From K.W. Morton and M.J. Baines, eds.,
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Lax-stability vs. eigenvalue stability of spectral methods

Lloyd N. Trefethen[1]
Department of Mathematics,
Massachusetts Institute of Technology, USA.

1. Background

It is well known that spectral methods for non-periodic geometries lead to matrices that are not normal. We show that in such situations there may be a wide gap between eigenvalue stability and Lax-stability, especially for first-order problems. For example, in some cases eigenvalue analysis predicts a stability restriction $\Delta t \leq CN^{-1}$, whereas the Lax-stability restriction is $\Delta t = O(N^{-2})$. When such anomalies occur, the results of spectral calculations may be highly sensitive to rounding errors and to the smoothness of the initial and boundary data.

ABSTRACT

INTRO-
DUCTION

At this conference spectral methods were mentioned in only two or three talks, but I believe this relative obscurity is temporary. Spectral methods are nothing more than finite difference or finite element methods carried to unusually high orders of accuracy — typically by means of global trigonometric or polynomial interpolants which are differentiated to yield approximate derivatives of discrete data sequences. Since many fluid flows are smooth in at least part of the domain of interest, high-order methods are of natural utility.

It is not my purpose to present any applications of spectral methods in detail, but I will mention some representative references. One early paper of lasting interest is the report by Fornberg and Whitham (1978) on interacting solitons in the KdV equation and in other nonlinear models of water waves. Important advances in our understanding of instabilities in incompressible flows were achieved by the spectral calculations of Orszag and Kells (1980), and Orszag and Patera (1982), who showed numerically, for example, that the onset of turbulence in plane Poiseuille flow is triggered by a three-dimensional finite-amplitude instability at Reynolds number ≈1000.

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A recent paper by Fornberg (1988) illustrates the impressive capabilities of spectral methods for solving problems in elastic wave propagation, as occur in geophysics, even in the presence of discontinuous interfaces. Finally, examples of "spectral element" calculations in fluid dynamics, in which a complicated domain is subdivided into smaller domains where spectral formulas are applied, can be found in the papers of Patera and his colleagues (Maday and Patera 1988).

Recently a comprehensive monograph has appeared on this subject: *Spectral Methods in Fluid Dynamics,* by Canuto, Hussaini, Quarteroni and Zang (1988). For details and further references the reader can do no better than consult that book. Another work well worth looking at is the earlier monograph by Gottlieb and Orszag (1977).

2. The stability problem

The purpose of this paper is to discuss the numerical stability of fully discrete spectral methods for time-dependent problems with boundaries — a subject that is incompletely understood. With finite difference or finite element methods, the process of computing derivatives is local and at least approximately translation-invariant, so the matrices that arise are typically close to normal[1] and the behaviour of the process as a whole can often be approximated by Fourier techniques ("von Neumann analysis"), possibly supplemented by an investigation of wave reflection at boundaries ("GKS analysis"). For details see Richtmyer and Morton (1967), Gustafsson et al. (1972), and Trefethen (1983). By contrast, spectral differentiation is a global process, and the matrices involved may be far from normal. This paper will explore some consequences of that fact.

In particular, for spectral methods on bounded domains, Lax-stability time-step restrictions may be much tighter than the restrictions associated with various weaker definitions of stability (related, but not all identical) which go by a number of names, including "eigenvalue stability," "time-stability," "von Neumann stability," "practical stability," "s-stability," and "stability in the sense of Forsythe and Wasow." In outline, the two ideas to be contrasted here are as follows:

Lax-stability: stability for fixed t as $\Delta x, \Delta t \to 0$.

Eigenvalue stability: stability for fixed Δx and Δt as $t \to \infty$.

To make the definitions precise, one has to look at norms of operators. Let $\|\;\|$ be an appropriate norm, and let $S_{N,\Delta t}$ be the discrete solution operator

[1]A normal matrix is one that possesses a complete orthogonal system of eigenvectors. Equivalently, A is normal if and only if $A^H A = AA^H$, where A^H is the conjugate transpose. Symmetric, skew-symmetric, orthogonal, and circulant matrices fall in this category.

on a grid of N points with time step Δt:

$$S_{N,\Delta t} : v^n \mapsto v^{n+1}, \qquad v^n = S^n_{N,\Delta t} v^0, \tag{2.1}$$

where v^n represents the computed solution at time step n. (For an m-step discretization, v^n is replaced by $w^n = (v^n, \ldots, v^{n+1-m})^T$ and $S_{N,\Delta t}$ becomes a block companion matrix of dimension mN.) Lax-stability is defined by

Lax-stability: $\|S^n_{N,\Delta t}\| \le C$ for all N, n such that $n\Delta t \le T$

for any T and some constant $C = C(T)$, where we assume a fixed relationship between the space and time discretizations:

$$\Delta t = \Delta t(N). \tag{2.2}$$

Eigenvalue stability is defined for ***fixed*** N and Δt by

Eigenvalue stability: $\|S^n_{N,\Delta t}\| \le C$ for all n,

for some constant C. This is equivalent to $\rho(S_{N,\Delta t}) \le 1$, if ρ denotes the spectral radius (largest eigenvalue in absolute value), together with the condition that any eigenvalues on the unit circle should be nondefective.

For some purposes these definitions must be weakened by logarithmic or algebraic factors in n or N, but since my concern here is general phenomena rather than precise theorems, I will not worry about such details. What is important is that Lax-stability is a uniform bound for all matrices in a certain class corresponding to finer and finer meshes, whereas eigenvalue stability is a bound on the powers of a single matrix corresponding to a fixed mesh.

For the common situation in which a spectral discretization consists of a spectral differentiation operator D_N in x coupled with a standard o.d.e. formula in t, there is a further equivalent statement of eigenvalue stability:

Eigenvalue stability: The eigenvalues of D_N lie within the stability region for the time-integrator.

(To be precise, we again permit only non-defective eigenvalues on the boundary.) An advantage of eigenvalue stability is that it is a relatively elementary concept, for stability regions are a familiar tool among numerical analysts (Gear 1971).

The points to be made in this paper can be summarized as follows. In comparison with finite difference and finite element methods, instabilities in spectral methods are

(A) Less well understood, and (B) More troublesome.

In particular, the stability condition for an explicit spectral method is typically both harder to predict and more restrictive than for explicit finite differences (assuming a uniform mesh). Furthermore, because a gap may arise between Lax-stability and eigenvalue stability, instabilities in spectral methods are sometimes

(C) More sensitive to rounding errors

and

(D) More sensitive to smoothness of the solution,

especially for first-order problems. The origin of (C) and (D) is the characteristic virtue of spectral methods: their high order of accuracy. For smooth problems, the truncation errors in a spectral calculation are often zero or negligible, so that any instability present is excited only by rounding errors. Consequently there are circumstances in which a Lax-unstable spectral method may give accurate answers. However, the accuracy may be destroyed by rounding errors, non-smooth data, or other perturbations such as variable coefficients or lower-order terms.

The distinctions between various notions of stability have been investigated for many years in the theoretical literature of finite difference methods; perhaps the central point of this paper is that certain spectral methods provide examples in which these distinctions are of paramount importance. A classic paper on convergence of unstable formulas with analytic initial data was written by Dahlquist (1954). The Lax stability theory appeared shortly thereafter in the survey by Lax and Richtmyer (1956). The sensitivity to perturbations of certain eigenvalue stable but Lax-unstable formulas was explored in an important paper by Kreiss (1962), which unfortunately has had limited influence because it was written in German. Discussions of various weaker definitions of stability can be found in Strang (1960) and Gottlieb and Orszag (1977), among others. The book by Richtmyer and Morton (1967) remains an excellent source on many of these topics.

3. 2nd-order differentiation

Consider the model problem

$$u_t = u_{xx}, \quad x \in [-1,1], \quad u(\pm 1, t) = 0, \tag{3.1}$$

and for concreteness let

$$1 = x_0 > x_1 > \cdots > x_{N-1} > x_N = -1 \tag{3.2}$$

be the *Chebyshev extreme points* (= *Gauss-Lobatto-Chebyshev points*) defined by $x_j = \cos(j\pi/N)$. Like most grids for nonperiodic spectral methods, this grid has spacing $O(N^{-1})$ in the interior but $O(N^{-2})$ near the

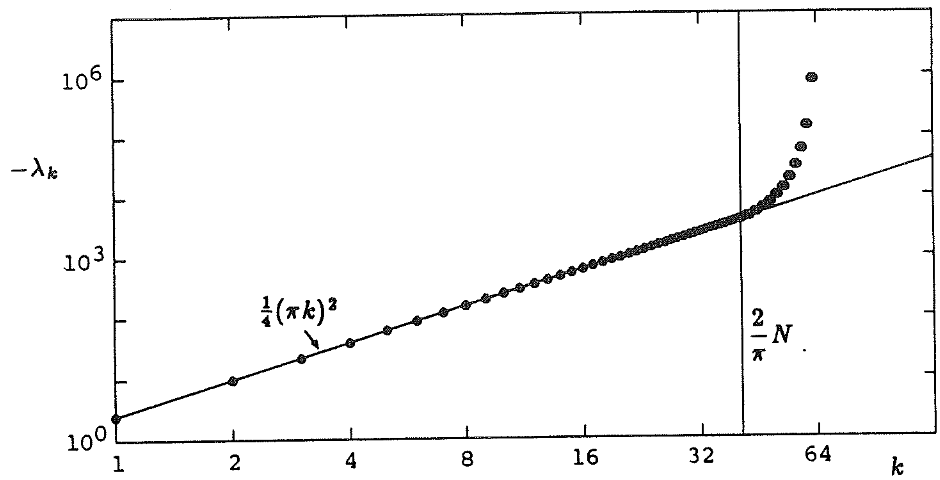


Fig. 1: Eigenvalues of the second-order spectral differentiation matrix for Chebyshev extreme points, $N = 64$ (log-log scale). The eigenvalues with $k > 2N/\pi$ occur in approximate pairs

boundary; such nonuniformity is essential if one is to avoid the Runge phenomenon of wild oscillations (Trefethen and Weideman 1989). If $v = (v_1, \ldots, v_{N-1})^T$ denotes a vector of data at positions $x_1, \ldots, x_{N-1}$, then the second-order spectral differentiation matrix D_N is the $(N-1) \times (N-1)$ matrix

$$D_N : v \mapsto w \tag{3.3}$$

defined implicitly as follows (explicit matrix entries can be derived from the first-order matrix given by Canuto et al. (1988), p. 69:

(1) Interpolate v by a polynomial p of degree N with $p(\pm 1) = 0$;
(2) Set $w_j = (D_N v)_j = p''(x_j)$, $1 \le j \le N-1$.

This differentiation process is global and, for "smooth" vectors v, highly accurate. A semidiscrete spectral approximation to (3.1) is now provided by the system of $N-1$ ordinary differential equations

$$v_t = D_N v, \tag{3.4}$$

and if the time derivative is replaced by a linear multistep or Runge–Kutta formula, the approximation becomes fully discrete.

The eigenvalues of D_N are real and negative — and as illustrated in Fig. 1 for $N = 64$, some of them are huge. A proportion $2/\pi$ are of order $O(N^2)$, and approximate closely the eigenvalues $\frac{1}{4}(\pi k)^2$ of the associated differential operator; the corresponding eigenvectors are very nearly the sines and cosines one gets for the latter, represented by at least two points per wavelength throughout the spatial grid. But the remaining "outlier" eigenvalues are of order $O(N^4)$, with the largest being about $0.0474N^4$ (Ouazzani et al. 1982, Trefethen and Weideman 1989). A rigorous upper bound as $N \to \infty$ is $\sqrt{11/4725}N^4 \approx 0.0482N^4$.

Because the largest eigenvalues are so big, any explicit time integration of (3.4) will be subject to an eigenvalue stability restriction

$$\Delta t \leq CN^{-4} \tag{3.5}$$

for some constant C that depends on how much of the negative real axis is contained in the stability region for the time-integrator. For Runge–Kutta formulas of orders 1–4 the constants are $C \approx 42$, 42, 53, 59, respectively; for the corresponding Adams–Bashforth formulas they are $C \approx 42$, 21, 11, 6.3.

Although the eigenvalues of D_N are real, D_N is not symmetric, or normal; but it is close to normal. One measure of this is a comparison between the spectral radius (largest eigenvalue) and the 2-norm (largest singular value):

$$\rho(D_N) \approx 0.0474N^4, \qquad \|D_N\| \approx 0.0483N^4. \tag{3.6}$$

Another is the size of the commutator:

$$\frac{\|D_N^T D_N - D_N D_N^T\|}{\|D_N^T D_N\|} \approx 0.163. \tag{3.7}$$

(These are empirical results for large N, based on the 2-norm.) A more useful measure of closeness to normality is the modest size of the condition number $\kappa(V) = \|V\|\,\|V^{-1}\|$ of V, the matrix of normalized eigenvectors of D_N:

$$\kappa(V) \approx 1.72,\ 4.06,\ 11.31 \quad \text{for} \quad N = 8,\ 32,\ 128. \tag{3.8}$$

(If D_N were normal $\kappa(V)$ would be 1.) So far as I am aware, because D_N is close to normal, important differences do not arise in this second-order problem between Lax-stability and eigenvalue stability, and the same condition (3.5) is at least approximately valid for both. This conclusion appears to extend also to other Chebyshev and Legendre meshes.

Here is a summary of the stability of second-order spectral differentiation in terms of statements (A)–(D) of the last section. First, (A) stability restrictions for explicit formulas are not as easy to predict as with finite differences, but reasonable estimates can be obtained in some cases by looking at coefficients of characteristic polynomials (Weideman and Trefethen 1988). Second, (B) these stability limits are extremely restrictive. Third, the matrices are close enough to normal that Lax-stability and eigenvalue stability correspond closely, and there is no exaggerated sensitivity to (C) rounding errors or (D) smoothness of the solution.

Because the stability limits for explicit second-order spectral formulas are so restrictive, every effort should be made to employ implicit schemes instead. The efficient implementation of implicit spectral methods is a topic of active current research, and impressive results have been achieved by iterative methods with suitably chosen preconditioners; see Chapter 5 of Canuto et al. (1988).

4. 1st-order differentiation in Chebyshev points

Now consider the first-order problem

$$u_t = u_x, \quad x \in [-1,1], \quad u(1,t) = 0, \tag{4.1}$$

again on the Chebyshev grid (3.2); to make the problem well-posed only the inflow boundary condition has been specified. We now deal with a vector $v = (v_1, \ldots, v_N)^T$ of data values at $x_1, \ldots, x_N$, and the first-order spectral differentiation matrix

$$D_N : \; v \mapsto w \tag{4.2}$$

is an $N \times N$ matrix defined implicitly as follows (for explicit entries see Canuto et al. (1988), p. 69):

(1) Interpolate v by a polynomial p of degree N with $p(1) = 0$;
(2) Set $w_j = (D_N v)_j = p'(x_j)$, $1 \leq j \leq N$.

The eigenvalues of D_N are complex, and they are plotted in Fig. 2 for $N = 32$. As in the second-order case of the last section, most of them (about 82%) are of reasonable size, $O(N)$, but the remainder are much larger: $O(N^2)$. (Details can be found in the recent paper by Dubiner (1987); the exact proportion of "outlier" eigenvalues approaches $1/2 - 1/\pi \approx 0.1817$ as $N \to \infty$.) Therefore, any explicit time integration of the spectral semidiscretization of (4.1) will be subject to an eigenvalue stability restriction

$$\Delta t \leq CN^{-2} \tag{4.3}$$

for some C that depends on the stability region for the time-integrator.

As pointed out first by Trefethen and Trummer (1987), the eigenvalues of D_N are highly sensitive to small perturbations such as rounding errors. To illustrate this, Fig. 2 shows eigenvalues computed in 16-digit precision together with eigenvalues of the same matrix after rounding each entry to 8 digits (relative to the largest element of the matrix). This figure shows that D_N must be far from normal, since the eigenvalues of normal matrices

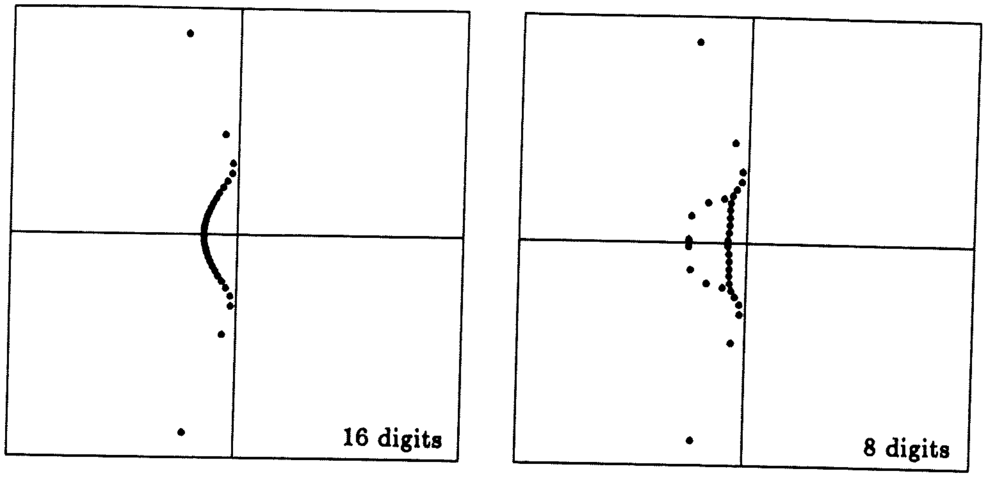


Fig. 2: Eigenvalues in the complex plane of the first-order spectral differentiation matrix for Chebyshev extreme points, N=32. The region shown is the square bounded by $\pm 100 \pm 100i$

are well-conditioned functions of the matrix entries. In analogy to (3.6)–(3.8), here are some measures of non-normality: the spectral radius and 2-norm differ considerably,

$$\rho(D_N) \approx 0.0886 N^2, \qquad \|D_N\| \approx 0.5498 N^2, \tag{4.4}$$

the commutator is larger than before,

$$\frac{\|D_N^T D_N - D_N D_N^T\|}{\|D_N^T D_N\|} \approx 0.870, \tag{4.5}$$

and the condition number of the matrix of eigenvectors is huge:

$$\kappa(V) \approx 5.8e2,\ 7.1e5,\ 1.1e12 \ \text{ for } \ N = 8,\ 16,\ 32. \tag{4.6}$$

What are the consequences of the non-normality of D_N? Are Lax-stability and eigenvalue stability quite distinct for this problem? So far as I am aware the answer is no, except possibly if an unusual time-integration formula is chosen. Experiments indicate that Lax-stability as well as eigenvalue stability is essentially determined by the need to fit the largest of the eigenvalues in a stability region; Fig. 2 suggests that the outlier eigenvalues, after all, are insensitive to perturbations. Since space is limited, therefore, we shall leave this example without further discussion and turn to a problem in which the difference between Lax-stability and eigenvalue stability is pronounced.

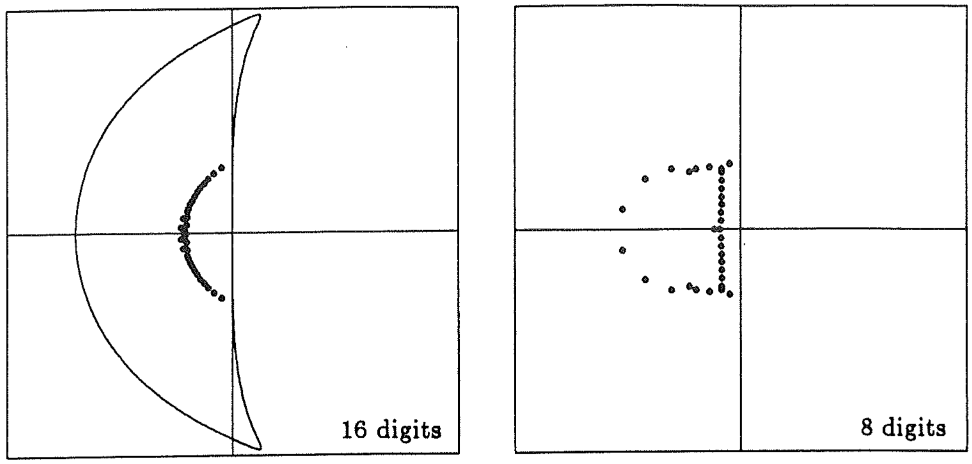


Fig. 3: Eigenvalues in the complex plane of the first-order spectral differentiation matrix for Legendre points, $N = 32$ (same scale as in Fig. 2). The stability region plotted on the left applies to Fig. 7, below

5. 1st-order differentiation in Legendre points

We now consider the most interesting example of this paper: the same problem as in the last section, but with $x_1, \ldots, x_N$ replaced by the zeros of the Legendre polynomial $P_N(x)$. These Legendre points are not far different from Chebyshev points, but according to a discovery of Dubiner (1987), the eigenvalues of the corresponding matrix D_N are much smaller than before: $O(N)$ rather than $O(N^2)$. Therefore should it not be possible to replace (4.3) by a more favorable stability restriction $\Delta t \le CN^{-1}$?

Following upon Dubiner's theoretical work, Tal-Ezer (1986) carried out experiments that indicated that this optimistic expectation is indeed justified, at least for certain scalar problems in one dimension. However, our own view is that eigenvalue analysis can be misleading, and the optimism must be qualified. Legendre spectral methods with large time steps may be sensitive to non-smooth data and to other perturbations of the problem such as the introduction of variable coefficients or lower-order terms. This sensitivity is of just the kind Kreiss warned of in 1962.

To begin the numerical illustrations, Fig. 3 repeats Fig. 2 for Legendre points. The prediction that the eigenvalues are $O(N)$ is clearly valid, in exact arithmetic, but as in the Chebyshev case, they are highly sensitive to perturbations. Therefore D_N must again be far from normal, and our measures of normality come out as follows:

$$\rho(D_N) \approx N, \qquad \|D_N\| \approx N^2, \tag{5.1}$$

$$\frac{\|D_N^T D_N - D_N D_N^T\|}{\|D_N^T D_N\|} \approx 1, \tag{5.2}$$

$$\kappa(V) \approx 8.5e3,\ 3.1e8,\ 2.5e17 \text{ for } N = 8,\ 16,\ 32. \tag{5.3}$$

Evidently the matrices D_N are even farther from normal than in the Chebyshev case. What is more important, the large outlier eigenvalues are no longer present to mask the non-normality.

Let us now restrict attention to the 3rd-order Adams–Bashforth formula

$$v^{n+1} = v^n + \frac{\Delta t}{12} D_N(23v^n - 16v^{n-1} + 5v^{n-2}). \tag{5.4}$$

For eigenvalue stability, the stability region of the time-integrator must enclose the eigenvalues of D_N, and computations show that the condition for this in exact arithmetic is approximately

$$\textit{Eigenvalue stability:}\quad \Delta t \le 0.734 N^{-1}. \tag{5.5}$$

In the presence of rounding errors, however, Fig. 3 suggests that this condition would have to be tightened, as is confirmed by experiments (Trefethen and Trummer 1987). It follows that (5.5) cannot be enough to ensure Lax-stability, for rounding errors are small perturbations, and thus by definition (together with the discrete Duhamel principle), Lax-stability would entail insensitivity to them.

Instead, I conjecture that the Lax-stability restriction for this model problem is approximately

$$\textit{Lax-stability:}\quad \Delta t = O(N^{-2}). \tag{5.6}$$

The use of the notation $O(N^{-2})$ rather than CN^{-2} is deliberate: (5.6) means that the spectral method is Lax-stable as $N \to \infty$ and $\Delta t(N) \to 0$ if and only if there exists *any* constant a such that

$$\Delta t \le a N^{-2} \tag{5.7}$$

for all N. Equation (5.6) may not be the exact condition for Lax-stability of this model problem, since rigorous analysis of the stability of spectral methods is typically complicated by small factors, as mentioned in Section 2; in any case an exact result may depend on the choice of norm. But I believe it is at least close to correct. The "only if" half of the statement is proved in the next section.

For empirical justification of (5.6), consider the solution operator $S_{N,\Delta t}$: $(v^n, v^{n-1}, v^{n-2})^T \mapsto (v^{n+1}, v^n, v^{n-1})^T$ for the Adams–Bashforth formula (5.4), which is a matrix of dimension $3N$:

$$S_{N,\Delta t} = \begin{pmatrix} I & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix} + \frac{\Delta t}{12} \begin{pmatrix} 23D_N & -16D_N & 5D_N \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.8}$$

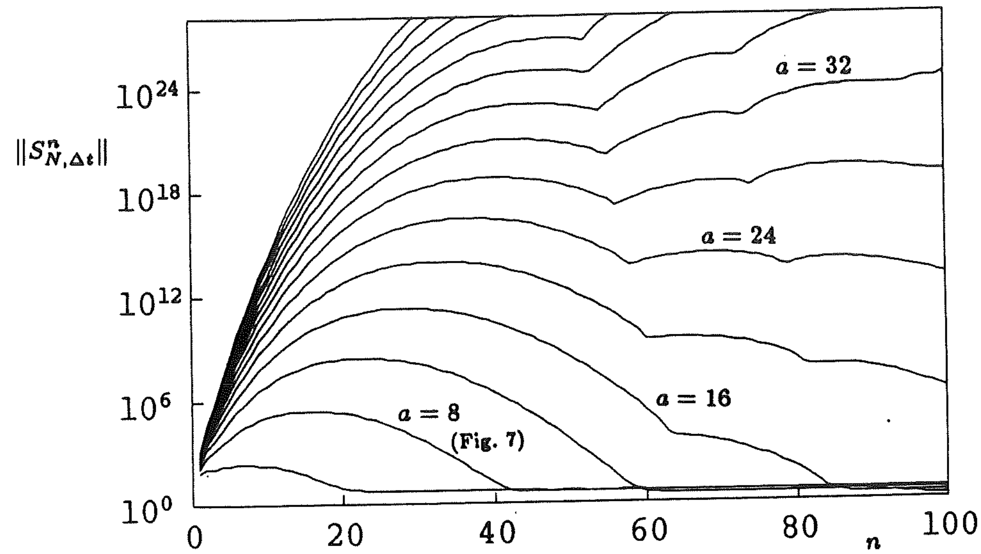


Fig. 4: Legendre grid with $N = 32$: $\|S^n_{N,\Delta t}\|$ as a function of n for $\Delta t = aN^{-2}$, various a. $S_{N,\Delta t}$ is power-bounded approximately for $a \le 23.5$

Fig. 4 shows numerically computed norms $\|S^n_{N,\Delta t}\|$ as functions of n with $N = 32$ for various values a in (5.7). (From now on $\| \ \|$ denotes $\| \ \|_\infty$, which can be computed much faster than $\| \ \|_2$.) For $a \le 0.734 \times 32 \approx 23.5$ (see (5.5)), $S_{N,\Delta t}$ is power-bounded and thus eigenvalue stable, but the figure shows that tremendous growth of the powers $S^n_{N,\Delta t}$ takes place for much smaller a, before the eventual decay.

Now for any N and Δt, let $\gamma_{N,\Delta t}$ be the power-boundedness constant

$$\gamma_{N,\Delta t} = \sup_n \|S^n_{N,\Delta t}\|, \tag{5.9}$$

or in other words, the maximum value of one of the curves in a plot like Fig. 4. If $\rho(S_{N,\Delta t}) > 1$ (ρ being again the spectral radius), or if $\rho(S_{N,\Delta t}) = 1$ with a defective eigenvalue on the unit circle, then $\gamma_{N,\Delta t} = \infty$ and the formula is eigenvalue unstable; otherwise it is eigenvalue stable. Fig. 5 shows $\gamma_{N,\Delta t}$ as a function of N for various values of a in (5.7). (To suppress the distracting irregularity that would otherwise result from the discreteness of N, the corner of each curve has been shifted slightly to the position corresponding to (5.5).) Each curve is eventually approximately constant, no matter how large a is, but the constants grow exponentially with a. Clearly Lax-stability is in jeopardy if (5.7) does not hold for some a.

A different view of the same data is provided by Fig. 6, which is a rough contour plot, on a log-log scale, of $\gamma_{N,\Delta t}$ as a function of N and Δt^{-1}. Below the line $\Delta t = 0.734N^{-1}$, with slope 1, the prediction $\gamma_{N,\Delta t} = \infty$ is verified. The interesting behaviour lies above this line, where $\gamma_{N,\Delta t}$ is finite and we

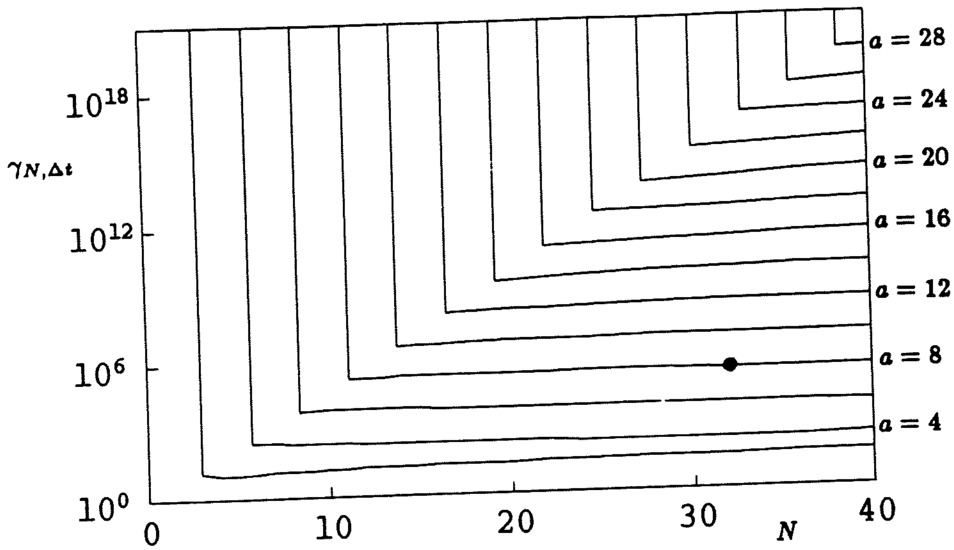


Fig. 5: Legendre grid: power-boundedness constants $\gamma_{N,\Delta t}$ (5.9) for $\Delta t = aN^{-2}$, various a. The solid dot marks the computation of Fig. 7

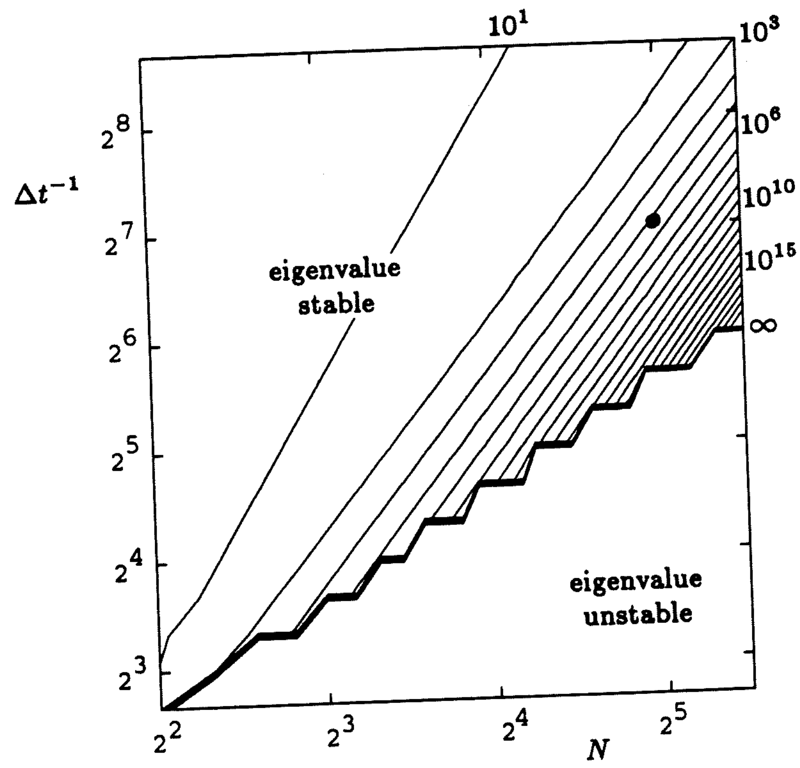


Fig. 6: Legendre grid: contour plot of the power-boundedness constants $\gamma_{N,\Delta t}$ (5.9). The solid dot marks the computation of Fig. 7

see a set of straight parallel contours with slope 2. For Lax-stability it is permissible to traverse this slope at a constant altitude, however high, and evidently (5.7) does just that. But any relationship of Δt to N that violates (5.7) as $N \to \infty$ will lead to an ascent of the slope rather than a traversal: Lax-instability.

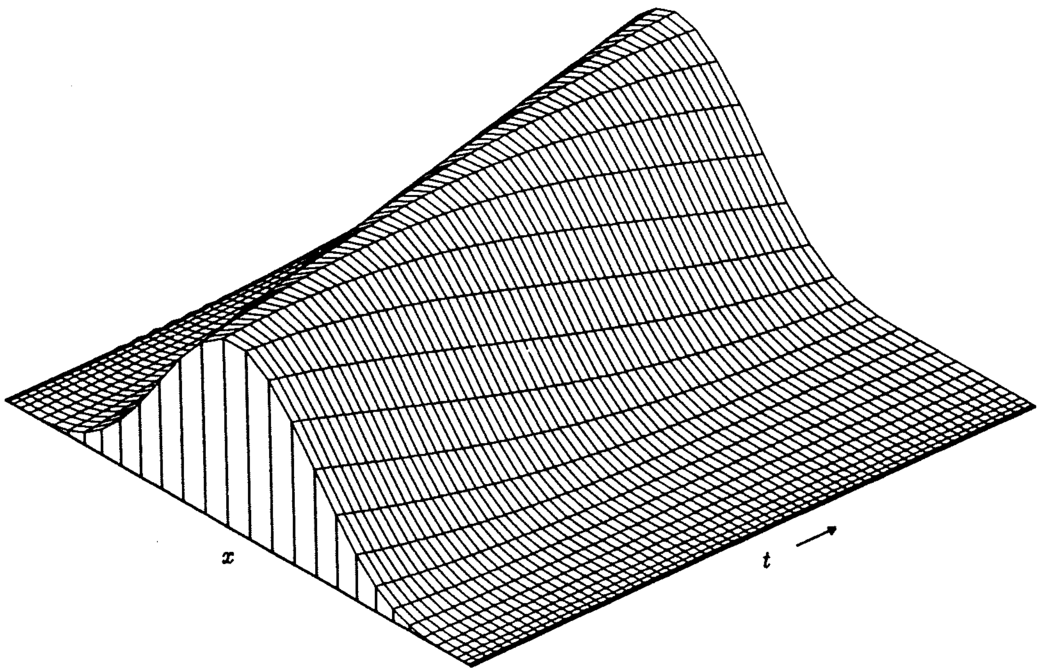
After several pages devoted to norms of operators, our final figure represents an actual spectral calculation selected to show that Lax-instability may reveal itself in some experiments but not others. Fig. 7 is based on computations with $N = 32$ and $\Delta t = aN^{-2}$, $a = 8$, for which the Adams–Bashforth formula is eigenvalue stable even in low-precision arithmetic; the stability region was shown in Fig. 3. However, Figs. 4–6 indicate that $\gamma_{N,\Delta t} \approx 10^{5.4}$, so the potential for unstable growth is present. But in Fig. 7a the initial function is $u(x,0) = \cos^4(\pi x/2)$; when extended by zeros outside $[-1,1]$, this function is three times continuously differentiable, which is evidently smooth enough that the computation is successful, but in Fig. 7b the initial function is $\cos^2(\pi x/2)$, whose extension outside $[-1,1]$ is only once continuously differentiable, and now errors appear at the boundary that grow considerably before dying away. The computation is unsuccessful, at least if transient phenomena are of importance.

Fig. 7b represents a mild, borderline example chosen to make the plots interesting; in general the instability may be far worse. With $a = 12$ or 16, for example, the error at the boundary grows to 3.1×10^3 and 2.5×10^6, respectively, before eventually decaying away. Trefethen and Trummer (1987) used this same initial data with $a = 16$, but examined the results only at $t = 1$ and failed to notice the instability — an indication that errors appearing in at least some unstable spectral calculations may indeed be transient.

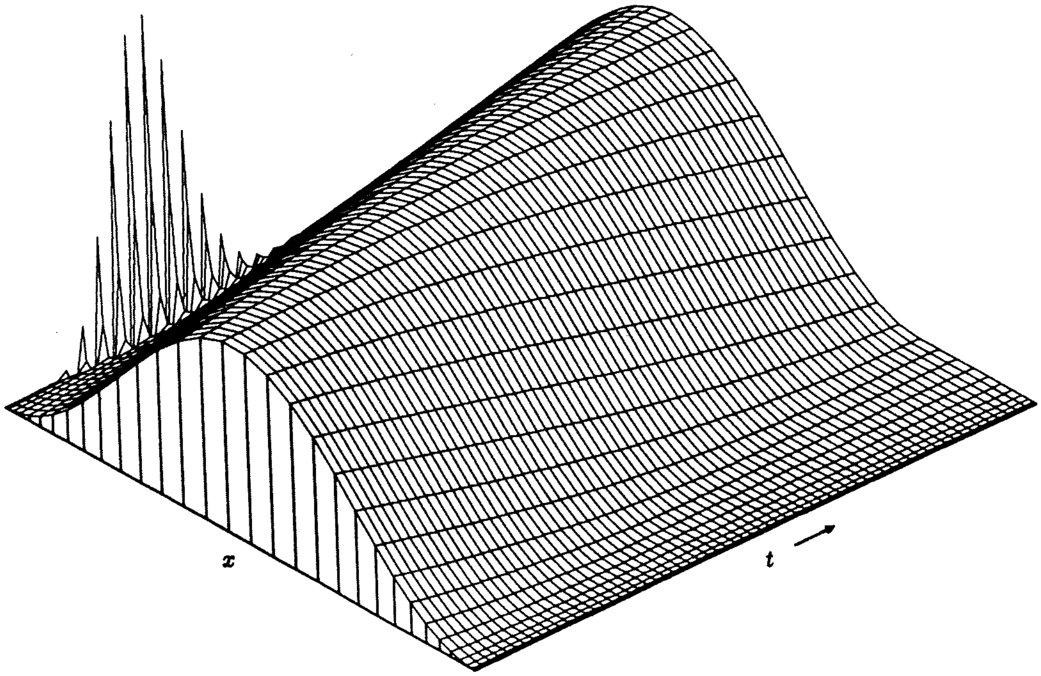
6. Conclusion; a pseudo CFL condition

Although many spectral calculations exhibit none of the stability problems highlighted in the example of the last section, that example is by no means unique. For example, another model problem with similar properties (but easier to analyze) is the equation $u_t = -xu_x$ on a Chebyshev or Legendre grid in $[-1,1]$ with no boundary conditions (Solomonoff and Turkel 1988, Trefethen and Trummer 1987). Since the papers in the literature that address the question of stability mainly restrict their examples to constant coefficients and analytic initial data, the limitations of eigenvalue analysis have not received as much notice as they deserve.

This paper has presented experiments; what is needed now is theory. A starting point would be to prove that the example of the last section is indeed Lax-stable if and only if (5.6) holds (perhaps in a slightly strengthened form). But deeper and more general questions suggest themselves



(a) $u(x,0) = \cos^4(\pi x/2)$ (smooth)



(b) $u(x,0) = \cos^2(\pi x/2)$ (less smooth)

Fig. 7: Legendre grid: two calculations for $0 \leq t \leq 0.5$ with $N = 32$, $\Delta t = 8N^{-2}$

too. What general techniques can be devised for predicting stability limits of fully discrete spectral methods? When is Lax-stability needed in practice? For steady-state rather than time-accurate calculations, under what circumstances is it safe to settle for less than Lax-stability and ignore transient blow-ups like that of Fig. 7b?

I will close with an observation that goes half-way towards justifying (5.6). That condition must be *necessary* for Lax-stability, for the simple reason that D_N contains elements of order $O(N^2)$, which implies that $S_{N,\Delta t}$ contains elements of order $O(\Delta t N^2)$; if (5.6) is violated, $S_{N,\Delta t}$ cannot itself be bounded uniformly in N, to say nothing of its powers. This argument generalizes to a "pseudo CFL condition" for explicit spectral methods: small gaps between mesh points imply correspondingly tight stability restrictions. (The usual CFL argument, based on domains of dependence, is vacuous for spectral methods because the spectral differentiation process is global.) Here is one example of how this assertion can be made into a theorem:

THEOREM. *Consider the pseudospectral approximation to (4.1) based on any grid (3.2) and any consistent, explicit time-integration formula, and define* $\Delta x = x_{N-1} - x_N$. *A necessary condition for Lax-stability of this approximation, in any matrix norm subordinate to a vector norm, is*

$$\Delta t = O(\Delta x) \text{ as } N \to \infty.$$

Proof. Let $p(x) = (x - x_0)\cdots(x - x_{N-1})/(x_N - x_0)\cdots(x_N - x_{N-1})$ be the polynomial interpolant to the values $0,\ldots,0,1$ at $x_0,\ldots,x_{N-1},x_N$, respectively. Then $(D_N)_{NN} = p'(x_N) = \sum_{j=0}^{N-1}(x_N - x_j)^{-1} < -1/\Delta x$, and thus $S_{N,\Delta t}$ contains an element of order at least $\Delta t/\Delta x$ as $N \to \infty$. For Lax-stability this element must be $O(1)$ as $N \to \infty$, hence the condition $\Delta t = O(\Delta x)$. ∎

In later work, I hope to make the observations of this paper more general and precise. Many fundamental problems of numerical analysis, not only in the area of spectral methods, depend upon a proper treatment of functions of non-normal matrices. Meanwhile, I remind readers that despite all of this attention to pathologies, spectral methods can be extraordinarily accurate; we shall see more of them in the years ahead.

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