Notes of a Numerical Analyst

Random Smoothies

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In 1872 Weierstrass presented his example of a continuous function that is nowhere differentiable. This was defined by a lacunary Fourier series,

$$f(x) = \mathrm{Re} \sum_{k=-\infty}^{\infty} a_k e^{ikx}, \qquad (1)$$

where most of the coefficients $\{a_k\}$ are zero but the nonzero ones decay more slowly than $|k|^{-1}$. Fifty years later continuous, nowhere-differentiable functions began to take on a new importance with the development of the mathematical theory of Brownian motion. Wiener showed that a Brownian path can be written as a Fourier series (1) where $\{a_k\}$ are independent normal variates with mean 0 and standard deviation proportional to $|k|^{-1}$ [3].

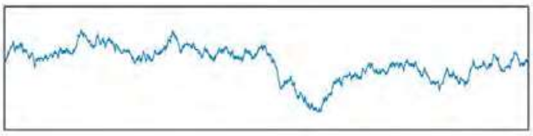

Fig. 1. A Brownian path, continuous but nowhere differentiable, can be defined by a random Fourier series with coefficients $O(|k|^{-1})$.

My students and I have been having fun with continuous functions at the other end of the smoothness spectrum [4]. What if f is C^∞ but nowhere analytic? With apologies to the fruit beverage industry, we call such a function a *smoothie*. Smoothies too can be defined by Fourier series, the key now being that the coefficients decrease at a rate faster than the reciprocal of any polynomial but slower than exponential. For example, a random smoothie can be defined by taking $\{a_k\}$ to be independent normal variates with mean 0 and standard deviation $\exp(-|k|^{1/2})$.

There is a small literature on C^∞, nowhere-analytic functions beginning with du Bois-Reymond, Cellérier and Borel in the Belle Époque [1, 5].

If you differentiate a smoothie, the result is another smoothie. Similarly if f is a smoothie and g is analytic, then $f + g$ and fg are smoothies, provided g is not identically zero. Adding two independent random smoothies gives another random smoothie, with probability 1. On the other hand the sum of two arbitrary smoothies need not be a smoothie, since, for example, one might be the negative of the other.

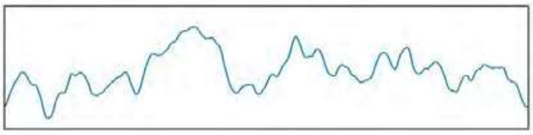

Fig. 2. A random smoothie, C^∞ but nowhere analytic, can be defined by a random Fourier series with coefficients decaying root-exponentially.

In Chebfun, the command `smoothie` generates a random smoothie on an interval, just as `randnfun` generates a smooth random function defined by a finite random Fourier series [2]. Apart from fun, the importance of smoothies for me is educational. My mission is to make sure every student knows the difference between C^∞ and analytic.

FURTHER READING

[1] G. G. Bilodeau, The origin and early development of non-analytic infinitely differentiable functions, *Arch. Hist. Exact Sci.*, 27 (1982), 115–135.
[2] S. Filip, A. Javeed and L. N. Trefethen, Smooth random functions, random ODEs, and Gaussian processes, *SIAM Rev.* 61 (2019), 185–205.
[3] J.-P. Kahane, *Some Random Series of Functions*, 2nd ed., Cambridge, 1985.
[4] T. Park, C^∞ but nowhere analytic functions, diss., MSc. Math. Sci., Oxford, 2021.
[5] "Smoothies: nowhere analytic functions," Chebfun example at `www.chebfun.org`.

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