Trees

(with a view towards Outer space)

- Geometric Group Theory

understand groups via actions (on geometric/topological spaces

algebra (subgroups?)

cohomology

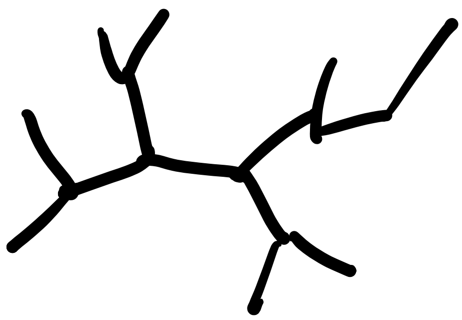
size

algorithms

This Course

(Usually discrete) group actions on trees

Combinatorial



'trees as graphs'

$\mathbb{R}$-trees

'trees as metric spaces'

In terms of research

Spaces of simplicial actions on trees

Understand how groups can act on $\mathbb{R}$-trees

<u>Graphs (á la Serre)</u>

Serre - Trees (Arbres)

$$\Gamma = (E\Gamma, V\Gamma, \bar{\ }: E\Gamma \to E\Gamma$$
$$\iota: E\Gamma \to V\Gamma)$$
$$\tau: E\Gamma \to V\Gamma$$

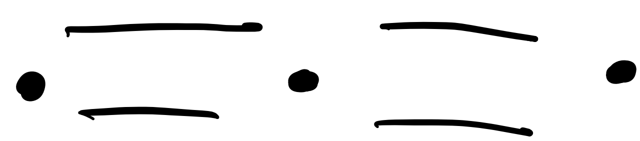
edge set

vertex set

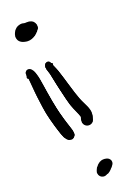
$\bar{e} \neq e \qquad \bar{\bar{e}} = e$

Define $\boxed{t(e) = \iota(\bar{e})}$

This defn has twice as many edges as one might expect.



$\iota(\bar{e})$



Two 'combinatorial' edges for each 'geometric' edge.

Def An orientation of a graph is a subset $O \subset E\Gamma$ containing exactly one element from each set $\{e, \bar{e}\}$.

Topological realization of a graph - simplicial complex with $|V\Gamma|$ vertices and $\frac{|E\Gamma|}{2}$ edges.

An *edge path* is a sequence

$$p = e_1, e_2, \ldots, e_n$$

such that $\iota(e_{i+1}) = \tau(e_i)$ for all i. It is **reduced** if $e_{i+1} \neq \overline{e_i}$. ~~Set the of reduced~~ for all i.

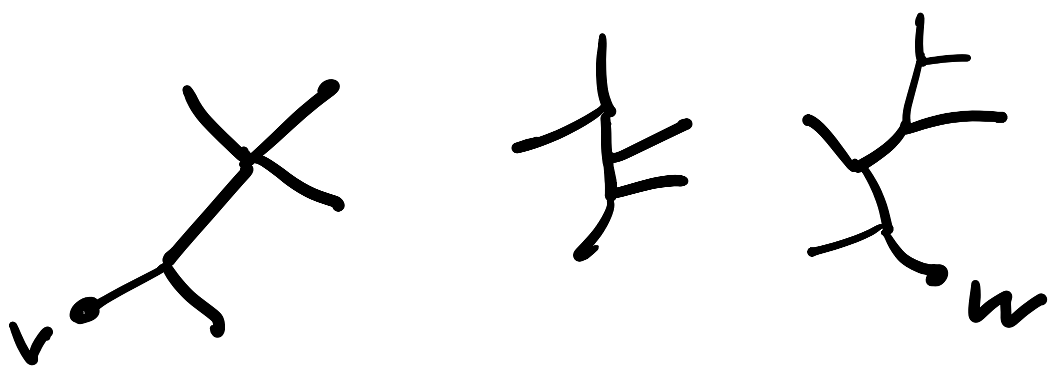
Each path has a unique reduction, the set of reduced loops from v to v form a group $\pi_1(\Gamma, v)$

$$l_1 \cdot l_2 := \text{reduction of the concatenation of } l_1 + l_2.$$

Def A simplicial/combinatorial tree is a connected, simply connected graph.

there is a path between any two points

$\pi_1(T, v) = 1$ for some (equiv. any) $v \in T$.



Def Forest is a union

Exercise Let Γ be a connected graph

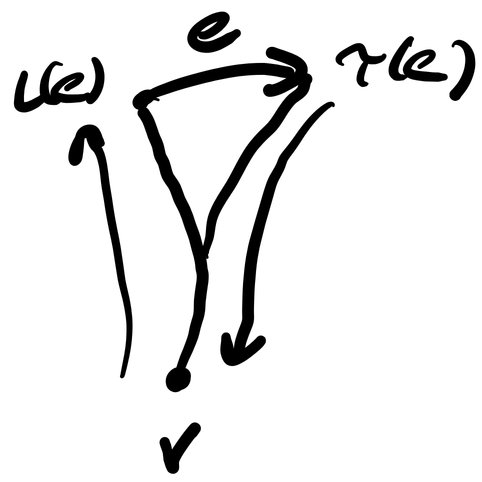
- Pick a maximal subtree $T \subset \Gamma$
- Pick an orientation $\mathcal{O}$ of $\Gamma - T$

Then $\pi_1(\Gamma, v) \cong F(\mathcal{O})$

① There is a unique reduced path $p_{v,w}$ from v to any other vertex $w \in V\Gamma$.

② The isomorphism $F(\mathcal{O}) \to \pi_1(\Gamma, v)$ is induced by

$$e \longmapsto p_{v\,\iota(e)} \cdot e \cdot \overline{p_{v\,\tau(e)}}$$



<u>Subgraph</u> $\Delta \subset \Gamma$ is a pair

$\&$ $V\Delta \subseteq V\Gamma$

$E\Delta \subset E\Gamma$

closed under $\bar{\ }$, ι.

A subtree is a subgraph that is a tree.

Any graph contains a subforest that contains all vertices.

One thing:

- $\bar{\ }$ operation, ι, τ operations extend to paths.

Allow path to be trivial, $p = v$. In which case

$$\bar{p} = \iota(p) = \tau(p) = v.$$

$\mathbb{R}$-trees

Def (Unique path defn)

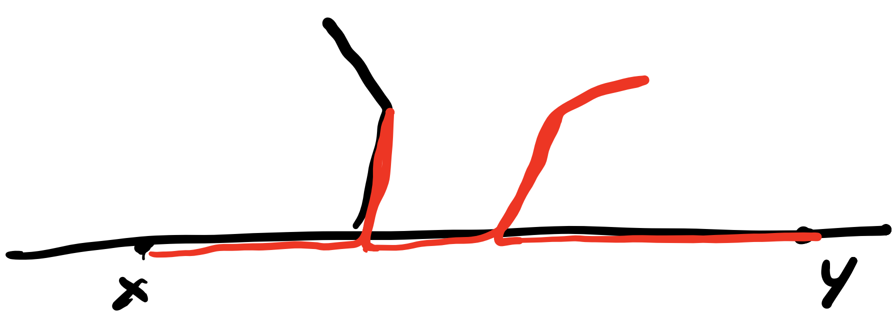
An **$\mathbb{R}$-tree** is a geodesic metric space T such that

- there is a unique geodesic $[x,y]$ between two points x + y

- If $f:[0,1] \to T$ is an <u>embedded</u> path from x to y then $f([0,1]) = [x,y]$.

<u>Qu</u> Is it the same if you swap the second condition for 'if you have a path f from x to y then the image of f contains the geodesic $[x,y]$?'

(Yes - 'Bottleneck condition')

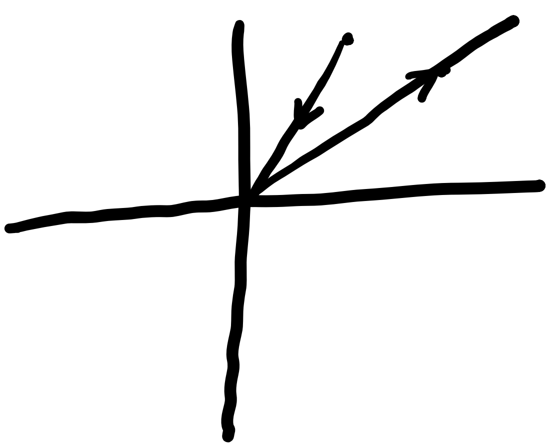


Def 2 An $\mathbb{R}$-tree is a 0-hyp geodesic metric space.
(δ-thin / δ-slim)

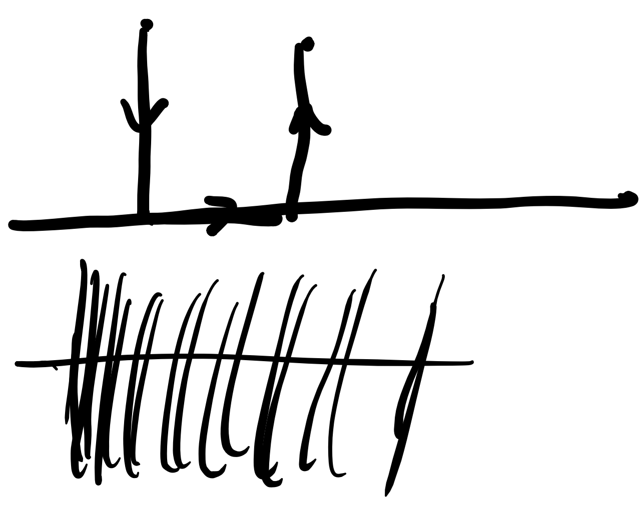
Exercise Prove ~~D~~ defs 1+2 are equivalent.

Examples **Any** interval in $\mathbb{R}$ (in particular, $\mathbb{R}$-trees need not be complete).

$\mathbb{R}^2$ Paris metric



Infinite comb



simplicial

not simplicial

- Any path $f: [0,1] \to \mathbb{R}$ between x & y contains $[x,y]$
- Any point p on $[x,y]$ separates: x, y are in distinct components of $T - p$.

Basic properties

Local-to-global:

Piecewise geodesics are geodesics.

$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$

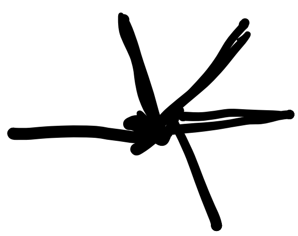
- non-degenerate

- $[x_{i-1}, x_i] \cap [x_i, x_{i+1}] = \{x_i\}$

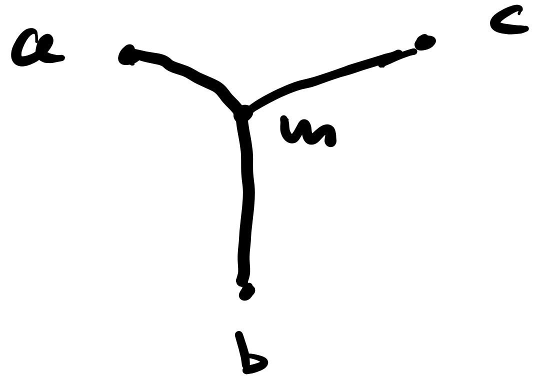
then $[x_0, x_n] = \bigcup_{i=0}^{n-1} [x_{n-1}, x_n]$.

<u>Why non-degenerate?</u>

I was worried about



- Tripod property:



$a, b, c \in T$

$\exists! m \in T$

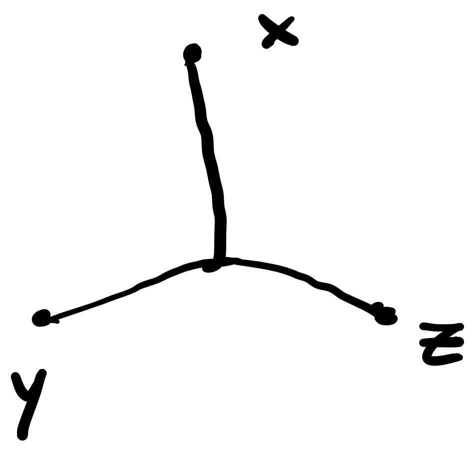
$[a,b] = [a,m] \cup [m,b]$
$[b,c] = [b,m] \cup [m,c]$
$[c,a] = [c,m] \cup [m,a]$

- Closed subtree projections

$S \subset T$ is a closed subtree

'closest point projection'

$\pi : T \longrightarrow S$



Any connected subset of a tree is a tree (with the restricted metric).

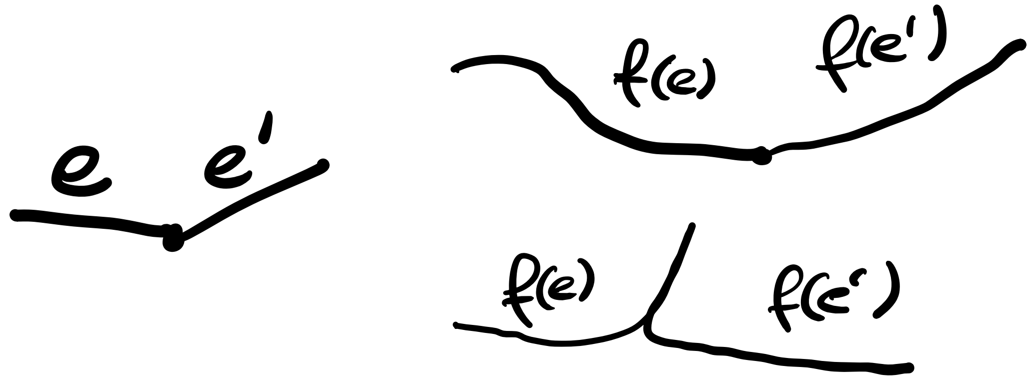
- Subtrees intersect in trees.

<u>Morphisms</u>

<u>Combinatorial</u>

$$f : T \to T'$$

is a map sending each edge e to an edge path $f(e)$



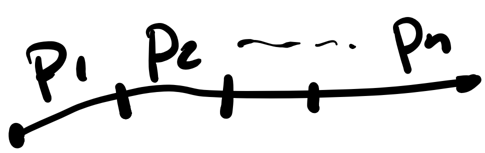
so that if $c(e)=c(e')$ then $c(f(e))=c(f(e'))$.

$$f(\bar{e})=\overline{f(e)}.$$

Rmk Allow edges sent to points.

Metric version?

$f: T \to T'$ is a map such that for any finite $P \subseteq T$ there exists a subdivision



such that there exist $\lambda_1, \dots, \lambda_n \geq 0$ such that

$$d_{T'}(f(x), f(y)) = \lambda_i \, d_T(x,y)$$

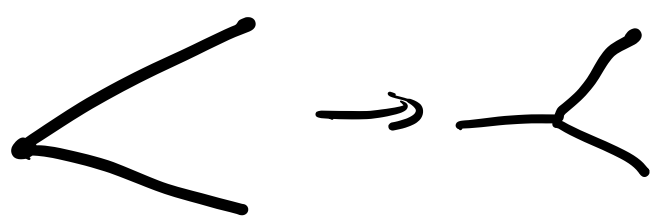
for all $x, y \in p_i$:

These are non-degenerate if no nondegenerate arcs are collapsed to points.

$$
\begin{tikzcd}
f : T \arrow[rr] \arrow[dr, "\text{forest collapse}"'] & & T' \\
& T'' \arrow[ur, "\text{non degenerate map.}"'] &
\end{tikzcd}
$$

Rmk Non-degenerate is not the same as injective

Eg folding



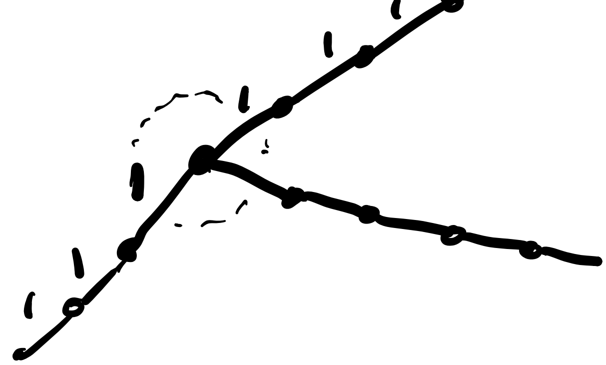
Def A <u>direction</u> at p is a component of $T-p$. p is a <u>branch point</u> of T if ~~...~~ $T-p$ has ≥ 3 components, p is a leaf/end of T if $T-p$ has exactly one component.

$DT = \{\text{directions in } T\}$

Rmk Non-degenerate morphism f induces

$$f: DT \to DT'.$$

Def An $\mathbb{R}$-tree is simplicial if it is obtained by giving lengths to edges in a simplicial tree, with the edges lengths bounded below by some $\varepsilon > 0$.
Alternatively the set of distances between branch points and leaves is bounded below by some $\varepsilon > 0$. + T is complete.



Three Tree Topoligies

On an $\mathbb{R}$-tree there are two possible topologies

- Metric
- Visual – weakest/coarsest topology such that every direction is open.

For simplicial tree there is also the simplicial topology

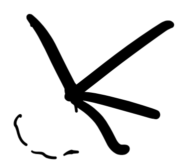
U open $\iff$ $U \cap$ each ~~simplex~~ edge is open.

Exercise If T is a locally finite simplicial tree (# directions at each point is finite) then all three topologies coincide

Otherwise we have:

$$\text{visual topology} \subsetneq \text{metric topology} \subsetneq \text{simplicial topology}.$$

(think about this with an infinite star).



Taster Group actions

$$f: G \to \mathrm{Isom}(X)$$

In general:

- What does an individual isometry look like?
- How do actions behave in general?

One approach: displacement

$$d(g) = \inf\{x \in X : d(gx, x)\}$$

This gives four types of isometries

$$d(g)=0 \begin{cases} \text{If } g \text{ fixes a point} \\ \quad g \ \underline{\text{elliptic}} \\ \text{Otherwise } g \text{ is} \\ \quad \underline{\text{neutral}} \ \underline{\text{parabolic}} \end{cases}$$

$$d(g)>0 \begin{cases} \text{If } g \ \cancel{\text{does}} \text{ realizes} \\ d(g) \text{ then } g \text{ is} \\ \quad \text{hyperbolic / loxodromic} \\ \text{otherwise } g \text{ is a} \\ \quad \underline{\text{non-neutral}} \ \text{para}\underline{\text{bolic}} \end{cases}$$

In the cases where $d(g)$ is realized let

$$C_g = \{ x \in X : d(x, gx) = d(g) \}$$

"characteristic set"