Eigenvalue Repulsion

Lloyd N. Trefethen *

March 25, 2026

The recent death of Peter Lax reminded me of the cover image of his wonderfully deep textbook of linear algebra [4] (Figure 1), which alerted many of us to the phenomenon of *eigenvalue repulsion*. (The mathematics goes back to von Neumann and Wigner in the early days of quantum mechanics [10], and other names for the effect are *level repulsion* and *eigenvalue avoided crossings*.) Suppose the entries of a real symmetric matrix $A(t)$ are continuous functions of t and we track the eigenvalues as t is varied. Then, unless there are some further symmetries in the problem, the curves will probably never cross. One free parameter is normally not enough to produce an eigenvalue degeneracy. For generalizations, see [9].

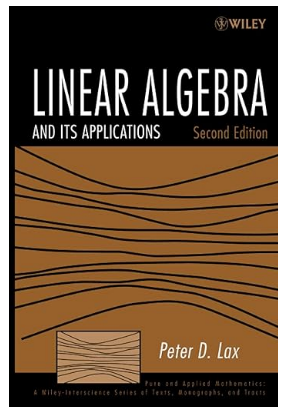

Figure 1: The cover of Lax's textbook [4] shows the eigenvalue avoidance phenomenon for real symmetric matrices, following the eigenvalues of a 12×12 example as a parameter is varied.

Many arguments can be given to explain this effect, and here is my favorite. The set of real symmetric 2×2 matrices is of dimension 3, whereas the set of real symmetric 2×2 matrices whose eigenvalues are equal is of dimension just 1:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \text{vs.} \quad \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Thus the codimension is 2, not 1 as one might have imagined. For complex Hermitian matrices the effect is even stronger, with a (real) codimension of 3:

$$\begin{pmatrix} a & b+ci \\ b-ci & d \end{pmatrix} \quad \text{vs.} \quad \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

For $n \times n$ matrices the dimensions generalize to $(n^2+n)/2$ vs. $(n^2+n)/2-2$ and n^2 vs. n^2-3, giving codimensions 2 and 3 again, respectively.

Numerical linear algebraists may enjoy the following alternative intuitive argument. Any real symmetric matrix can be orthogonally tridiagonalized without changing the eigenvalues. For a degenerate eigenvalue to be possible, the tridiagonal matrix must have a pair of off-diagonal zeros, so that it decouples into two pieces, and achieving this uses up one real parameter. But then it takes another real parameter to make an eigenvalue of one of the pieces match an eigenvalue of the other.

Applications go far beyond matrices. One problem beloved of mathematicians and physicists alike is that of *quantum billiards*, that is, eigenvalues of the Laplacian operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$ in a planar domain with zero Dirichlet boundary conditions [7]. This is a real symmetric configuration, and if the domain is one with a geometric symmetry like a rectangle, then certain parameter choices may lead to

*Professor of Applied Mathematics in Residence, School of Engineering and Applied Sciences, Harvard University

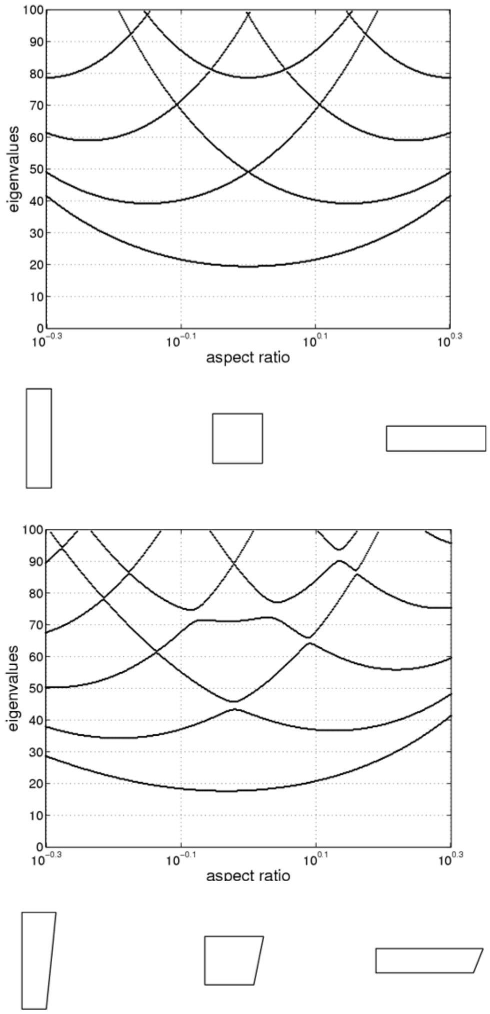

Figure 2: If a tall skinny rectangle is morphed into a short fat one, the symmetry is such that there are eigenvalue degeneracies along the way (images from [2]). Tilting one side (Vermont → Arkansas → Tennessee) breaks the symmetry and eliminates the degeneracies, so that the curves never intersect. (There may appear to be some intersections in this figure, but that is a matter of finite plotting resolution.)

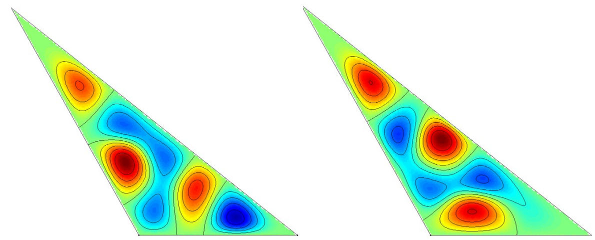

Figure 3: An example from [1] of a triangle whose Laplace eigenvalue problem has an "accidental" degeneracy, i.e., unrelated to symmetry (figure from [11]). Both eigenmodes correspond to the same eigenvalue.

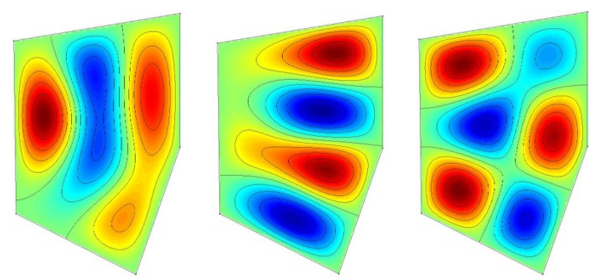

Figure 4: An example from [11] of a pentagon with an accidental *double* degeneracy. All three eigenmodes correspond to the same eigenvalue.

eigenvalue degeneracies (Figure 2). If the symmetry is broken, however, perhaps by snipping a corner or tilting one of the sides, then the degeneracies vanish.

Berry and Wilkinson considered the space of triangular billiard domains, where the spectral problem is determined by two real parameters since we can fix two vertices and then vary the third [1]. This gives enough degrees of freedom for "accidental" degeneracies to appear, unrelated to symmetries, and Figure 3 shows an example of a triangle with such a degeneracy that they determined numerically. If you want a double degeneracy, that is, three identical eigenvalues, then the codimension jumps to 5 and you need to turn to pentagons. Figure 4 shows one such shape as determined by Simon Wojczyszyn [11].

The phenomenon of eigenvalues repelling one another is part of the bedrock of the theory of random matrices, where the codimensions govern the proba-

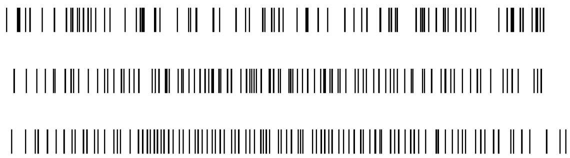

Figure 5: Top: 100 uniformly distributed random points in $[-1, 1]$. Middle: eigenvalues of a 100×100 random real symmetric matrix. Bottom: eigenvalues of a 100×100 random complex Hermitian matrix.

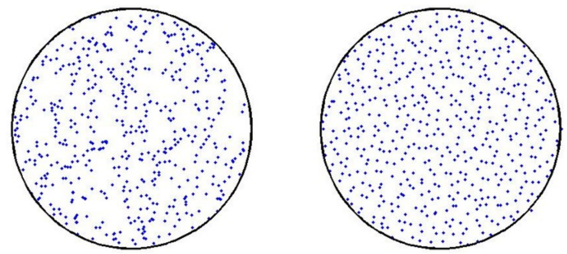

Figure 6: Left: 400 uniformly distributed random points in the unit disk. Right: eigenvalues of a 400×400 random complex non-Hermitian matrix. The difference in the images illustrates how eigenvalues tend to repel one another.

bility density function for eigenvalue separations in the zero-separation limit. Figure 5 illustrates the real case, showing the different degrees of uniformity that arise with random points in an interval, eigenvalues of a real symmetric random matrix (GOE = Gaussian orthogonal ensemble), and eigenvalues of a complex Hermitian random matrix (GUE = Gaussian unitary ensemble). Figure 6 illustrates the complex case, comparing uniformly distributed random points in a disk against eigenvalues of a random complex non-Hermitian matrix (complex Gaussian or Ginibre ensemble). Figure 7 combines some of these ideas in an example related to *Dyson Brownian motion* [8], showing the eigenvalues of a 2×2 complex Hermitian matrix whose upper-triangular entries are independent Brownian paths. These trajectories can also be interpreted as Brownian paths conditioned on the property of non-intersection for all time.

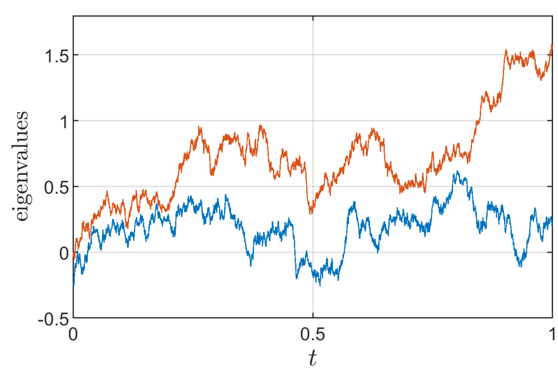

Figure 7: Dyson Brownian motion: eigenvalues of a 2×2 complex Hermitian matrix $A(t)$ with Brownian path entries [3]. With probability 1, there are no crossings. If the dimension were $n \times n$, we would have n paths with no crossings.

We finish with an example of a random flavor where the data are so excellent as to look deterministic. An old idea (the *Hilbert-Pólya conjecture*) is that the Riemann hypothesis would be settled if we could prove that the nontrivial zeros of the zeta function $\zeta(s)$, after subtracting $1/2$ and dividing by i, are the eigenvalues of a complex Hermitian operator. Powerful evidence in support of this idea is provided by the fact that high up on the critical line $\mathrm{Re}(s) = 1/2$, the distribution of nearest-neighbor spacings of the zeros of $\zeta(s)$ is matched very well by the distribution of nearest-neighbor spacings of eigenvalues of large random complex Hermitian matrices [5]. Figure 8 reproduces a plot from [6] showing such agreement

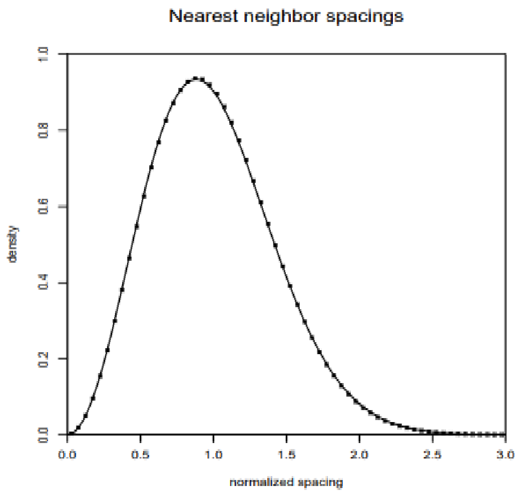

Figure 8: Computed nearest-neighbor spacings of Riemann zeros (dots) compared with predictions based on random complex Hermitian matrices (curve); figure from [6].

looking perfect to plotting accuracy based on computations by Andrew Odlyzko of a billion zeros of $\zeta(s)$ on the critical line close to zero number $1.3 \cdot 10^{16}$. If there were no eigenvalue repulsion, the probability density curve would attain a nonzero limit at 0. If the repulsion were that associated with random real symmetric matrices, the curve would reach 0 with a nonzero slope. Instead, it hits 0 with a zero slope, showing the stronger repulsion associated with random complex Hermitian matrices.

Acknowledgments. I am grateful for helpful suggestions from Sir Michael Berry, Jonathan Keating, and Jan Wiersig.

References

[1] M. V. Berry and M. Wilkinson, Diabolical points in the spectra of triangles, *Proc. R. Soc. Lond. A,* 392 (1984), 15–43.

[2] T. Betcke and L. N. Trefethen, Computations of eigenvalue avoidance in planar domains, *Proc. Appl. Math. Mech.,* 4 (2004), 634–635.

[3] F. J. Dyson, A Brownian-motion model for the eigenvalues of a random matrix, *J. Math. Phys.,* 3 (1962), 1191–1198.

[4] P. D. Lax, *Linear Algebra and its Applications,* 2nd ed., John Wiley & Sons Inc. (2007).

[5] H. L. Montgomery, The pair correlation of zeros of the zeta function, in H. G. Diamond, ed., *Analytic Number Theory,* Amer. Math. Soc. (1973), 181–193.

[6] A. M. Odlyzko, The 10^{22}-nd zero of the Riemann zeta function, *Contemp. Math.,* 290 (2001), 139–144.

[7] H.-J. Stöckmann, *Quantum Chaos: An Introduction,* Cambridge U. Press (1999).

[8] T. Tao, *Topics in Random Matrix Theory,* AMS (2023).

[9] M. Teytel, How rare are multiple eigenvalues?, *Comm. Pure Appl. Math.,* 52 (1999), 917–934.

[10] J. von Neumann and E. Wigner, Über das Verhalten von Eigenwerten bei Adiabatischen Prozessen, *Phys. Z.,* 30 (1929), 467–470.

[11] S. Wojczyszyn and L. N. Trefethen, Doubly diabolical points in the spectra of pentagons, unpublished manuscript, Oxford U. Computing Laboratory, 2007. `https://people.maths.ox.ac.uk/trefethen/wojczyszyn.pdf`