. 14.

Unlike in the flute, in a clarinet the tone holes are not large enough relative to the bore size to cut off the tube at their location. The open holes of a clarinet change the character of the sound in a more complicated way. The tube effectively ends at a point somewhat past the first open tone hole, flattening the fundamental. The acoustic length of the tube due to open tone holes increases with frequency, flattening the upper harmonics relative to the fundamental by an appreciable fraction of a semitone. However, such effects are generally countered by appropriate shaping and placement of the holes in the construction of real clarinets, so the eigenvalue picture is actually much the same as in the ideal case [3].

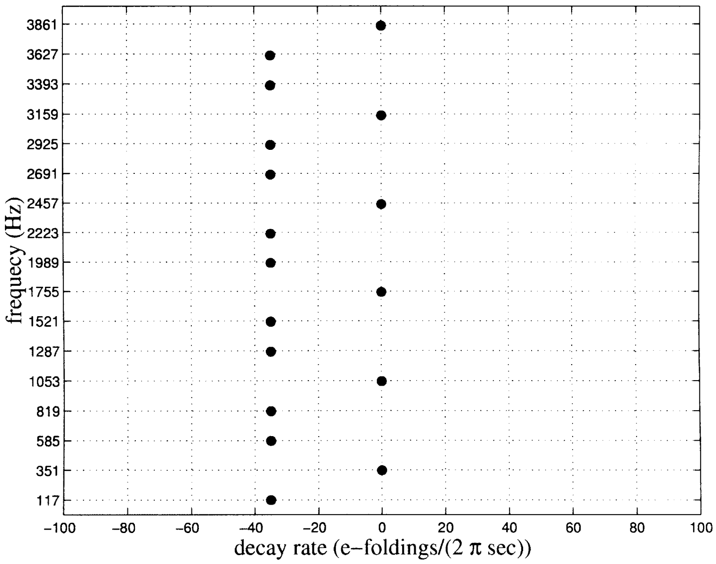

Fig. 14. The effect of opening the register hole on a clarinet — schematic. The two-thirds of the harmonics that did not already have a node at the register hole are greatly attenuated, and the pitch goes up a twelfth. Compare Fig. 2.

Like the flute, the main causes of energy loss in a clarinet are wall losses and sound radiation. Other than having a lower fundamental and only having the odd harmonics strongly present, the eigenvalues after incorporating losses look similar to those for the flute.

5. Drums

Strings, flutes, and clarinets are all essentially one-dimensional resonators, and that is why their eigenvalues fall naturally into integer ratios, making the sound strongly musical. We turn now to instruments that are not essentially one dimensional, where more complicated physical effects have been exploited by designers over the years — largely by trial and error — to move the eigenvalues to favorable locations so as to achieve a musical effect.

We begin with a drum. The simplest mathematical description of a drum is as an ideal circular membrane with clamped edges. By ideal, we mean that the membrane is perfectly uniform, has no stiffness, and experiences no losses. In particular, in this simplest description we are neglecting the surrounding air and the body of the drum.

The equation of motion for a circular membrane is

$$\nabla^2 \eta = \frac{1}{c^2}\,\eta_{tt},$$

where $\eta(r,\theta)$ is the displacement of the membrane from its equilibrium at the point (r,θ) on its surface, c^2 is T/σ, T is the tension, and σ is the mass density per unit area. The eigenmodes are obtained by assuming a solution of the form $\eta(r,\theta) = R(r)\Theta(\theta)\mathrm{e}^{-\mathrm{i}\omega t}$. This leads to the solution

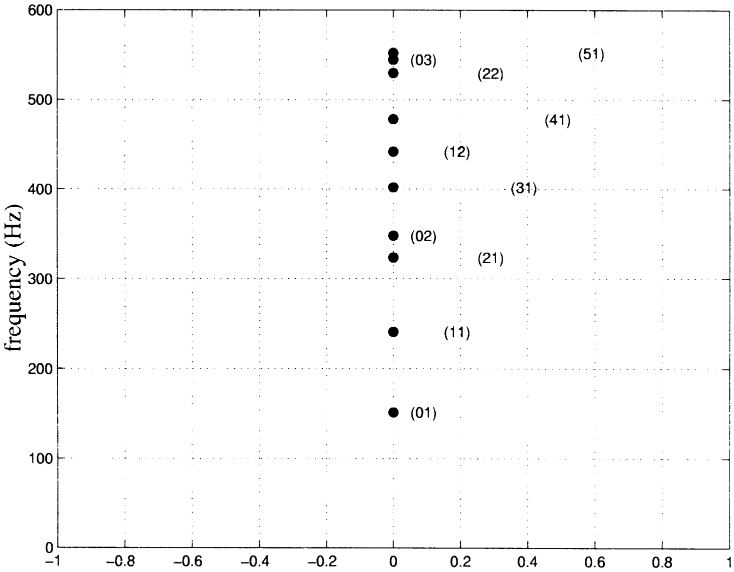

Fig. 15. Eigenvalues for an ideal drum, with angular (m) and radial (n) mode numbers indicated in parentheses (mn). These zeros of Bessel functions are not at all harmonically related; an ideal drum would not sound with any definite pitch.

$\eta(r,\theta) = AJ_m(z)\cos(m\theta)$, where $J_m(z)$ is the mth-order Bessel function of the first kind and $z = \omega r/c$. For a membrane of radius a with fixed edge, the allowed frequencies are the values ω such that $J_m(\omega a/c) = 0$ [9].

For example, given a membrane of radius 0.33 m, tension $T = 4415$ N/m, and mass density $\sigma = 0.26$ kg/m^2, which corresponds to a typical mylar membrane kettledrum, the eigenvalues up to 600 Hz are shown in Fig. 15. The mode (mn) refers to the nth zero of $J_m(\omega a/c) = 0$. The eigenfunction corresponding to mode (mn) has m nodal diameter lines and n circular nodes.

Notice that the eigenvalues of the ideal membrane are not in the least harmonically related. A real kettledrum, on the other hand, has a definite pitch. To explain this effect we need to add some more realistic physical properties to the kettledrum. The two most important properties to consider are the membrane moving through the air and the presence of the kettle. Our discussion follows the laboratory experiments and highly readable articles of Rossing [15,16].

According to Rossing, the most important phenomenon in terms of making the kettledrum sound with a definite pitch is the interaction of the membrane with the air. This "air loading" has two important effects. It tends to lower certain modes, making them more closely harmonic [15, p. 280], and it tends to damp more quickly the modes that are not nearly harmonic. We will not go into the physics, but will sketch some of the consequences.

The (11), (21), (31), (41), and (51) modes are shifted in frequency so that they are more nearly harmonically related. The (01), (02), and (03) modes are not harmonic, even with the air loading, but these modes are damped out very quickly. See Fig. 16, which shows frequencies and decay rates for a real kettledrum head (without the kettle) as measured in [4, Table V]. The drum has the same physical properties and tension as the ideal membrane described above.

Notice that the ($0n$) modes, which are not harmonically related, decay very quickly. In addition, the (11), (21), (31), and (41) modes have shifted in frequency so that they are more nearly harmonically

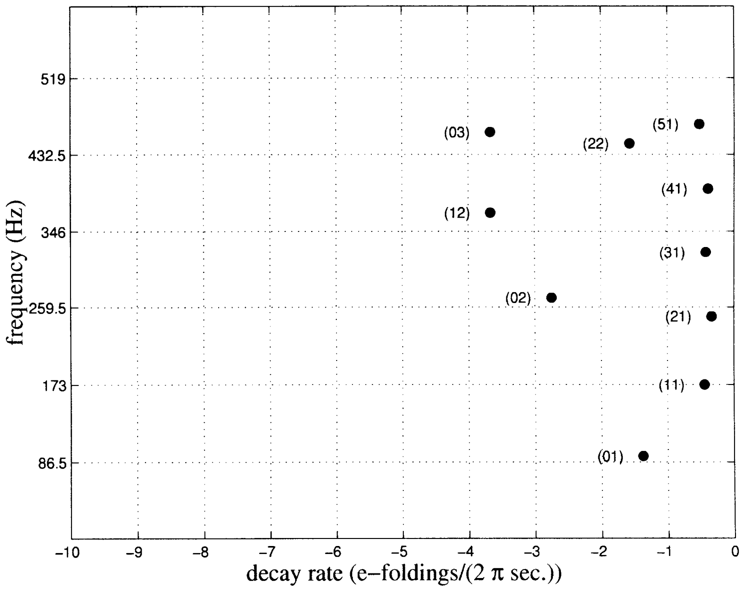

Fig. 16. Eigenvalues for a kettledrum head (without the kettle), estimated from experimental measurements [4, Table V]. Notice that the least damped frequencies are now roughly harmonic. (There are presumably other modes above 500 Hz that were not included in these measurements.)

related. Observations more than a century ago by Rayleigh were the first to show that the inharmonic but heavily damped $(0n)$ modes have little effect on the sound of the drum. He observed that touching the drum in the center, further damping these modes, has little effect on the sound [13, p. 348].

The effect of the kettle is to further tune the modes that are least damped. The modes that were made more nearly harmonic by including air loading are shifted even closer to harmonic by the presence of the kettle. This tuning is fairly small, but the kettle has a more important effect on the damping rates. It tends to make the harmonic modes, (11), (21), (31), (41), and (51), radiate less efficiently and therefore sustain longer, making the drum sound more musical [16, p. 177]. (Although these modes are harmonically related by a "missing fundamental" at one-half the frequency of the (11) mode, the pitch of the drum is usually heard as that of the (11) mode itself [16, p. 174].)

Fig. 17 shows modal frequencies and decay rates for a real kettledrum (with the kettle) as measured in [4, Table II]. Note that the tension on this drum is not the same as in the previous measurements. This drum has tension $T = 3710$ N/m.

There are other properties of a real kettledrum that affect its sound, such as bending stiffness and stiffness to shear in the membrane, but these effects are not as large [4, p. 1336]. Note also, that the sound of a kettledrum is greatly affected by the manner in which it is struck and by the characteristics of the mallet used to strike it, but we do not consider these affects.

Given that the effects of air loading and the presence of a kettle tend to make the modes approximately harmonic, why do not all drums sound with a definite pitch? Part of the answer is that years of design have led to timpani whose parameters (radius, tension, kettle size, etc.) have been favorably adjusted to achieve a musical effect. Drums constructed "at random" are hardly likely to sound the same. Another part of the answer is that many drums are intended to blend with any

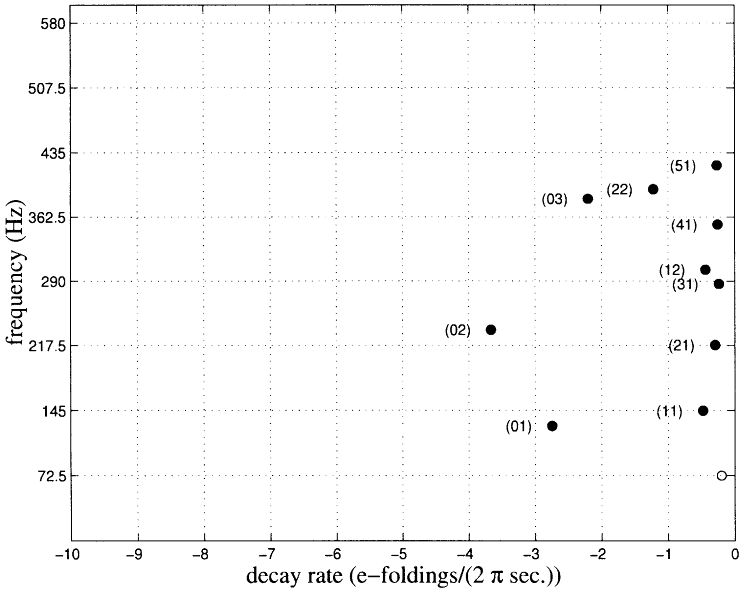

Fig. 17. Eigenvalues for a kettledrum head (with kettle), estimated from experimental measurements [4, Table II]. Modes (11), (21), and (31) are now almost exactly harmonically related. The open circle represents where the "missing fundamental" would be. (Again, some higher modes are omitted.)

key and are therefore intentionally designed not to have a definite pitch. One method is to use two drumheads, tuned to two different frequencies. This is the situation, for example, in a bass drum. The two membranes are typically tuned as much as a fourth apart and the numerous partial frequencies in combination give little sense of a pitch [7, p. 512].

Fig. 18 shows some modal frequencies and decay rates for a real bass drum as measured in [6, Table II].

6. Bells

Bells, like drums, are multi-dimensional objects, and a bell constructed "at random" will certainly not sound musical. Centuries of evolution, however, have done remarkable things to the eigenvalues of certain bells. We conclude our tour with Fig. 19, which shows the measured eigenvalues of an actual $A_4\sharp$ minor-third bell. The fundamental frequency is 456.8 Hz. In bells, there are two modes that have three nodal diameters and one nodal line. In one, the nodal circle is at the waist of the bell, and at the other, it is nearer the mouth of the bell. The first, which corresponds to the minor third, is referred to in the figure as the (31) mode. The second corresponds to the perfect fifth and is referred to as the (31#) mode. Similarly, the (21#) mode has a nodal diameter near the mouth of the bell instead of around the waist of the bell. The (21#) mode is the fundamental frequency, and the (20) mode an octave lower is known as the hum.

Eigenvalues of bells such as these are discussed in the beautiful dissertation of Roozen–Kroon [14]. For our purposes, it is enough to note the astonishingly satisfying imaginary parts of the first

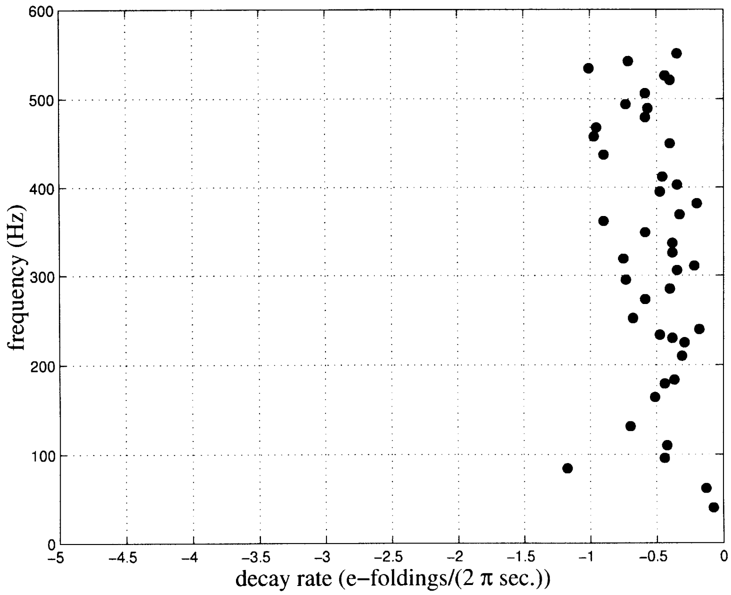

Fig. 18. Eigenvalues for a bass drum as measured in [6, Table II]. For a drum of this kind, no attempt has been made to achieve harmonic relationships among the frequencies.

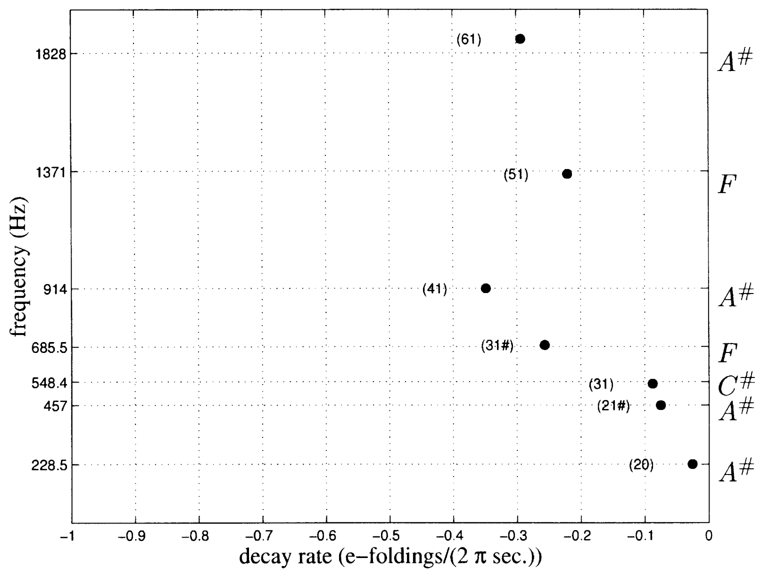

Fig. 19. Eigenvalues of a minor third $A_4\sharp$ bell, measured in [18], as given in [14, Table 5.3.1]. The grid lines show the positions of the frequencies corresponding to a minor third chord at 456.8 Hz, together with two octaves above the fundamental and one below. All six of these modes are closely matched by eigenvalues of the bell, a tribute to how far bell design has evolved over the centuries to achieve a musical effect. The eigenvalue picture for an unmusical bell, such as one worn by a cow, would look utterly different.

six eigenvalues in Fig. 19. These six notes line up like a chord played on a piano, and with decay rates as low as about half an e-folding per second, you can almost hear this clean bell ring.

Acknowledgements

It is a pleasure to thank Prof. Allard Jongman for providing the facilities of the Cornell Phonetics Laboratory for digitizing some guitar sounds, which enabled us to confirm some of the decay rates of Figs. 1–8. (Of course, real data are never quite as clean as in our figures.) The work of both authors was supported in part by NSF Grant DMS-9500975CS.

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