Notes of a Numerical Analyst

Unstable Instabilities

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If a linear process has an exponentially unstable mode, it will probably show up. For example, let A be a $1{,}000 \times 1{,}000$ matrix with 999 eigenvalues equal to 0.5 and one eigenvalue 1.01, and compute $x_{n+1} = Ax_n$ for a random vector x_0. Pretty soon exponential growth at the rate $(1.01)^n$ will take over. Another famous example is the instability of finite difference discretisations of linear PDEs. As von Neumann discovered, if there is an unstable Fourier mode, it will destroy the simulation.

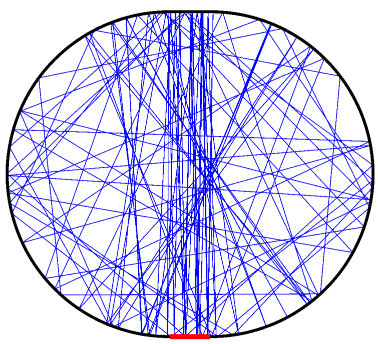

Figure 1. 100 steps of the "pinball model": a chaotic billiards problem where energy doubles each time the red segment is hit but multiplies by 0.93 at other impacts. Trajectories near the midline experience exponential energy growth but are themselves unstable.

Nonlinear dynamical systems are different. Worst case unstable modes may fail to cause trouble in practice, as we can illustrate by a "pinball model", an abstraction of a classic pinball machine. Let Γ be a closed contour for which the classical billiards problem has chaotic trajectories. To be concrete, take a Bunimovich stadium consisting of two semicircles of radius 1 separated by parallel line segments of length $1/4$. The trajectories will be chaotic, as in Fig. 1. Now imagine that each time a particle hits the red segment, its energy doubles, whereas when it hits anywhere else, its energy is multiplied by 0.93. This dynamics is exponentially unstable in the worst case, with energy increasing at the rate about $(1.36)^n$ for trajectories that bounce up and down between the segments. However, Fig. 2 confirms that this instability is unstable.

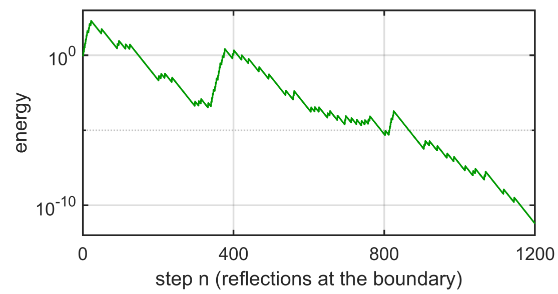

Figure 2. Long-term pinball model trajectories show energy decay

Unstable instabilities may be all around us. Are the Navier–Stokes equations ill-posed? Some think the answer is yes in principle but no in practice, because fluid flows tend to deviate away from configurations that might lead to singularities [3]. Why does Gaussian elimination solve $Ax = b$ reliably despite rounding errors being exponentially amplified in the worst case [2]? It seems that LU factorisations bend away from the patterns that generate exponential growth of matrix entries, but this has never been proved. Why are NP-complete problems like 3-SAT and travelling salesman often tractable despite their exponential complexity [1]? Somehow, typical instances often do not excite the worst case.

Do widely varied systems like these share analogous mechanisms of unstable instability?

FURTHER READING

[1] L. Fortnow, Fifty years of P vs. NP and the possibility of the impossible, *Commun. ACM* 65(1) (2021) 76–85.
[2] L.N. Trefethen, D. Bau, III, *Numerical Linear Algebra*, SIAM, 2022.
[3] Y. Wang et al., Discovery of unstable singularities. Preprint, 2025, arxiv.org/abs/2509.14185.

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