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REAL VS. COMPLEX RATIONAL CHEBYSHEV APPROXIMATION ON COMPLEX DOMAINS

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Let S be a Jordan region symmetric about the real axis, and consider best maximum norm approximation on S of an analytic (or merely continuous) function f satisfying $f(\bar{z}) = \overline{f(z)}$ by a rational function of type (m,n) with either real or complex coefficients. For $m = 0$ and $n \geq 4$, the error in complex approximation can be arbitrarily much smaller than the error in real approximation. In contrast, for $(m,n) = (0,1)$ the complex error can be better by at most a constant factor.

1. Introduction; statement of the result

Let $S \subset \mathbb{C}$ be a compact point set symmetric about the real axis, let C^r be the set of complex functions f defined and continuous on S and satisfying $f(\bar{z}) = \overline{f(z)}$, let $A^r \subset C^r$ be the subset of functions analytic in the interior of S, let $E^c_{mn}(f)$ be the error in best approximation (with respect to the maximum norm) of f on S from the set R^c_{mn} of complex rational functions with numerator degree at most m and denominator degree

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at most n, and let $E^r_{mn}(f)$ denote the corresponding error for the subset $R^r_{mn} \subset R^c_{mn}$ with real coefficients. Finally, set

$$\gamma^S_{mn} := \inf_{f \in A^r \setminus R^r_{mn}} \frac{E^c_{mn}(f)}{E^r_{mn}(f)} .$$

For the case where $S = I$ is an <u>interval</u> of the real line (and hence $A^r = C^r$), A.A. Gončar seems to have been the first to notice that for some $f \in C^r$, complex approximations are better than real ones. From the work of K.N. Lungu, E.B. Saff and R.S. Varga [4, 6], and A. Ruttan [3] various conditions (necessary or sufficient) are known for $E^c_{mn}(f) < E^r_{mn}(f)$, and in particular, it is known that $\gamma^I_{mn} < 1$ for all $m \geq 0$, $n \geq 1$. However, no triple (f,m,n) with $E^c_{mn}(f) < \frac{1}{2} E^r_{mn}(f)$ and no positive lower bound for any γ^I_{mn} have been known until recently, when we established the results

$$\gamma^I_{mn} = 0 \quad \text{for} \quad n \geq m+3 , \tag{1}$$

$$\gamma^I_{01} > 0 \tag{2}$$

[2, 5]. At the same time we showed that for the closed <u>unit disk</u> Δ

$$\gamma^\Delta_{0n} = 0 \quad \text{for} \quad n \geq 4 \tag{1'}$$

[2, 5], and we mentioned that our arguments for (1') and (2) can be extended to the case where S is the closure of a Jordan region symmetric about $\mathbb{R}$ whose boundary is differentiable at the two points of intersection with the real axis:

$$\gamma^S_{0n} = 0 \quad \text{for} \quad n \geq 4 , \tag{1"}$$

$$\gamma^S_{01} > 0 . \tag{2"}$$

The proof of (1") is short, and it is described in [5]. In contrast, the proof of (2") is tedious, and the similar proof of (2) was only outlined in [5]. Here, we now want to state the whole proof for the case of a symmetric Jordan region. Analyticity of f turns out to be inessential, i.e. we prove a slightly more general result:

THEOREM. *Let $S \subset \mathbb{C}$ be the closure of a Jordan region which is symmetric about the real axis and whose boundary is differentiable at the two points of intersection with the real axis. Then*

$$\inf_{f \in C^r \setminus R^r_{01}} \frac{E^c_{01}(f)}{E^r_{01}(f)} > 0 . \tag{3}$$

A fortiori, $\gamma^S_{01} > 0$.

We do not claim that (1), (1'), and (1") have much significance for practical approximation problems. In a realistic application, $E^c_{mn}(f)$ will rarely be much less than $E^r_{mn}(f)$, and even if it is, an increase in degree of a real approximation will probably be more cost-effective than a switch to complex coefficients. In fact, if $f \in C^r$, than for any $c \in R^c_{mn}$ the associated function $r(z) := \frac{1}{2}[c(z) + \overline{c(\overline{z})}]$ lies in $R^r_{m+n,2n}$ and

$$||f - r|| = \frac{1}{2} \max_{z \in S} \left|[f(z) - c(z)] + [f(z) - \overline{c}(z)]\right| \leq ||f - c|| .$$

In particular, $E^r_{m+n,2n}(f) \leq E^c_{mn}(f)$, cf. [4, Prop. 1].

2. Proof

Let $f \in C^r$ be arbitrary, and let c^* be a best approximation of f from R^c_{mn} . For any $c \in R^c_{mn}$ set

$$\overline{c}(z) := \overline{c(\overline{z})} , \quad \hat{c}(z) := \frac{1}{2}[c(z) - \overline{c}(z)] .$$

Then

$$||\hat{c^*}|| = \frac{1}{2} \max_{z \in S} \left|[f(z) - c^*(z)] - [f(z) - \overline{c}^*(z)]\right|$$

$$\leq \|f - c^*\| = E^c_{mn}(f) .$$

Since $E^r_{mn}(f) \leq E^c_{mn}(f) + \|c^* - r\|$ for every $r \in R^r_{mn}$, it follows that

$$E^r_{mn}(f) \leq E^c_{mn}(f) + \|\hat{c^*}\| \frac{\|c^* - r\|}{\|\hat{c^*}\|} \leq E^c_{mn}(f) \left[1 + \frac{\|c^* - r\|}{\|\hat{c^*}\|} \right] \tag{4}$$

if $\|\hat{c^*}\| \neq 0$, i.e. $c^* \notin R^r_{mn}$.

Now suppose that for any $c \in R^c_{mn} \setminus R^r_{mn}$ with no poles on S we can find $r_c \in R^r_{mn}$ (depending on c) such that

$$\frac{\|\hat{c}\|}{\|c - r_c\|} \geq \delta \tag{5}$$

for some fixed $\delta > 0$. According to (4) this implies

$$\frac{E^c_{mn}(f)}{E^r_{mn}(f)} \geq \frac{1}{1 + 1/\delta} \tag{6}$$

whenever $c^* \notin R^r_{mn}$. But otherwise $E^c_{mn}(f) = E^r_{mn}(f)$ trivially. Therefore if (5) holds, (6) is true for all $f \in C^r \setminus R^r_{mn}$.

Our proof for the case $(m,n) = (0,1)$ consists in defining a suitable mapping $c \mapsto r_c$ and in verifying (5) for this mapping.

First, without loss of generality we may assume that $\pm 1 \in \partial S$. Then, as in [5], $S^c := \bar{\mathbb{C}} \setminus S$ (where $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$) is split up into

$$A^{\pm} := \{z \in \mathbb{C} ; |\arg(-1 \pm z)| < \theta\} \cup \{\infty\} ,$$

$$C := \{z \in \mathbb{C} ; |z| \geq \rho\} \setminus (A^+ \cup A^-) ,$$

$$B := \mathbb{C} \setminus (A^+ \cup A^- \cup C \cup S) ,$$

as indicated in Fig. 1. $\theta \in (0, \pi/4)$ and ρ are assumed to be chosen such that

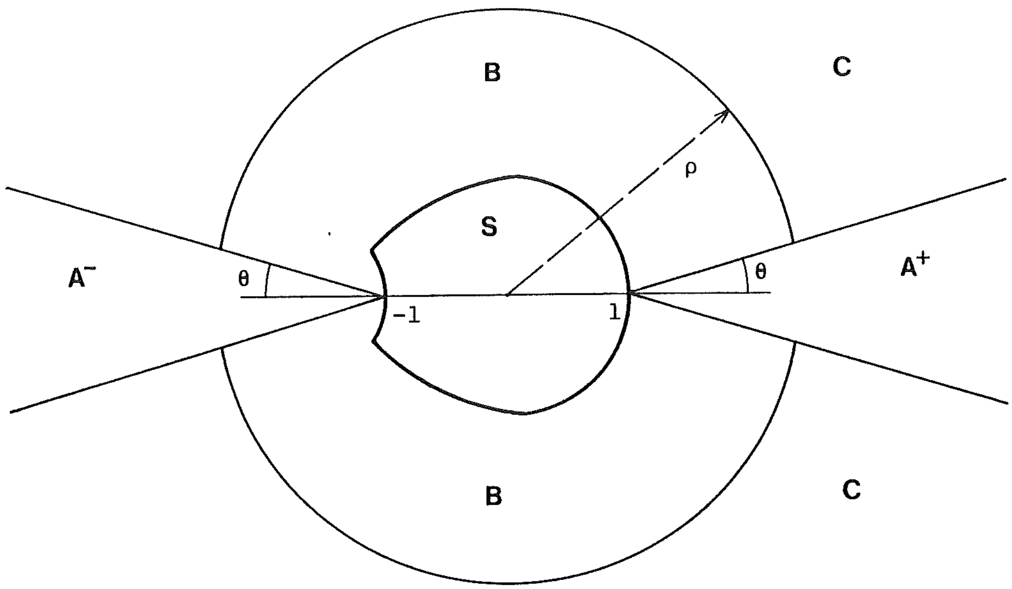

Fig. 1

$$2S \cap C = \emptyset\,, \quad \text{i.e.} \quad \frac{\rho}{2} > \rho_S := \max\{|z|;\ z \in S\}, \tag{7}$$

$$S \cap \{z \in \mathbb{C}\,;\ |\arg(-1+\sigma z)| < 2\theta\} = \emptyset\,, \quad \sigma = \pm 1\,, \tag{8}$$

$$\rho \geq 4/\sin\theta\ . \tag{9}$$

For any $c \in R^c_{01} \setminus R^c_{00}$,

$$c(z) := \frac{a}{1 - z/z_0}\,,$$

we define r_c depending on the position of the pole z_0 by

$$r_c(z) := \begin{cases} \dfrac{1 - 1/|z_0|}{1 \mp z/|z_0|}\ \mathrm{Re}\ c(\pm 1) & \text{if} \quad z_0 \in A^{\pm}, \\ 0 & \text{if} \quad z_0 \in B\,, \\ \mathrm{Re}\ a & \text{if} \quad z_0 \in C\,. \end{cases} \tag{10}$$

Correspondingly, our verification of (5) is split into three cases. We may restrict a to have unit modulus since c, $\hat{c}$, and r_c can all be multiplied by an arbitrary real scale factor. The simplest case is

<u>$z_0 \in B$</u>. Let $z^* \in S$ be a point closest to z_0, so that $|c(z^*)| = \|c\|$. Then

$$\|\hat{c}\| \geq |\hat{c}(z^*)| = \frac{1}{2}|c(z^*)|\left|1 - \frac{\bar{c}(z^*)}{c(z^*)}\right|$$
$$= \frac{1}{2}\|c\|\left|1 - \frac{\bar{a}}{a}\frac{\overline{z_0}}{z_0}\frac{z_0 - z^*}{\overline{z_0} - z^*}\right| . \qquad (11)$$

Due to the differentiability of ∂S at ± 1,

$$\left|\frac{z_0 - z^*}{\overline{z_0} - z^*}\right| \leq \cos\theta < 1 \qquad (12)$$

whenever $z_0 \in B$ is close enough to ± 1, say, $z_0 \in U_B^{\pm}$. On the other hand, since $B \setminus (U_B^+ \cup U_B^-)$ has a positive distance from the real axis, there is a fixed $\gamma_B < 1$ such that $|z_0 - z^*|/|\overline{z_0} - z^*| \leq \gamma_B$ for all z_0 in this set. Therefore, in view of (10)-(12) we get for all $z_0 \in B$

$$\|\hat{c}\| \geq \|c - r_c\|\,\delta_B \quad \text{with} \quad \delta_B := \frac{1}{2}\min\{1-\cos\theta,\ 1-\gamma_B\}. \qquad (13)$$

<u>$z_0 \in C$</u>. In view of

$$c(z) = a + \frac{az}{z_0}[1 + h(z)] \quad \text{with} \quad h(z) := \frac{z/z_0}{1 - z/z_0},$$

we get

$$\|\hat{c}\| \geq \max_{z=\pm 1}|\hat{c}(z)| = \max_{z=\pm 1}|\mathrm{Im}\, c(z)|$$
$$= \max_{\pm}\left|\mathrm{Im}\left[a \pm \frac{a}{z_0}\left(1 + h(\pm 1)\right)\right]\right| .$$

Now, by (9), $|z_0| \geq \rho \geq 4/\sin\theta \geq 4$ for $z_0 \in C$, hence

$$|h(\pm 1)| \leq \frac{1}{|z_0|(1-|z_0|^{-1})} \leq \frac{2}{|z_0|} \leq \frac{2}{\rho} \leq \frac{1}{2}\sin\theta \tag{15}$$

A fortiori,

$$\|\hat{c}\| \geq |\mathrm{Im}\, a| - \frac{2}{|z_0|} . \tag{16}$$

Likewise, using (7) we get

$$\begin{aligned} \|c - r_C\| &= \max_{z\in S} \left| i\, \mathrm{Im}\, a + \frac{az}{z_0}\left[1 + \frac{z}{z_0} + \left(\frac{z}{z_0}\right)^2 + \ldots\right]\right| \\ &\leq |\mathrm{Im}\, a| + 2\,\frac{\rho_S}{|z_0|} . \end{aligned} \tag{17}$$

From (16) and (17) we conclude that for $\mathrm{Im}\, a \neq 0$ and $3/|\mathrm{Im}\, a| \leq |z_0| \leq \infty$

$$\frac{\|\hat{c}\|}{\|c - r_C\|} \geq \frac{|\mathrm{Im}\, a| - 2/|z_0|}{|\mathrm{Im}\, a| + 2\rho_S/|z_0|} \geq \frac{1}{3 + 2\rho_S} . \tag{18}$$

For $\mathrm{Im}\, a \neq 0$ and $\rho \leq |z_0| \leq 3/|\mathrm{Im}\, a|$ inequality (17) implies $\|c - r_C\| \leq (3 + 2\rho_S)/|z_0|$, and therefore using (14) we get for $|z_0|$ in this range

$$\frac{\|\hat{c}\|}{\|c - r_C\|} \geq \frac{1}{3 + 2\rho_S} \max_{\pm} \left| \mathrm{Im}\left[a|z_0| \pm \frac{a\,\overline{z_0}}{|z_0|}\,(1 + h(\pm 1))\right]\right| \tag{19}$$

Now for $z_0 \in C$ $(z_0 \neq \infty)$ the two disks

$$D^{\pm} := \left\{ a|z_0| \pm \frac{a\,\overline{z_0}}{|z_0|}\,(1 + \zeta)\ ;\ |\zeta| \leq \frac{1}{2}\sin\theta \right\}$$

lie on opposite sides of the ray $a\,\mathbb{R}^+$ at a distance of at least $\frac{1}{2}\sin\theta$ from it, and also outside the disk $|z| \leq 2$, cf. Fig. 2. Therefore, at least one of them has a distance $\geq \frac{1}{2}\sin\theta$ from the real axis, and (18) and (19) imply

$$\frac{\|\hat{c}\|}{\|c - r_C\|} \geq \frac{\sin\theta}{6 + 4\rho_S} \quad \text{if} \quad z_0 \in C \ \text{and}\ \mathrm{Im}\, a \neq 0 . \tag{20}$$

By continuity, the same bound also holds if $\mathrm{Im}\, a = 0$. (Note

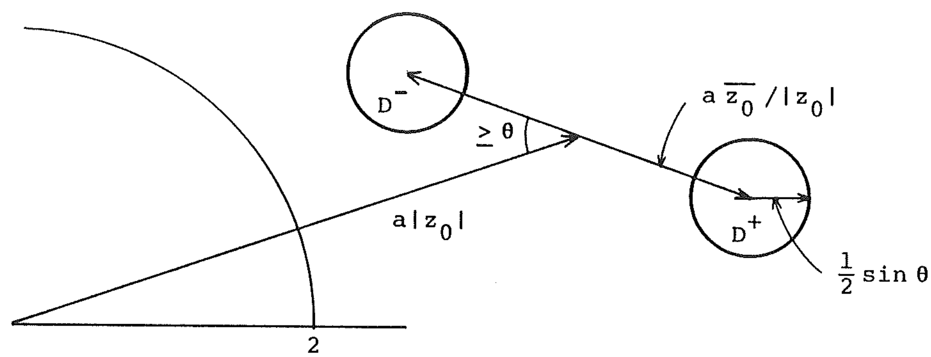

Fig. 2

that $\hat{c} \neq 0$ for $z_0 \in C$.)

$z_0 \in A^+$. Here we again use the estimate $\|\hat{c}\| \geq \max\{|\hat{c}(1)|, |\hat{c}(-1)|\}$; now the case where $\hat{c}(1) = \operatorname{Im} c(1) = 0$, i.e. $a = \pm a_0$ with

$$a_0 := \frac{1 - 1/z_0}{|1 - 1/z_0|}, \tag{21}$$

will require special attention. In view of the invariance with respect to real factors of a mentioned above, we may restrict ourselves to the plus sign and set

$$a = a_0\, e^{i\alpha} \quad (|\alpha| \leq \pi/2), \qquad d := \tfrac{1}{2}(e^{-2i\alpha} - 1),$$

so that $\bar{a} = \overline{a_0}\, e^{-i\alpha} = \overline{a_0}\,(1 + 2d)\, e^{i\alpha}$ and

$$\operatorname{Re} c(1) = \operatorname{Re} \frac{e^{i\alpha} a_0}{1 - 1/z_0} = \frac{\cos\alpha}{|1 - 1/z_0|} = \frac{e^{i\alpha}(1 + d)}{|1 - 1/z_0|}. \tag{22}$$

Then

$$\hat{c}(z) = \frac{1}{2}\, \frac{e^{i\alpha}}{|1 - 1/z_0|} \left[\frac{z_0 - 1}{z_0 - z} - (1 + 2d)\frac{\overline{z_0} - 1}{\overline{z_0} - z} \right],$$

or,

$$\hat{c}(z) = \frac{e^{i\alpha}|z_0|}{|z_0 - 1|} \cdot \frac{i\,(1-z)\,\mathrm{Im}\,z_0 - d\,(z_0 - z)(\overline{z_0} - 1)}{(z_0 - z)(\overline{z_0} - z)}\,, \tag{23}$$

and by (22),

$$\begin{aligned} c(z) - r_c(z) &= c(z) - \frac{|z_0| - 1}{|z_0| - z}\,\mathrm{Re}\,c(1) \\ &= \frac{e^{i\alpha}}{|1 - 1/z_0|}\left[\frac{z_0 - 1}{z_0 - z} - (1+d)\frac{|z_0| - 1}{|z_0| - z}\right] \\ &= \frac{e^{i\alpha}|z_0|}{|z_0 - 1|} \cdot \frac{(z_0 - |z_0|)(1 - z) - d(z_0 - z)(|z_0| - 1)}{(z_0 - z)(|z_0| - z)}\,. \end{aligned} \tag{24}$$

In particular,

$$|\hat{c}(1)| = \frac{|z_0||d|}{|z_0 - 1|}\,. \tag{25}$$

Now for $z_0 > 1$ (24) yields

$$\|c - r_c\| = \frac{z_0\,|d|}{|z_0 - 1|}\,\sup_{z \in S}\frac{z_0 - 1}{|z_0 - z|} \le \frac{z_0\,|d|}{|z_0 - 1|} \cdot \frac{1}{\sin 2\theta}\,,$$

and, since $d \neq 0$ if $c \notin R^r_{mn}$, we get, using (25),

$$\frac{\|\hat{c}\|}{\|c - r_c\|} \ge \sin 2\theta \quad \text{if} \quad z_0 > 1\,. \tag{26}$$

Likewise, using additionally (8) and the relation

$$z_0 - |z_0| = \frac{i\,e^{i\phi/2}}{\cos(\phi/2)}\,\mathrm{Im}\,z_0\,, \quad \text{where} \quad \phi := \arg(z_0 - 1)\,, \tag{27}$$

we obtain asymptotically for $|\mathrm{Im}\,z_0| = \beta|d| \to 0$ with fixed $\beta > 0$

$$\frac{\|\hat{c}\|}{\|c - r_c\|} \ge \sin 2\theta\left[1 + \frac{\beta(1+\rho)}{\sin\theta\,\cos(\theta/2)\,(|z_0| - 1)^2}\right]^{-1} + o(1)\,. \tag{28}$$

On the other hand, if $d = 0$, we get from (23) and (24)

$$\frac{\|\hat{c}\|}{\|c - r_c\|} \geq \inf_{z \in S} \left| \frac{\hat{c}(z)}{c(z) - r_c(z)} \right| = \frac{|\mathrm{Im}\, z_0|}{|z_0 - |z_0||} \inf_{z \in S} \left| \frac{|z_0| - z}{\overline{z_0} - z} \right|$$

$$\geq \frac{1}{2} \cos \frac{\theta}{2} \quad \text{if} \quad z_0 \in A^+ , \quad \alpha = d = 0 . \tag{29}$$

Now, clearly, $\|\hat{c}\| > 0$ unless $\alpha = 0$ and $\mathrm{Im}\, z_0 = 0$, which means that $c = r_c \in R^r_{01}$. Conversely, $c = r_c$ implies $\alpha = 0$, $\mathrm{Im}\, z_0 = 0$, $\|\hat{c}\| = 0$. Hence $\|\hat{c}\| / \|c - r_c\|$ is a positive continuous function of

$$(z_0, \alpha) \in \left((\overline{A^+} \setminus \{1, \infty\}) \times [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \right) \setminus \mathbb{R} \times \{0\} . \tag{30}$$

Since $\overline{A^+} \times [-\pi/2, \pi/2]$ is a compact subset of $\overline{\mathbb{C}} \times \mathbb{R}$, it suffices to establish positive lower bounds for $\|\hat{c}\| / \|c - r_c\|$ that hold in the three cases

$$(z_0, \alpha) \to \mathbb{R} \times \{0\} , \qquad z_0 \to 1 , \qquad z_0 \to \infty .$$

But for $1 < |z_0| < \infty$ the first case has just been treated, and (26), (28), (29) are appropriate bounds, except that z_0 should not approach 1 in (28). Hence we are left with the two other cases and may assume $z_0 \notin \mathbb{R}$:

For $z_0 \to 1$ and $\phi = \arg(z_0 - 1) \neq 0$, one can deduce from (23), (24), and (27) that for fixed $z \neq 1$ asymptotically

$$|\hat{c}(z)| \sim \frac{|i \sin \phi - d\, e^{-i\phi}|}{|1 - z|} , \quad |c(z) - r_c(z)| \sim \frac{\left| 2\, i \sin \frac{\phi}{2} e^{i\phi/2} - d \right|}{|1 - z|} .$$

Hence $\|\hat{c}\| \sim |\hat{c}(1)|$, $\|c - r_c\| \sim |c(1) - r_c(1)|$; but $\hat{c}(1) = c(1) - r_c(1)$ $(\forall z_0)$ as is easily checked, and therefore

$$\frac{\|\hat{c}\|}{\|c - r_c\|} \to 1 . \tag{31}$$

Finally, for $z_0 \to \infty$, (23) and (24) imply

$$|\hat{c}(z)| \to |d|, \qquad |c(z) - r_c(z)| \to |d| \tag{32}$$

uniformly for all $z \in S$. Hence (31) holds again if $d \neq 0$. The case $d = 0$ is covered by (29).

This concludes the proof. □

Except for the case $z_0 \in B$ this proof remains valid if S is replaced by an interval $I \subset \mathbb{R}$. In fact, in the two other cases we have used very few properties of S except its symmetry, namely (7), (8), and $\pm 1 \in \partial S$.

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