Lectures on Nonlinear Wave Equations

Qian Wang

Mathematical Institute, University of Oxford

April 28, 2015

Contents

Assessment

There are 2 or 3 problem sets.

References

- Hömander, Lars, *Lectures on nonlinear hyperbolic differential equations*, Mathématiques & Applications, 26, Springer, 1997.
- Sogge, Christopher D, *Lectures on nonlinear wave equations*, Monographs in Analysis, II. International Press, 1995.

- More references will be added during lectures.

1. Preliminaries

1.1. Conventions.

In this course we only consider the Cauchy problems of nonlinear wave equations. We will consider functions $u(t,x)$ defined on

$$\mathbb{R}^{1+n} := \{(t,x) : t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n\},$$

where t denotes the time and $x := (x^1, \cdots, x^n)$ the space variable. We sometimes write $t = x^0$ and use

$$\partial_0 = \frac{\partial}{\partial t} \quad \text{and} \quad \partial_j := \frac{\partial}{\partial x^j} \text{ for } j = 1, \cdots, n.$$

For any multi-index $\alpha = (\alpha_0, \cdots, \alpha_n)$ and any function $u(t,x)$ we write

$$|\alpha| := \alpha_0 + \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad \partial^\alpha u := \partial_0^{\alpha_0} \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} u.$$

Given any function $u(t,x)$, we use

$$|\partial_x u|^2 := \sum_{j=1}^{n} |\partial_j u|^2 \quad \text{and} \quad |\partial u|^2 := |\partial_0 u|^2 + |\partial_x u|^2.$$

We will use Einstein summation convention: *any term in which an index appears twice stands for the sum of all such terms as the index assumes all of a preassigned range of values.*

- A Greek letter is used for index taking values $0, \cdots, n$.
- A Latin letter is used for index taking values $1, \cdots, n$.

For instance

$$b^\mu \partial_\mu u = \sum_{\mu=0}^{n} b^\mu \partial_\mu u \quad \text{and} \quad b^j \partial_j u = \sum_{j=1}^{n} b^j \partial_j u.$$

1.2. Gronwall's inequality.

Lemma 1 (Gronwall's inequality)

Let E, A and b be nonnegative functions defined on $[0, T]$ with A being increasing. If

$$E(t) \leq A(t) + \int_0^t b(\tau)E(\tau)d\tau, \quad 0 \leq t \leq T,$$

then there holds

$$E(t) \leq A(t) \exp\left(\int_0^t b(\tau)d\tau\right), \quad 0 \leq t \leq T.$$

Proof. Let $0 < t_0 \leq T$ be a fixed but arbitrary number. Consider

$$V(t) := A(t_0) + \int_0^t b(\tau)E(\tau)d\tau.$$

Since A is increasing, we have $E(t) \leq V(t)$ for $0 \leq t \leq t_0$. Thus

$$\frac{d}{dt}V(t) = b(t)E(t) \leq b(t)V(t)$$

which implies that $V(t) \leq V(0)\exp\left(\int_0^t b(\tau)d\tau\right)$. Therefore, by using $V(0) = A(t_0)$, we have

$$E(t) \leq V(t) \leq A(t_0)\exp\left(\int_0^t b(\tau)d\tau\right), \quad 0 \leq t \leq t_0.$$

By taking $t = t_0$ we obtain the desired inequality for $t = t_0$. Since t_0 is arbitrary, we complete the proof. □

1.3. The Sobolev spaces H^s.

For any fixed $s \in \mathbb{R}$, $H^s := H^s(\mathbb{R}^n)$ denotes the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{H^s} := \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2},$$

where $\hat{f}$ denotes the Fourier transform of f, i.e.

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x) dx.$$

We list some properties of H^s as follows:

- H^s is a Hilbert space and $H^0 = L^2$.

- If $s \geq 0$ is an integer, then $\|f\|_{H^s} \approx \sum_{|\alpha| \leq s} \|\partial^\alpha f\|_{L^2}$.
- $H^{s_2} \subset H^{s_1}$ for any $-\infty < s_1 \leq s_2 < \infty$.
- H^{-s} is the dual space of H^s for any $s \in \mathbb{R}$.
- Let $\Delta := \sum_{j=1}^n \partial_j^2$ be the Laplacian on $\mathbb{R}^n$. Then for any $s, t \in \mathbb{R}$, $(I - \Delta)^{t/2} : H^s \to H^{s-t}$ is an isometry.
- If $s > k + n/2$ for some integer $k \geq 0$, then $H^s \hookrightarrow C^k(\mathbb{R}^n)$ compactly and

$$\sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^\infty} \leq C_s \|f\|_{H^s}, \quad \forall f \in H^s,$$

  where C_s is a constant independent of f.
- There are many other deeper results on H^s which will be introduced later on.

- Given integer $k \geq 0$, $C^k([0,T], H^s)$ consists of functions $f(t,x)$ such that

$$\sum_{j=0}^{k} \max_{0 \leq t \leq T} \|\partial_t^j f(t,\cdot)\|_{H^s} < \infty.$$

- Given $1 \leq p < \infty$, $L^p([0,T], H^s)$ consists of functions $f(t,x)$ such that

$$\int_0^T \|f(t,\cdot)\|_{H^s}^p d\tau < \infty.$$

  $L^\infty([0,T], H^s)$ can be defined similarly.
- Both $C^k([0,T], H^s)$ and $L^p([0,T], H^s)$ are Banach spaces.

1.4. Standard linear wave equations.

The classical wave operator on $\mathbb{R}^{1+n}$ is

$$\Box := \partial_t^2 - \Delta,$$

where $\Delta = \sum_{j=1}^n \partial_j^2$ is the Laplacian on $\mathbb{R}^n$. Given functions f and g, the Cauchy problem

$$\begin{aligned} \Box u = 0 \quad & \text{on } [0,\infty) \times \mathbb{R}^n, \\ u(0,\cdot) = f, \quad & \partial_t u(0,\cdot) = g \end{aligned} \tag{1}$$

has been well-understood. We summarize some well-known results as follows:

- **Uniqueness:** *(1) has at most one solution* $u \in C^2([0,\infty) \times \mathbb{R}^n)$.

  This follows from the general energy estimates derived later.

- **Existence:** *If* $f \in C^{[n/2]+2}(\mathbb{R}^n)$ *and* $g \in C^{[n/2]+1}(\mathbb{R}^n)$, *then (1) has a unique solution* $u \in C^2([0,\infty) \times \mathbb{R}^n)$.

  In fact, the solution can be given explicitly. For instance, when $n = 1$ the solution is given by the D'Alembert formula

$$u(t,x) = \frac{1}{2}\left(f(x+t) + f(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(\tau)d\tau;$$

  when $n = 2$ we have

$$u(t,x) = \partial_t \left(\frac{t}{2\pi} \int_{|y|<1} \frac{f(x+ty)}{\sqrt{1-|y|^2}} dy \right) + \frac{t}{2\pi} \int_{|y|<1} \frac{g(x+ty)}{\sqrt{1-|y|^2}} dy;$$

and for $n = 3$ we have

$$u(t,x) = \frac{1}{4\pi t^2} \int_{|y-x|=t} [f(y) - \langle \nabla f(y), x-y \rangle + tg(y)] \, d\sigma(y).$$

- **Finite speed of propagation:** *Given* $(t_0, x_0) \in (0,\infty) \times \mathbb{R}^n$, $u(t_0, x_0)$ *is completely determined by the values of* f *and* g *in the ball* $B(x_0, t_0) := \{x \in \mathbb{R}^n : |x - x_0| \leq t_0\}$, *i.e.* $B(x_0, t_0)$ *is the domain of dependence of* (t_0, x_0).

  We will obtain a more general result by the energy method.

- **Huygens' principle:** *Given* $(t_0, x_0) \in (0,\infty) \times \mathbb{R}^n$. *When* $n \geq 3$ *is odd,* $u(t_0, x_0)$ *depends only on the values of* f, *and* g *(and derivatives) on the sphere* $|x - x_0| = t_0$.

- **Decay estimates:** *When $f, g \in C_0^\infty(\mathbb{R}^n)$, $u(t,x)$ satisfies the decay estimate*

$$|u(t,x)| \lesssim \begin{cases} (1+t)^{-\frac{n-1}{2}}, & n \text{ is odd}, \\ (1+t)^{-\frac{n-1}{2}}(1+|t-|x||)^{-\frac{n-1}{2}}, & n \text{ is even}. \end{cases}$$

We will derive these estimates from the Klainerman-Sobolev inequality without using the explicit formula of solutions.

These decay estimates are crucial in proving global and long time existence results for nonlinear wave equations.

2. Energy Estimates

2.1. Energy estimates in $[0, T] \times \mathbb{R}^n$

We first consider the linear wave operator

$$\Box_g u := \partial_t^2 u - g^{jk}(t,x)\partial_j\partial_k u, \tag{2}$$

where $(g^{jk}(t,x))$ is a C^∞ symmetric matrix function defined on $[0, T] \times \mathbb{R}^n$ and is elliptic in the sense that there exist positive constants $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda|\xi|^2 \leq g^{jk}(t,x)\xi_j\xi_k \leq \Lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^n \tag{3}$$

for all $(t,x) \in [0, T] \times \mathbb{R}^n$.

Lemma 2

Let $\Box_g$ be defined by (2) with g^{jk} satisfying (3). Then for any $u \in C^2([0,T] \times \mathbb{R}^n)$ there holds

$$\|\partial u(t,\cdot)\|_{L^2} \le C_0 \left(\|\partial u(0,\cdot)\|_{L^2} + \int_0^t \|\Box_g u(\tau,\cdot)\|_{L^2} d\tau \right) \times \exp\left(C_1 \int_0^t \sum_{j,k=1}^n \|\partial g^{jk}(\tau,\cdot)\|_{L^\infty} d\tau \right)$$

for $0 \le t \le T$, where C_0 and C_1 are positive constants depending only on the ellipticity constants λ and Λ.

Proof. We consider the "energy"

$$E(t) := \int_{\mathbb{R}^n} \left(|\partial_t u|^2 + g^{jk} \partial_j u \partial_k u \right) dx.$$

It follows from the ellipticity of (g^{jk}) that

$$E(t) \approx \|\partial u(t, \cdot)\|_{L^2}^2. \tag{4}$$

Direct calculation shows that

$$\partial_t \left(|\partial_t u|^2 + g^{jk} \partial_j u \partial_k u \right) = 2\partial_t u \partial_t^2 u + 2g^{jk} \partial_j \partial_t u \partial_k u + \partial_t g^{jk} \partial_j u \partial_k u$$

$$= 2\partial_t u \Box_g u + 2\partial_j \left(g^{jk} \partial_t u \partial_k u \right) - 2\partial_j g^{jk} \partial_t u \partial_k u + \partial_t g^{jk} \partial_j u \partial_k u.$$

Therefore, by using the divergence theorem we can obtain

$$\frac{d}{dt}E(t) = 2\int_{\mathbb{R}^n} \partial_t u \Box_g u dx + \int_{\mathbb{R}^n} \left(-2\partial_j g^{jk}\partial_t u \partial_k u + \partial_t g^{jk}\partial_j u \partial_k u\right) dx.$$

This implies, with $\Phi(t) := \sum_{j,k=1}^{n} \|\partial g^{jk}\|_{L^\infty}$, that

$$\frac{d}{dt}E(t) \leq 2\|\Box_g u(t,\cdot)\|_{L^2}\|\partial_t u(t,\cdot)\|_{L^2} + 2\Phi(t)\int_{\mathbb{R}^n} |\partial u(t,\cdot)|^2 dx.$$

In view of (4), it follows that

$$\frac{d}{dt}E(t) \leq 2\|\Box_g u(t,\cdot)\|_{L^2} E(t)^{1/2} + C\Phi(t)E(t).$$

This gives

$$\frac{d}{dt}E(t)^{1/2} \leq \|\Box_g u(t,\cdot)\|_{L^2} + C\Phi(t)E(t)^{1/2}.$$

Consequently

$$\frac{d}{dt}\left\{E(t)^{1/2}\exp\left(-C\int_0^t \Phi(\tau)d\tau\right)\right\}$$
$$\leq \|\Box_g u(t,\cdot)\|_{L^2}\exp\left(-C\int_0^t \Phi(\tau)d\tau\right) \leq \|\Box_g u(t,\cdot)\|_{L^2}.$$

Integrating with respect to t gives

$$E(t)^{1/2} \exp\left(-C \int_0^t \Phi(\tau) d\tau\right) \leq E(0)^{1/2} + \int_0^t \|\Box_g u(\tau, \cdot)\|_{L^2} d\tau.$$

This together with (4) gives the desired inequality. $\Box$

The energy estimate in Lemma 2 can be extended for more general linear operator

$$Lu := \partial_t^2 u - g^{jk} \partial_j \partial_k u + b \partial_t u + b^j \partial_j u + cu,$$

where g^{jk}, b^j, b and c are smooth functions on $[0, T] \times \mathbb{R}^n$ with bounded derivatives, and (g^{jk}) is elliptic in the sense of (3).

Theorem 3

Let $0 < T < \infty$ *and* $s \in \mathbb{R}$, *Then for any*

$$u \in C([0,T], H^{s+1}) \cap C^1([0,T], H^s) \quad \text{with} \quad Lu \in L^1([0,T], H^s)$$

there holds

$$\sum_{|\alpha|\le 1} \|\partial^\alpha u(t,\cdot)\|_{H^s} \le C \left(\sum_{|\alpha|\le 1} \|\partial^\alpha u(0,\cdot)\|_{H^s} + \int_0^t \|Lu(\tau,\cdot)\|_{H^s} d\tau \right)$$

for $0 \le t \le T$, *where* C *is a constant depending only on* T, s, *and the* L^∞ *bounds of* g^{jk}, b^j, b, c *and their derivatives.*

Proof. For simplicity we consider only $s \in \mathbb{Z}$. By an approximation argument, it suffices to assume that $u \in C_0^\infty([0, T] \times \mathbb{R}^n)$. We consider three cases.

Case 1: $s = 0$. We need to establish

$$\sum_{|\alpha|\le 1} \|\partial^\alpha u(t,\cdot)\|_{L^2} \lesssim \sum_{|\alpha|\le 1} \|\partial^\alpha u(0,\cdot)\|_{L^2} + \int_0^t \|Lu(\tau,\cdot)\|_{L^2} d\tau. \quad (5)$$

To see this, we first use Lemma 2 to obtain

$$\|\partial u(t,\cdot)\|_{L^2} \lesssim \|\partial u(0,\cdot)\|_{L^2} + \int_0^t \|\Box_g u(\tau,\cdot)\|_{L^2} d\tau$$

From the definition of L it is easy to see that

$$\|\Box_g u(\tau,\cdot)\|_{L^2} \lesssim \|Lu(\tau,\cdot)\|_{L^2} + \sum_{|\alpha|\leq 1} \|\partial^\alpha u(\tau,\cdot)\|_{L^2}.$$

Therefore

$$\|\partial u(t,\cdot)\|_{L^2} \lesssim \|\partial u(0,\cdot)\|_{L^2} + \int_0^t \|Lu(\tau,\cdot)\|_{L^2} d\tau + \int_0^t \sum_{|\alpha|\leq 1} \|\partial^\alpha u(\tau,\cdot)\|_{L^2} d\tau. \quad (6)$$

By the fundamental theorem of Calculus we can write

$$u(t,x) = u(0,x) + \int_0^t \partial_t u(\tau,x) dt.$$

Thus it follows from the Minkowski inequality that

$$\|u(t,\cdot)\|_{L^2} \leq \|u(0,\cdot)\|_{L^2} + \int_0^t \|\partial_t u(\tau,\cdot)\|_{L^2} d\tau.$$

Adding this inequality to (6) gives

$$\sum_{|\alpha|\leq 1} \|\partial^\alpha u(t,\cdot)\|_{L^2} \lesssim \sum_{|\alpha|\leq 1} \|\partial^\alpha u(0,\cdot)\|_{L^2} + \int_0^t \|Lu(\tau,\cdot)\|_{L^2} d\tau$$
$$+ \int_0^t \sum_{|\alpha|\leq 1} \|\partial^\alpha u(\tau,\cdot)\|_{L^2} d\tau.$$

An application of the Gronwall inequality then gives (5).

Case 2: $s \in \mathbb{N}$. Let β be any multi-index β satisfying $|\beta| \le s$. We apply (5) to $\partial_x^\beta u$ to obtain

$$\begin{aligned} \sum_{|\alpha|\le 1} \|\partial_x^\beta \partial^\alpha u(t,\cdot)\|_{L^2} &\lesssim \sum_{|\alpha|\le 1} \|\partial_x^\beta \partial^\alpha u(0,\cdot)\|_{L^2} + \int_0^t \|L\partial_x^\beta u(\tau,\cdot)\|_{L^2} d\tau \\ &\lesssim \sum_{|\alpha|\le 1} \|\partial_x^\beta \partial^\alpha u(0,\cdot)\|_{L^2} + \int_0^t \|\partial_x^\beta L u(\tau,\cdot)\|_{L^2} d\tau \\ &\quad + \int_0^t \|[L, \partial_x^\beta] u(\tau,\cdot)\|_{L^2} d\tau, \qquad (7) \end{aligned}$$

where $[L, \partial_x^\beta] := L\partial_x^\beta - \partial_x^\beta L$ denotes the commutator. Direct calculation shows that

$$[L, \partial_x^\beta]u = \Big(\partial_x^\beta(g^{jk}\partial_j\partial_k u) - g^{jk}\partial_x^\beta\partial_j\partial_k u\Big) + \Big(b\partial_x^\beta\partial_t u - \partial_x^\beta(b\partial_t u)\Big) + \Big(b^j\partial_x^\beta\partial_j u - \partial_x^\beta(b^j\partial_j u)\Big) + \Big(c\partial_x^\beta u - \partial_x^\beta(cu)\Big)$$

from which we can see $[L, \partial_x^\beta]$ is a differential operator of order $\le |\beta| + 1 \le s + 1$ involving no t-derivatives of order > 1. Thus

$$\Big|[L, \partial_x^\beta]u\Big| \lesssim \sum_{|\gamma|\le s} \left(|\partial_x^\gamma \partial u| + |\partial_x^\gamma u|\right).$$

Consequently

$$\Big\|[L, \partial_x^\beta]u\Big\|_{L^2} \lesssim \sum_{|\gamma|\le s} \left(\|\partial_x^\gamma \partial u\|_{L^2} + \|\partial_x^\gamma u\|_{L^2}\right) \lesssim \sum_{|\alpha|\le 1} \|\partial^\alpha u\|_{H^s}.$$

Combining this inequality with (7) gives

$$\sum_{|\alpha|\le 1} \|\partial_x^\beta \partial^\alpha u(t,\cdot)\|_{L^2} \lesssim \sum_{|\alpha|\le 1} \|\partial_x^\beta \partial^\alpha u(0,\cdot)\|_{L^2} + \int_0^t \|\partial_x^\beta Lu(\tau,\cdot)\|_{L^2} d\tau + \int_0^t \sum_{|\alpha|\le 1} \|\partial^\alpha u(\tau,\cdot)\|_{H^s} d\tau,$$

Summing over all β with $|\beta| \le s$ we obtain

$$\sum_{|\alpha|\le 1} \|\partial^\alpha u(t,\cdot)\|_{H^s} \lesssim \sum_{|\alpha|\le 1} \|\partial^\alpha u(0,\cdot)\|_{H^s} + \int_0^t \|Lu(\tau,\cdot)\|_{H^s} d\tau + \int_0^t \sum_{|\alpha|\le 1} \|\partial^\alpha u(\tau,\cdot)\|_{H^s} d\tau.$$

By the Gronwall inequality we obtain the estimate for $s \in \mathbb{N}$.

Case 3: $s \in -\mathbb{N}$. We consider

$$v(t,\cdot) := (I - \Delta_x)^s u(t,\cdot).$$

Since $-s \in \mathbb{N}$, we can apply the estimate established in Case 2 to v to derive that

$$\sum_{|\alpha|\le 1} \|\partial^\alpha v(t,\cdot)\|_{H^{-s}} \lesssim \sum_{|\alpha|\le 1} \|\partial^\alpha v(0,\cdot)\|_{H^{-s}} + \int_0^t \|Lv(\tau,\cdot)\|_{H^{-s}} d\tau.$$

We can write

$$\begin{aligned} Lv(\tau,\cdot) &= (I-\Delta)^s Lu(\tau,\cdot) + [L,(I-\Delta)^s]u(\tau,\cdot) \\ &= (I-\Delta)^s Lu(\tau,\cdot) + (I-\Delta)^s[(I-\Delta)^{-s}, L]v(\tau,\cdot). \end{aligned}$$

Therefore

$$\|Lv(\tau,\cdot)\|_{H^{-s}} \le \|Lu(\tau,\cdot)\|_{H^s} + \|[(I-\Delta)^{-s}, L]v(\tau,\cdot)\|_{H^s}.$$

Consequently

$$\sum_{|\alpha|\le 1} \|\partial^\alpha v(t,\cdot)\|_{H^{-s}} \lesssim \sum_{|\alpha|\le 1} \|\partial^\alpha v(0,\cdot)\|_{H^{-s}} + \int_0^t \|Lu(\tau,\cdot)\|_{H^s} d\tau$$
$$+ \int_0^t \|[(I-\Delta)^{-s}, L]v(\tau,\cdot)\|_{H^s} d\tau. \qquad (8)$$

It is easy to check $[(I-\Delta)^{-s}, L]$ is a differential operator of order $\le -2s+1$ involving no t-derivatives of order > 1. We can write

$$[(I-\Delta)^{-s}, L]v = \sum_{|\alpha|\le 1} \sum_{|\beta|,|\gamma|\le -s} \partial_x^\gamma (\Gamma_{\alpha\beta\gamma} \partial_x^\beta \partial^\alpha v),$$

where $\Gamma_{\alpha\beta\gamma}$ are smooth bounded functions. Therefore

$$\left\|[(I-\Delta)^{-s}, L]v\right\|_{H^s} \lesssim \sum_{|\alpha|\le 1}\sum_{|\beta|\le -s} \|\partial_x^\beta \partial^\alpha v\|_{L^2} \lesssim \sum_{|\alpha|\le 1} \|\partial^\alpha v\|_{H^{-s}}.$$

Combining this inequality with (8), we obtain

$$\sum_{|\alpha|\le 1} \|\partial^\alpha v(t,\cdot)\|_{H^{-s}} \lesssim \sum_{|\alpha|\le 1} \|\partial^\alpha v(0,\cdot)\|_{H^{-s}} + \int_0^t \|Lu(\tau,\cdot)\|_{H^s} d\tau + \int_0^t \sum_{|\alpha|\le 1} \|\partial^\alpha v(\tau,\cdot)\|_{H^{-s}} d\tau$$

An application of the Gronwall inequality gives

$$\sum_{|\alpha|\le 1} \|\partial^\alpha v(t,\cdot)\|_{H^{-s}} \lesssim \sum_{|\alpha|\le 1} \|\partial^\alpha v(0,\cdot)\|_{H^{-s}} + \int_0^t \|Lu(\tau,\cdot)\|_{H^s} d\tau.$$

Since $\|\partial^\alpha v(t,\cdot)\|_{H^{-s}} = \|\partial^\alpha u(t,\cdot)\|_{H^s}$, the proof is complete. □

2.2. Finite Speed of Propagation

We consider the wave equation

$$\Box u := \partial_t^2 u - \Delta u = F(t, x, u, \partial u, \partial^2 u) \quad \text{in } [0, \infty) \times \mathbb{R}^n, \tag{9}$$

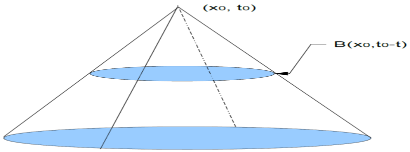
where $F(t, x, u, \mathbf{p}, \mathbf{A})$ is a smooth function with

$$F(t, x, 0, 0, \mathbf{A}) = 0 \quad \text{for all } t, x, \text{ and } \mathbf{A}.$$

For any fixed $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^n$, we introduce

$$C_{t_0, x_0} := \{(t, x) : 0 \leq t \leq t_0 \text{ and } |x - x_0| \leq t_0 - t\} \tag{10}$$

which is called the backward light cone through (t_0, x_0).



The following result says that any "disturbance" originating outside

$$B(x_0, t_0) := \{x \in \mathbb{R}^n : |x - x_0| \le t_0\}$$

has no effect on the solution within C_{t_0,x_0}.

Theorem 4 (finite speed of propagation)

Let u be a C^2 solution of (9) in C_{t_0,x_0}. If $u \equiv \partial_t u \equiv 0$ on $B(x_0, t_0)$, then $u \equiv 0$ in C_{t_0,x_0}.

Proof. Consider for $0 \le t \le t_0$ the function

$$\begin{aligned} E(t) &:= \int_{B(x_0,t_0-t)} \left(u^2 + |u_t(t,x)|^2 + |\nabla u(t,x)|^2\right) dx \\ &= \int_0^{t_0-t} \int_{\partial B(x_0,\tau)} \left(u^2 + |u_t|^2 + |\nabla u|^2\right) d\sigma d\tau. \end{aligned}$$

We have

$$\begin{aligned}
\frac{d}{dt}E(t) &= 2\int_{B(x_0,t_0-t)} (uu_t + u_t u_{tt} + \nabla u\cdot\nabla u_t)\,dx \\
&\qquad - \int_{\partial B(x_0,t_0-t)} \left(u^2+|u_t|^2+|\nabla u|^2\right)d\sigma \\
&= 2\int_{B(x_0,t_0-t)} u_t\,(u+\Box u)\,dx + 2\int_{B(x_0,t_0-t)} \operatorname{div}(u_t\nabla u)dx \\
&\qquad - \int_{\partial B(x_0,t_0-t)} \left(u^2+|u_t|^2+|\nabla u|^2\right)d\sigma.
\end{aligned}$$

Using $\Box u = F(t,x,u,\partial u,\partial^2 u)$ and the divergence theorem we have

$$\begin{aligned}
\frac{d}{dt}E(t) &= 2\int_{B(x_0,t_0-t)} u_t\left(u+F(t,x,u,\partial u,\partial^2 u)\right)dx \\
&\quad + 2\int_{\partial B(x_0,t_0-t)} u_t\nabla u\cdot\nu\,d\sigma - \int_{\partial B(x_0,t_0-t)} \left(u^2+|u_t|^2+|\nabla u|^2\right)d\sigma,
\end{aligned}$$

where ν denotes the outward unit normal to $\partial B(x_0, t_0 - t)$. We have

$$2|u_t \nabla u \cdot \nu| \leq 2|u_t||\nabla u| \leq |u_t|^2 + |\nabla u|^2.$$

Consequently

$$\frac{d}{dt}E(t) \leq 2 \int_{B(x_0,t_0-t)} u_t \left(u + F(t,x,u,\partial u,\partial^2 u)\right) dx.$$

Since $F(t,x,0,0,\partial^2 u) = 0$, we have

$$F(t,x,u,\partial u,\partial^2 u) = F(t,x,u,\partial u,\partial^2 u) - F(t,x,0,0,\partial^2 u)$$
$$= \int_0^1 \frac{\partial}{\partial s} F(t,x,su,s\partial u,\partial^2 u) ds$$
$$= \int_0^1 \left(\frac{\partial F}{\partial u}(t,x,su,s\partial u,\partial^2 u)u + \mathbf{D_p} F(t,x,su,s\partial u,\partial^2 u) \cdot \partial u \right) ds.$$

This gives

$$|F(t,x,u,\partial u,\partial^2 u)| \le \int_0^1 \left|\frac{\partial F}{\partial u}(t,x,su,s\partial u,\partial^2 u)\right| ds|u| + \int_0^1 \left|\mathbf{D_p} F(t,x,su,s\partial u,\partial^2 u)\right| ds|\partial u|.$$

Let $C = \max\{C_0, C_1\}$, where

$$C_0 := \max_{(t,x)\in C_{t_0,x_0}} \int_0^1 \left|\frac{\partial F}{\partial u}(t,x,su(t,x),s\partial u(t,x),\partial^2 u(t,x))\right| ds,$$

$$C_1 := \max_{(t,x)\in C_{t_0,x_0}} \int_0^1 \left|\mathbf{D_p} F(t,x,su(t,x),s\partial u(t,x),\partial^2 u(t,x))\right| ds.$$

Then

$$|F(t,x,u,\partial u,\partial^2 u)| \le C\left(|u| + |\partial u|\right).$$

Therefore

$$\frac{d}{dt}E(t) \leq 2(1+C)\int_{B(x_0,t_0-t)} |u_t|\,(|u|+|\partial u|)\,dx \leq 2(1+C)E(t).$$

Since $u(0,\cdot) \equiv u_t(0,\cdot) \equiv 0$ on $B(x_0,t_0)$ implies that $E(0)=0$, we have $E(t) \equiv 0$ for $0 \leq t \leq t_0$. Therefore $u \equiv 0$ in C_{t_0,x_0}. □

3. Local Existence Results

We prove the local existence for Cauchy problem of quasi-linear wave equations. The proof is based on existence result of linear equations and the energy estimates.

3.1. Existence result for linear wave equations

Consider first the linear wave equation

$$\begin{aligned} &Lu = F \qquad \text{on } [0, T] \times \mathbb{R}^n, \\ &u|_{t=0} = f, \quad \partial_t u|_{t=0} = g, \end{aligned} \tag{11}$$

where L is a linear differential operator defined by

$$Lu := \partial_t^2 u - g^{jk} \partial_j \partial_k u + b \partial_t u + b^j \partial_j u + cu$$

in which g^{jk}, b^j, b and c are smooth functions on $[0, T] \times \mathbb{R}^n$ and (g^{jk}) is elliptic in the sense of (3).

The adjoint operator L^* of L is defined by

$$\int_0^T \int_{\mathbb{R}^n} \varphi L\psi dxdt = \int_0^T \int_{\mathbb{R}^n} \psi L^*\varphi dxdt, \quad \forall \varphi, \psi \in C_0^\infty((0, T)\times\mathbb{R}^n).$$

A straightforward calculation shows that

$$L^*\varphi = \partial_t^2\varphi - \partial_j\partial_k(g^{jk}\varphi) - \partial_t(b\varphi) - \partial_j(b^j\varphi) + c\varphi.$$

If $u \in C^2([0, T] \times \mathbb{R}^n)$ is a classical solution of (11), then by integration by parts we have for $\varphi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n)$ that

$$\int_0^T \int_{\mathbb{R}^n} F\varphi dxdt = \int_0^T \int_{\mathbb{R}^n} uL^*\varphi dxdt - \int_{\mathbb{R}^n} \varphi(0, x)g(x)dx + \int_{\mathbb{R}^n} [\varphi_t(0, x) - (b\varphi)(0, x)]\, f(x)dx. \quad (12)$$

Conversely, we can show, if $u \in C^2([0,T] \times \mathbb{R}^n)$ satisfies (12) for all $\varphi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n)$, then u is a classical solution of (11).

We will call a less regular u a weak solution of (11) if it satisfies (12), where the involved integrals might be understood as duality pairing in appropriate spaces.

Theorem 5

Let $s \in \mathbb{R}$ and $T > 0$. Then for any $f \in H^{s+1}(\mathbb{R}^n)$, $g \in H^s(\mathbb{R}^n)$ and $F \in L^1([0,T], H^s(\mathbb{R}^n))$, the linear wave equation (11) has a unique weak solution

$$u \in C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$$

in the sense that (12) holds for all $\varphi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n)$.

Proof.

1. The uniqueness follows immediately from Theorem 3.
2. We first consider the case that

$$f = g = 0 \quad \text{and} \quad F \in C_0^\infty([0,T] \times \mathbb{R}^n).$$

Let $s \in \mathbb{R}$ be any fixed number. we may apply Theorem 3 to L^* with t replaced by $T - t$ to derive that

$$\|\varphi(t,\cdot)\|_{H^{-s}} \lesssim \int_0^T \|L^*\varphi(\tau,\cdot)\|_{H^{-s-1}} d\tau$$

for any $\varphi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n)$

Using F we can define on $\mathcal{V} := L^* C_0^\infty((-\infty, T) \times \mathbb{R}^n)$ a linear functional $\ell_F(\cdot)$ by

$$\ell_F(L^*\varphi) = \int_0^T \int_{\mathbb{R}^n} F\varphi dxdt, \quad \varphi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n).$$

Then we have

$$\begin{aligned} |\ell_F(L^*\varphi)| &\leq \int_0^T \|F(t,\cdot)\|_{H^s} \|\varphi(t,\cdot)\|_{H^{-s}} dt \\ &\lesssim \int_0^T \|L^*\varphi(t,\cdot)\|_{H^{-s-1}} dt, \end{aligned}$$

i.e.,

$$|\ell_F(\psi)| \leq \int_0^T \|\psi(t,\cdot)\|_{H^{-s-1}} dt, \quad \forall \psi \in \mathcal{V}.$$

We can view $\mathcal{V}$ as a subspace of $L^1([0,T],H^{-s-1})$. Then, by Hahn-Banach theorem, ℓ_F can be extended to a bounded linear functional on $L^1([0,T],H^{-s-1})$. Thus, we can find $u \in L^\infty([0,T],H^{s+1})$, the dual space of $L^1([0,T],H^{-s-1})$, such that

$$\ell_F(\psi) = \int_0^T \int_{\mathbb{R}^n} u\psi dxdt, \quad \forall \psi \in L^1([0,T],H^{-s-1}).$$

Therefore, for all $\varphi \in C_0^\infty((-\infty,T)\times\mathbb{R}^n)$ there holds

$$\int_0^T \int_{\mathbb{R}^n} F\varphi dxdt = \ell_F(L^*\varphi) = \int_0^T \int_{\mathbb{R}^n} uL^*\varphi dxdt$$

So u is a weak solution.

By using $Lu = F$ we have

$$\partial_t(\partial_t u) - b\partial_t u = g^{jk}\partial_j\partial_k u - b^j\partial_j u - cu + F \in L^\infty([0, T], H^{s-1}).$$

This implies that $\partial_t u \in L^\infty([0, T], H^{s-1})$ and

$$\partial_t^2 u \in L^\infty([0, T], H^{s-1}) \subset L^\infty([0, T], H^{s-2}).$$

Consequently $u \in C^1([0, T], H^{s-1})$. Since s can be arbitrary, we have

$$u \in C^1([0, T], C^\infty(\mathbb{R}^n)).$$

Using this and $Lu = F$ we can improve the regularity of u to $u \in C^\infty([0, T] \times \mathbb{R}^n)$.

3. For the case $f, g \in C_0^\infty(\mathbb{R}^n)$ and $F \in C_0^\infty([0,T] \times \mathbb{R}^n)$, we can reduce it to the previous case by considering $\tilde{u} = u - (f + tg)$.
4. We finally consider the general case by an approximation argument. We may take sequences $\{f_m\}$, $\{g_m\} \subset C_0^\infty(\mathbb{R}^n)$ and $\{F_m\} \subset C_0^\infty([0,T) \times \mathbb{R}^n)$ such that

$$\|f_m - f\|_{H^{s+1}} + \|g_m - g\|_{H^s} + \int_0^T \|F_m(t,\cdot) - F(t,\cdot)\|_{H^s} dt \to 0$$

as $m \to \infty$. Let u_m be the solution of (11) with data f_m, g_m and F_m. Then $u_m \in C^\infty([0,T] \times \mathbb{R}^n)$ and

$$u_m \in X_T := C([0,T], H^{s+1}) \cap C^1([0,T], H^s)$$

Since for any m and l there holds

$$L(u_m - u_l) = F_m - F_l \quad \text{on } [0, T] \times \mathbb{R}^n,$$
$$(u_m - u_l)(0, \cdot) = f_m - f_l, \quad \partial_t(u_m - u_l)(0, \cdot) = g_m - g_l,$$

we can use Theorem 3 to derive that

$$\sum_{|\alpha|\le 1} \|\mathbf{D}^\alpha(u_m - u_l)\|_{H^s} \lesssim \|f_m - f_l\|_{H^{s+1}} + \|g_m - g_l\|_{H^s}$$
$$+ \int_0^T \|F_m(t,\cdot) - F_l(t,\cdot)\|_{H^s} dt.$$

Thus $\{u_m\}$ is a Cauchy sequence in X_T and there is $u \in X_T$ such that $\|u_m - u\|_{X_T} \to 0$ as $m \to \infty$. Since u_m satisfies (12) with f, g and F replaced by f_m, g_m and F_m, we can see that u satisfies (12) by taking $m \to \infty$. □

3.2. Local existence for quasi-linear wave equations

We next consider the quasi-linear wave equation

$$\begin{aligned} &\partial_t^2 u - g^{jk}(u, \partial u)\partial_j\partial_k u = F(u, \partial u), \\ &u(0, \cdot) = f, \quad \partial_t u(0, \cdot) = g, \end{aligned} \tag{13}$$

where

- g^{jk} and F are C^∞ functions, and $F(0,0) = 0$;
- (g^{jk}) is elliptic in the sense that

$$C_0(u, \mathbf{p})|\xi|^2 \leq g^{jk}(u, \mathbf{p})\xi_j\xi_k \leq C_1(u, \mathbf{p})|\xi|^2, \quad \forall \xi \in \mathbb{R}^n,$$

where $C_0(u, p)$ and $C_1(u, p)$ are positive continuous functions with respect to (u, p).

Theorem 6

If $(f, g) \in H^{s+1} \times H^s$ *for* $s \geq n + 2$*, then there is a* $T > 0$ *such that (13) has a unique solution* $u \in C^2([0, T] \times \mathbb{R}^n)$*; moreover*

$$u \in L^\infty([0, T], H^{s+1}) \cap C^{0,1}([0, T], H^s).$$

Proof. 1. We first prove uniqueness. Let u and $\tilde{u}$ be two solutions. Then $v := u - \tilde{u}$ satisfies

$$\partial_t^2 v - g^{jk}(u, \partial u)\partial_j\partial_k v = R, \quad v(0, \cdot) = 0, \quad \partial_t v(0, \cdot) = 0,$$

where

$$R := [F(u, \partial u) - F(\tilde{u}, \partial\tilde{u})] + \left[g^{jk}(u, \partial u) - g^{jk}(\tilde{u}, \partial\tilde{u})\right] \partial_j\partial_k \tilde{u}.$$

It is clear that

$$|R| \le C(|v| + |\partial v|),$$

where C depends on the bound on $|\partial^2 \tilde{u}|$ and the bounds on the derivatives of g^{jk} and F. In view of Theorem 3, we have

$$\sum_{|\alpha|\le 1} \|\partial^\alpha v(t,\cdot)\|_{L^2} \lesssim \int_0^t \|R(\tau,\cdot)\|_{L^2} d\tau \lesssim \int_0^t \sum_{|\alpha|\le 1} \|\partial^\alpha v(\tau,\cdot)\|_{L^2} d\tau.$$

By Gronwall inequality, $\sum_{|\alpha|\le 1} \|\partial^\alpha v\|_{L^2} = 0$. Thus $v = 0$, i.e. $u = \tilde{u}$.

2. Next we prove the existence. By an approximation argument as in the proof of Theorem 5 we may assume that $f, g \in C_0^\infty(\mathbb{R}^n)$.

We use the Picard iteration. Let $u_{-1} = 0$ and define u_m, $m \geq 0$, successively by

$$\begin{aligned} &\partial_t^2 u_m - g^{jk}(u_{m-1}, \partial u_{m-1})\partial_j \partial_k u_m = F(u_{m-1}, \partial u_{m-1}), \\ &u_m(0, \cdot) = f, \quad \partial_t u_m(0, \cdot) = g. \end{aligned} \tag{14}$$

By Theorem 5, all u_m are in $C^\infty([0, \infty) \times \mathbb{R}^n)$. In what follows, we will show that $\{u_m\}$ converges and the limit is a solution.

Step 1. Consider

$$A_m(t) := \sum_{|\alpha| \leq s+1} \|\partial^\alpha u_m(t, \cdot)\|_{L^2}.$$

We prove that $\{A_m(t)\}$ is uniformly bounded in m and $t \in [0, T]$ with small $T > 0$.

By using (14) it is easy to show that

$$A_m(0) \leq A_0, \quad m = 0, 1, \cdots$$

for some constant A_0 independent of m; in fact A_0 can be taken as the multiple of

$$\|f\|_{H^{s+1}} + \|g\|_{H^s}.$$

We claim that there exist $0 < T \leq 1$ and $A > 0$ such that

$$\sup_{0 \leq t \leq T} A_m(t) \leq A, \qquad m = 0, 1, \cdots . \tag{15}$$

We show it by induction on m. Since $F(0,0) = 0$, (15) with $m = 0$ follows from Theorem 3. with $A = CA_0$ for a large C.

Assume next (15) is true for some $m \geq 0$. By Sobolev embedding,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n} \sum_{|\alpha|\leq s+1-[(n+2)/2]} |\partial^\alpha u_m(t,x)| \leq CA_m(t) \leq CA.$$

Since $s \geq n+2$, we have $s+1-[(n+2)/2] \geq [(s+3)/2]$. Thus

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n} \sum_{|\alpha|\leq (s+3)/2} |\partial^\alpha u_m(t,x)| \leq CA. \tag{16}$$

By the definition of u_{m+1} we have for any $|\alpha| \leq s$ that

$$\begin{aligned} &\partial_t^2 \partial^\alpha u_{m+1} - g^{jk}(u_m, \partial u_m)\partial_j\partial_k\partial^\alpha u_{m+1} \\ &\quad = \partial^\alpha F(u_m, \partial u_m) - [\partial^\alpha, g^{jk}(u_m, \partial u_m)]\partial_j\partial_k u_{m+1}. \end{aligned} \tag{17}$$

Observation 1.
$[\partial^\alpha, g^{jk}(u_m, \partial u_m)]\partial_j\partial_k u_{m+1}$ is a linear combination of finitely many terms, each term is a product of derivatives of u_m or u_{m+1} in which at most one factor where u_m or u_{m+1} is differentiated more than $(|\alpha|+3)/2$ times.

To see this, we note that $[\partial^\alpha, g^{jk}(u_m, \partial u_m)]\partial_j\partial_k u_{m+1}$ is a linear combination of terms

$$a(u_m, \partial u_m)\partial^{\alpha_1} u_m \cdots \partial^{\alpha_k} u_m \partial^{\beta_1}\partial u_m \cdots \partial^{\beta_l}\partial u_m \partial^\gamma \partial^2 u_{m+1},$$

where $|\alpha_1| + \cdots + |\alpha_k| + |\beta_1| + \cdots + |\beta_l| + |\gamma| = |\alpha|$ and $|\gamma| \leq |\alpha| - 1$.

- If $|\gamma| \geq (|\alpha| - 1)/2$, then

$$|\alpha_1| + \cdots + |\alpha_k| + |\beta_1| + \cdots + |\beta_l| \leq (|\alpha| + 1)/2.$$

So $|\alpha_j| \leq (|\alpha| + 1)/2$ and $|\beta_j| \leq (|\alpha| + 1)/2$ for all α_j and β_j.

- If $|\gamma| < (|\alpha| - 1)/2$, then

$$|\alpha_1| + \cdots + |\alpha_k| + |\beta_1| + \cdots + |\beta_l| \leq |\alpha|.$$

So there is at most one index among $\{\alpha_1, \cdots, \beta_l\}$ whose length is $> |\alpha|/2$.

Since $|\alpha| \leq s$, we have $(|\alpha|+3)/2 \leq (s+3)/2$. Using Observation 1, it follows from (16) that

$$\left|[\partial^\alpha, g^{jk}(u_m, \partial u_m)]\partial_j\partial_k u_m\right| \leq C_A\Big(\sum_{|\beta|\leq|\alpha|+1}(|\partial^\beta u_m| + |\partial^\beta u_{m+1}|) + 1\Big)$$
$$\leq C_A\Big(\sum_{|\beta|\leq s+1}(|\partial^\beta u_m| + |\partial^\beta u_{m+1}|) + 1\Big).$$

where C_A is a constant depending on A but independent of m. So, by the induction hypothesis, we have

$$\left\|[\partial^\alpha, g^{jk}(u_m, \partial u_m)]\partial_j\partial_k u_{m+1}\right\|_{L^2} \leq C_A\left(A_{m+1}(t) + A_m(t) + 1\right)$$
$$\leq C_A(A_{m+1}(t) + 1), \qquad (18)$$

Observation 2.

$\partial^\alpha F(u_m, \partial u_m)$ is a linear combination of finitely many terms, each term is a product of derivatives of u_m in which at most one factor where u_m is differentiated more than $|\alpha|/2+1$ times.

Indeed, we note that $\partial^\alpha F(u_m, \partial u_m)$ is a linear combination of terms

$$a(u_m, \partial u_m)\partial^{\beta_1} u_m \cdots \partial^{\beta_k} u_m \partial^{\gamma_1}\partial u_m \cdots \partial^{\gamma_l}\partial u_m$$

where $|\beta_1| + \cdots + |\beta_k| + |\gamma_1| + \cdots + |\gamma_l| = |\alpha|$. Thus $|\beta_j| \le |\alpha|/2$ and $|\gamma_j| \le |\alpha|/2$ except one of the multi-indices.

Using Observation 2, we have from (16) that

$$|\partial^\alpha F(u_m, \partial u_m)| \le C_A\Big(\sum_{|\beta|\le|\alpha|+1} |\partial^\beta u_m| + 1\Big) \le C_A\Big(\sum_{|\beta|\le s+1} |\partial^\beta u_m| + 1\Big).$$

Therefore, by the induction hypothesis, we have

$$\|\partial^\alpha F(u_m, \partial u_m)\|_{L^2} \leq C_A\left(A_m(t) + 1\right) \leq C_A. \tag{19}$$

In view of Lemma 2, (18) and (19), we have from (17) that

$$\begin{aligned}
&\|\partial^\alpha u_{m+1}(t,\cdot)\|_{L^2} + \|\partial^\alpha \partial u_{m+1}(t,\cdot)\|_{L^2} \\
&\leq C_0\Big(\|\partial^\alpha u_{m+1}(0,\cdot)\|_{L^2} + \|\partial^\alpha \partial u_{m+1}(0,\cdot)\|_{L^2} \\
&\qquad\qquad + C_A \int_0^t (A_{m+1}(\tau) + 1) d\tau\Big) \\
&\quad \times \exp\left(C_1 \int_0^t \sum_k \left\|\partial_j\left(g^{jk}(u_m, \partial u_m)\right)(\tau,\cdot)\right\|_{L^\infty} d\tau\right)
\end{aligned}$$

Using (16) we have

$$\sum_k \left\| \partial_j \left(g^{jk}(u_m, \partial u_m) \right) (\tau, \cdot) \right\|_{L^\infty} \lesssim A.$$

Summing over all α with $|\alpha| \leq s$, we therefore obtain

$$A_{m+1}(t) \leq Ce^{CAt} \left(A_{m+1}(0) + C_A t + C_A \int_0^t A_{m+1}(\tau) d\tau \right).$$

By Gronwall's inequality and $A_{m+1}(0) \leq A_0$ we obtain

$$A_{m+1}(t) \leq Ce^{CAt} \left(A_0 + C_A t \right) \exp \left(tCC_A e^{CA} \right)$$

So, if we set $A := 2CA_0$ and take $T > 0$ small but independent of m, we obtain $A_{m+1}(t) \leq A$ for $0 \leq t \leq T$. This completes the proof of the claim (15).

Step 2. We will show that $\{u_m\}$ converges to a function u in $C([0,T],H^1)\cap C^1([0,T],L^2)$. To this end, consider

$$E_m(t) := \sum_{|\alpha|\le 1} \|\partial^\alpha(u_m-u_{m-1})(t,\cdot)\|_{L^2}.$$

We have

$$\left(\partial_t^2 - g^{jk}(u_{m-1},\partial u_{m-1})\partial_j\partial_k\right)(u_m-u_{m-1}) = R_m,$$
$$(u_m-u_{m-1})(0,\cdot) = 0 = \partial_t(u_m-u_{m-1})(0,\cdot),$$

where

$$R_m := \left[g^{jk}(u_{m-1},\partial u_{m-1}) - g^{jk}(u_{m-2},\partial u_{m-2})\right]\partial_j\partial_k u_{m-1} + \left[F(u_{m-1},\partial u_{m-1}) - F(u_{m-2},\partial u_{m-2})\right]$$

Observing that

$$|R_m| \lesssim (|u_{m-1} - u_{m-2}| + |\partial u_{m-1} - \partial u_{m-2}|)(1 + |\partial^2 u_{m-1}|).$$

In view of Theorem 3 and (16), we can obtain

$$E_m(t) \le C \int_0^t E_{m-1}(\tau) d\tau, \quad m = 0, 1, \cdots .$$

Consequently

$$E_m(t) \le \frac{(Ct)^m}{m!} \sup_{0 \le t \le T} E_0(t), \quad m = 0, 1, \cdots .$$

This shows that $\sum_m E_m(t) \le C_0$. Thus $\{u_m\}$ is a Cauchy sequence and converges to some $u \in X_T := C([0, T], H^1) \times C^1([0, T], L^2)$.

Step 3. We prove that

$$u \in L^\infty([0,T], H^{s+1}) \cap C^{0,1}([0,T], H^s). \qquad (20)$$

In fact, from (15) we have

$$\|u_m(t,\cdot)\|_{H^{s+1}} + \|\partial_t u_m(t,\cdot)\|_{H^s} \le A.$$

So, for each fixed t, we can find a subsequence of $\{u_m\}$, say $\{u_m\}$ itself, such that

$$u_m(t,\cdot) \rightharpoonup \tilde{u} \quad \text{weakly in } H^{s+1},$$
$$\partial_t u_m(t,\cdot) \rightharpoonup \tilde{w} \quad \text{weakly in } H^s.$$

Since $u_m(t,\cdot) \to u(t,\cdot)$ in H^1 and $\partial_t u_m(t,\cdot) \to \partial_t u(t,\cdot)$ in L^2, we must have $u(t,\cdot) = \tilde{u}$ and $\partial_t u(t,\cdot) = \tilde{w}$.

By the weakly lower semi-continuity of norms we have

$$\|u(t,\cdot)\|_{H^{s+1}} \le \liminf_m \|u_m(t,\cdot)\|_{H^{s+1}} \le A,$$
$$\|\partial_t u(t,\cdot)\|_{H^s} \le \liminf_m \|\partial_t u_m(t,\cdot)\|_{H^s} \le A.$$

We thus obtain (20). By (15) and the same argument we can further obtain

$$\sum_{|\alpha|\le s+1} \|\partial^\alpha u(t,\cdot)\|_{L^2} \le A$$

This together with (15), the result in step 2, and the interpolation inequality gives

$$\sup_{0\le t\le T} \sum_{|\alpha|\le s} \|\partial^\alpha u_m(t,\cdot) - \partial^\alpha u(t,\cdot)\|_{L^2} \to 0.$$

By Sobolev embedding,

$$\max_{(t,x)\in[0,T]\times\mathbb{R}^n} \sum_{|\alpha|\le(s+1)/2} |\partial^\alpha u_m(t,x) - \partial^\alpha u(t,x)| \to 0$$

Therefore $u_m \to u$ in $C^2([0,T]\times\mathbb{R}^n)$ and u is a solution. $\square$

Remark. Theorem 6 holds when $(f,g)\in H^{s+1}\times H^s$ with $s > (n+2)/2$.

The interval of existence for quasi-linear wave equation could be very small.

Example. For any $\varepsilon > 0$, there exists $g \in C_c^\infty(\mathbb{R}^n)$ such that

$$\square u = (\partial_t u)^2, \quad u|_{t=0} = 0, \quad \partial_t u|_{t=0} = g \tag{21}$$

does not admit a C^2 solution past time ε.

To see this, we first note that $u(t,x) = -\log(1 - t/\varepsilon)$ solves (46) with $g \equiv 1/\varepsilon$, and $u \to \infty$ as $t \to \varepsilon$.

Next we fix an $R > \varepsilon$ and choose $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi(x) = 1$ for $|x| \leq R$. Consider (46) with $g(x) = \chi(x)/\varepsilon$, which has a solution on some interval $[0, T]$. We claim that the solution will blow up no later than $t = \varepsilon$.

In fact, let

$$\Omega = \{(t,x) : 0 \leq t < \varepsilon, |x| + t \leq R\}.$$

By the finite speed of propagation, u inside Ω is completely determined by the value of g on $B(0,R)$ on which $g \equiv 1$. Thus $u(t,x) = -\log(1 - t/\varepsilon)$ in Ω which blows up at $t = \varepsilon$.

The following theorem gives a criterion on extending solutions which is important in establishing global existence results.

Theorem 7

If $f, g \in C_0^\infty(\mathbb{R}^n)$, then there is $T > 0$ so that the Cauchy problem (13) has a unique solution $u \in C^\infty([0, T] \times \mathbb{R}^n)$. Let

$$T_* := \sup\{T > 0 : \text{(13) has a solution } u \in C^\infty([0, T] \times \mathbb{R}^n)\}.$$

If $T_* < \infty$, then

$$\sum_{|\alpha| \le (n+6)/2} |\partial^\alpha u(t, x)| \notin L^\infty([0, T_*) \times \mathbb{R}^n). \tag{22}$$

Proof. In the proof of Theorem 6, we have constructed a sequence $\{u_m\} \subset C^\infty([0, \infty) \times \mathbb{R}^n)$ by (14) with $u_{-1} = 0$ which converges in $C^2([0, T] \times \mathbb{R}^n)$ to a solution u.

We also showed that for each $s \geq n+2$ there exist $T_s > 0$ and $A_s > 0$ such that

$$\sum_{|\alpha| \leq s+1} \|\partial^\alpha u_m(t,\cdot)\|_{L^2} \leq A_s, \quad 0 \leq t \leq T_s \tag{23}$$

for all $m = 0, 1, \cdots$. Here the subtle point is that T_s depends on s.

If we could show that (23) holds for all s on $[0, T]$ with $T > 0$ independent of s, the argument of Step 3 in the proof of Theorem 6 implies that $\{u_m\}$ converges in $C^\infty([0, T] \times \mathbb{R}^n)$ to u.

We now fix $s_0 \geq n+3$ and let $T > 0$ be such that

$$\sup_{0 \leq t \leq T} \sum_{|\alpha| \leq s_0+1} \|\partial^\alpha u_m(t,\cdot)\|_{L^2} \leq C_0 < \infty, \quad m = 0, 1, \cdots .$$

and show that for all $s \geq s_0$ there holds

$$\sup_{0 \leq t \leq T} \sum_{|\alpha| \leq s+1} \|\partial^\alpha u_m(t, \cdot)\|_{L^2} \leq C_s < \infty, \quad \forall m. \tag{24}$$

We show (24) by induction on s. Assume that (24) is true for some $s \geq s_0$, we show it is also true with s replaced by $s+1$. By the induction hypothesis and Sobolev embedding,

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^n} \sum_{|\alpha| \leq s+1-[(n+2)/2]} |\partial^\alpha u_m(t,x)| \leq A_s < \infty, \quad \forall m.$$

Since $s \geq n+3$, we have $[(s+4)/2] \leq s+1-[(n+2)/2]$. So

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^n} \sum_{|\alpha| \leq (s+4)/2} |\partial^\alpha u_m| \leq C, \quad \forall m.$$

This is exactly (16) with s replaced by $s+1$. Same argument there can be used to derive that

$$\sup_{0\le t\le T}\sum_{|\alpha|\le s+2}\|\partial^\alpha u_m(t,\cdot)\|_{L^2}\le C_{s+1}<\infty,\quad \forall m.$$

We complete the induction argument and obtain a C^∞ solution.

Finally, we show that if $T_*<\infty$, then (22) holds. Otherwise, if

$$\sup_{[0,T_*)\times\mathbb{R}^n}\sum_{|\alpha|\le(n+6)/2}|\partial^\alpha u(t,x)|\le C<\infty,$$

then applying the above argument to u we have with $s_0=n+3$ that

$$\sup_{[0,T_*)\times\mathbb{R}^n}\sum_{|\alpha|\le s_0+1}\|\partial^\alpha u(t,\cdot)\|_{L^2}\le C_0<\infty$$

Repeating the above argument we obtain for all $s \geq s_0$ that

$$\sup_{[0,T_*)\times\mathbb{R}^n} \sum_{|\alpha|\leq s+1} \|\partial^\alpha u(t,\cdot)\|_{L^2} \leq C_s < \infty.$$

So u can be extend to $u \in C^\infty([0, T_*] \times \mathbb{R}^n)$.

Since $f, g \in C_0^\infty(\mathbb{R}^n)$, by the finite speed of propagation we can find a number R (possibly depending on T_*) such that $u(t,x) = 0$ for all $|x| \geq R$ and $0 \leq t < T_*$. Consequently

$$u(T_*, x) = \partial_t u(T_*, x) = 0 \quad \text{when } |x| \geq R.$$

Thus, $u(T_*, x)$ and $\partial_t u(T_*, x)$ are in $C_0^\infty(\mathbb{R}^n)$, and can be used as initial data at $t = T_*$ to extend u beyond T_* by the local existence result. This contradicts the definition of T_*. $\square$

4. Klainerman-Sobolev inequality

We turn to global existence of Cauchy problems for nonlinear wave equations

$$\Box u = F(u, \partial u).$$

This requires good decay estimates on $|u(t,x)|$ for large t. Recall the classical Sobolev inequality

$$|f(x)| \le C \sum_{|\alpha| \le (n+2)/2} \|\partial^\alpha f\|_{L^2}, \quad \forall x \in \mathbb{R}^n$$

which is very useful. However, it is not enough for the purpose. To derive good decay estimates for large t, one should replace ∂f by Xf with suitable vector fields X that exploits the structure of Minkowski space. This leads to Klainerman inequality of Sobolev type.

4.1. Invariant vector fields in Minkowski space

- We use $x = (x^0, x^1, \cdots, x^n)$ to denote the natural coordinates in $\mathbb{R}^{1+n}$, where $x^0 = t$ denotes time variable.
- We use Einstein summation convention. A Greek letter is used for index taking values $0, 1, \cdots, n$.
- A vector field X in $\mathbb{R}^{1+n}$ is a first order differential operator of the form

$$X = \sum_{i=0}^{n} X^\mu \frac{\partial}{\partial x^\mu} = X^\mu \partial_\mu,$$

  where X^μ are smooth functions. We will identify X with (X^μ).
- The collection of all vector fields on $\mathbb{R}^{1+n}$ is called the tangent space of $\mathbb{R}^{1+n}$ and is denoted by $T\mathbb{R}^{1+n}$.

- For any two vector fields $X = X^\mu \partial_\mu$ and $Y = Y^\mu \partial_\mu$, one can define the Lie bracket

$$[X, Y] := XY - YX.$$

Then

$$\begin{aligned}[X, Y] &= (X^\mu \partial_\mu)(Y^\nu \partial_\nu) - (Y^\nu \partial_\nu)(X^\mu \partial_\mu) \\ &= X^\mu Y^\nu \partial_\mu \partial_\nu + X^\mu (\partial_\mu Y^\nu)\partial_\nu - Y^\nu X^\mu \partial_\nu \partial_\mu - Y^\nu (\partial_\nu X^\mu)\partial_\mu \\ &= (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu)\partial_\nu = (X(Y^\mu) - Y(X^\mu))\partial_\mu.\end{aligned}$$

So $[X, Y]$ is also a vector field.

- A linear mapping $\eta : T\mathbb{R}^{1+n} \to \mathbb{R}$ is called a 1-form if

$$\eta(fX) = f\eta(X), \quad \forall f \in C^\infty(\mathbb{R}^{1+n}), X \in T\mathbb{R}^{1+n}.$$

For each $\mu = 0, 1, \cdots, n$, we can define the 1-form dx^μ by

$$dx^\mu(X) = X^\mu, \quad \forall X = X^\mu \partial_\mu \in T\mathbb{R}^{1+n}.$$

Then for any 1-form η we have

$$\eta(X) = X^\mu \eta(\partial_\mu) = \eta_\mu dx^\mu(X), \quad \text{where } \eta_\mu := \eta(\partial_\mu).$$

Thus any 1-form in $\mathbb{R}^{1+n}$ can be written as $\eta = \eta_\mu dx^\mu$ with smooth functions η_μ. We will identify η with (η_μ).

- A bilinear mapping $T : T\mathbb{R}^{1+n} \times T\mathbb{R}^{1+n} \to \mathbb{R}$ is called a (covariant) 2-tensor field if for any $f \in C^\infty(\mathbb{R}^{1+n})$ and $X, Y \in T\mathbb{R}^{1+n}$ there holds

$$T(fX, Y) = T(X, fY) = fT(X, Y).$$

It is called symmetric if $T(X, Y) = T(Y, X)$ for all vector fields X and Y.

- Let $(\mathbf{m}_{\mu\nu}) = \mathrm{diag}(-1, 1, \cdots, 1)$ be the $(1+n) \times (1+n)$ diagonal matrix. We define $\mathbf{m} : T\mathbb{R}^{1+n} \times T\mathbb{R}^{1+n} \to \mathbb{R}$ by

$$\mathbf{m}(X, Y) := \mathbf{m}_{\mu\nu} X^\mu Y^\nu$$

for all $X = X^\mu \partial_\mu$ and $Y = Y^\mu \partial_\mu$ in $T\mathbb{R}^{1+n}$. It is easy to check $\mathbf{m}$ is a symmetric 2-tensor field on $\mathbb{R}^{1+n}$. We call $\mathbf{m}$ the Minkowski metric on $\mathbb{R}^{1+n}$. Clearly

$$\mathbf{m}(X, X) = -\left(X^0\right)^2 + \left(X^1\right)^2 + \cdots + (X^n)^2.$$

- A vector field X in $(\mathbb{R}^{1+n}, \mathbf{m})$ is called space-like, time-like, or null if

$$\mathbf{m}(X, X) > 0, \quad \mathbf{m}(X, X) < 0, \quad \text{or} \quad \mathbf{m}(X, X) = 0$$

respectively.
- In $(\mathbb{R}^{1+n}, \mathbf{m})$ one can define the Laplace-Beltrami operator which turns out to be the D'Alembertian

$$\Box = \mathbf{m}^{\mu\nu} \partial_\mu \partial_\nu, \quad \text{where } (\mathbf{m}^{\mu\nu}) := (\mathbf{m}_{\mu\nu})^{-1}.$$

- The energy estimates related to $\Box u = 0$ can be derived by introducing the so called energy-momentum tensor. To see how to write down this tensor, we consider a vector field $X = X^\mu \partial_\mu$ with constant X^μ.

Then for any smooth function u we have

$$
\begin{aligned}
(Xu)\Box u &= X^\rho \partial_\rho u \, \mathbf{m}^{\mu\nu} \partial_\mu \partial_\nu u \\
&= \partial_\mu \left(X^\rho \mathbf{m}^{\mu\nu} \partial_\nu u \partial_\rho u \right) - X^\rho \mathbf{m}^{\mu\nu} \partial_\mu \partial_\rho u \partial_\nu u.
\end{aligned}
$$

Using the symmetry of $(\mathbf{m}^{\mu\nu})$ we can obtain

$$
X^\rho \mathbf{m}^{\mu\nu} \partial_\mu \partial_\rho u \partial_\nu u = \partial_\rho \left(\frac{1}{2} X^\rho \mathbf{m}^{\mu\nu} \partial_\mu u \partial_\nu u \right).
$$

Therefore $(Xu)\Box u = \partial_\nu \left(Q[u]_\mu^\nu X^\mu \right)$, where

$$
Q[u]_\mu^\nu = \mathbf{m}^{\nu\rho} \partial_\rho u \partial_\mu u - \frac{1}{2} \delta_\mu^\nu \left(\mathbf{m}^{\rho\sigma} \partial_\rho u \partial_\sigma u \right)
$$

in which δ_μ^ν denotes the Kronecker symbol, i.e. $\delta_\mu^\nu = 1$ when $\mu = \nu$ and 0 otherwise.

- This motivates to introduce the symmetric 2-tensor

$$Q[u]_{\mu\nu} := \mathbf{m}_{\mu\rho} Q[u]^{\rho}_{\nu} = \partial_\mu u \partial_\nu u - \frac{1}{2}\mathbf{m}_{\mu\nu}\left(\mathbf{m}^{\rho\sigma}\partial_\rho u \partial_\sigma u\right)$$

  which is called the energy-momentum tensor associated to $\Box u = 0$. Then for any vector fields X and Y we have

$$Q[u](X, Y) = (Xu)(Yu) - \frac{1}{2}\mathbf{m}(X, Y)\mathbf{m}(\partial u, \partial u)$$

- The divergence of the energy-momentum tensor can be calculated as

$$\begin{aligned}\mathbf{m}^{\mu\nu}\partial_\mu Q[u]_{\nu\rho} &= \mathbf{m}^{\mu\nu}\partial_\mu \left(\partial_\nu u \partial_\rho u - \frac{1}{2}\mathbf{m}_{\nu\rho}\left(\mathbf{m}^{\sigma\eta}\partial_\sigma u \partial_\eta u\right)\right)\\ &= \mathbf{m}^{\mu\nu}\partial_\mu \partial_\nu u \partial_\rho u = (\Box u)\partial_\rho u.\end{aligned}$$

- Let X be a vector field. Using $Q[u]$ we can introduce the 1-form

$$P_\mu := Q[u]_{\mu\nu} X^\nu.$$

Then we have

$$\begin{aligned}
\mathbf{m}^{\mu\nu}\partial_\mu P_\nu &= \mathbf{m}^{\mu\nu}\partial_\mu \left(\mathbf{Q}[u]_{\nu\rho} X^\rho\right) \\
&= \mathbf{m}^{\mu\nu}\partial_\mu Q[u]_{\nu\rho} X^\rho + \mathbf{m}^{\mu\nu} Q[u]_{\nu\rho}\partial_\mu X^\rho \\
&= \Box u \,\partial_\rho u \, X^\rho + \mathbf{m}^{\mu\nu} Q[u]_{\nu\rho}\mathbf{m}^{\rho\eta}\partial_\mu X_\eta \\
&= (\Box u) X u + \frac{1}{2} Q[u]^{\mu\rho}\left(\partial_\mu X_\rho + \partial_\rho X_\mu\right).
\end{aligned}$$

where $Q[u]^{\mu\nu} = \mathbf{m}^{\mu\rho}\mathbf{m}^{\sigma\nu} Q[u]_{\rho\sigma}$.

- For a vector field X, we define

$$^{(X)}\pi_{\mu\nu} := \partial_\mu X_\nu + \partial_\nu X_\mu$$

which is called the deformation tensor of X with respect to $\mathbf{m}$. Then we have

$$\partial_\mu(\mathbf{m}^{\mu\nu} P_\nu) = (\Box u) X u + \frac{1}{2} Q[u]^{\mu\nu}\, {}^{(X)}\pi_{\mu\nu}. \tag{25}$$

- Assume that u vanishes for large $|x|$ at each t. For any $t_0 < t_1$, we integrate $\partial_\mu(\mathbf{m}^{\mu\nu} P_\nu)$ over $[t_0, t_1] \times \mathbb{R}^n$ and note that ∂_t is the future unit normal to each slice $\{t\} \times \mathbb{R}^n$, we obtain

$$\iint_{[t_0,t_1]\times\mathbb{R}^n} \partial_\mu(\mathbf{m}^{\mu\nu} P_\nu) dx dt = \int_{\{t=t_1\}} Q[u](X, \partial_t) dx - \int_{\{t=t_0\}} Q[u](X, \partial_t) dx.$$

Therefore, we obtain the useful identity

$$\int_{\{t=t_1\}} Q[u](X,\partial_t)dx = \int_{\{t=t_0\}} Q[u](X,\partial_t)dx + \iint_{[t_0,t_1]\times\mathbb{R}^n} \Box u \cdot Xudxdt$$
$$+ \frac{1}{2}\iint_{[t_0,t_1]\times\mathbb{R}^n} Q[u]^{\mu\nu}\ {}^{(X)}\pi_{\mu\nu}dxdt. \qquad (26)$$

- By taking $X=\partial_t$ in (26), noting ${}^{(\partial_t)}\pi = 0$ and

$$Q[u](\partial_t,\partial_t) = \frac{1}{2}\left(|\partial_t u|^2 + |\nabla u|^2\right),$$

we obtain for $E(t) = \frac{1}{2}\int_{\{t\}\times\mathbb{R}^n}(|\partial_t u|^2 + |\nabla u|^2)dx$ the identity

$$E(t_1) = E(t_0) + \iint_{[t_0,t_1]\times\mathbb{R}^n} \Box u\, \partial_t udxdt.$$

Starting from here, we can easily derive the energy estimate.

- The identity (26) can be significantly simplified if ${}^{(X)}\pi = 0$. A vector field $X = X^\mu \partial_\mu$ in $(\mathbb{R}^{1+n}, \mathbf{m})$ is called a *Killing vector field* if ${}^{(X)}\pi = 0$, i.e.

$$\partial_\mu X_\nu + \partial_\nu X_\mu = 0 \qquad \text{in } \mathbb{R}^{1+n}.$$

We can determine all Killing vector fields in $(\mathbb{R}^{1+n}, \mathbf{m})$. Write $\pi_{\mu\nu} = {}^{(X)}\pi_{\mu\nu}$, Then

$$\begin{aligned}
\partial_\rho \pi_{\mu\nu} &= \partial_\rho \partial_\mu X_\nu + \partial_\rho \partial_\nu X_\mu, \\
\partial_\mu \pi_{\nu\rho} &= \partial_\mu \partial_\nu X_\rho + \partial_\mu \partial_\rho X_\nu, \\
\partial_\nu \pi_{\rho\mu} &= \partial_\nu \partial_\rho X_\mu + \partial_\nu \partial_\mu X_\rho.
\end{aligned}$$

Therefore

$$\partial_\mu \pi_{\nu\rho} + \partial_\nu \pi_{\rho\mu} - \partial_\rho \pi_{\mu\nu} = 2\partial_\mu \partial_\nu X_\rho.$$

If X is a Killing vector field, then $\partial_\mu \partial_\nu X_\rho = 0$ for all μ, ν, ρ. Thus each X_ρ is an affine function, i.e. there are constants $a_{\rho\nu}$ and b_ρ such that

$$X_\rho = a_{\rho\nu} x^\nu + b_\rho.$$

Using again $0 = \partial_\mu X_\nu + \partial_\nu X_\mu$, we obtain $a_{\mu\nu} = -a_{\nu\mu}$. Thus

$$\begin{aligned} X &= X^\mu \partial_\mu = \mathbf{m}^{\mu\nu} X_\nu \partial_\mu = \mathbf{m}^{\mu\nu} \left(a_{\nu\rho} x^\rho + b_\nu \right) \partial_\mu \\ &= \sum_{\nu=0}^{n} \left(\sum_{\rho<\nu} + \sum_{\rho>\nu} \right) a_{\nu\rho} x^\rho \mathbf{m}^{\mu\nu} \partial_\mu + \mathbf{m}^{\mu\nu} b_\nu \partial_\mu \\ &= \sum_{\nu=0}^{n} \sum_{\rho<\nu} a_{\nu\rho} x^\rho \mathbf{m}^{\mu\nu} \partial_\mu + \sum_{\rho=0}^{n} \sum_{\nu<\rho} a_{\nu\rho} x^\rho \mathbf{m}^{\mu\nu} \partial_\mu + \mathbf{m}^{\mu\nu} b_\nu \partial_\mu \\ &= \sum_{\nu=0}^{n} \sum_{\rho<\nu} \left(a_{\nu\rho} x^\rho \mathbf{m}^{\mu\nu} \partial_\mu + a_{\rho\nu} x^\nu \mathbf{m}^{\mu\rho} \partial_\mu \right) + \mathbf{m}^{\mu\nu} b_\nu \partial_\mu \end{aligned}$$

In view of $a_{\rho\nu} = -a_{\nu\rho}$, we therefore obtain

$$X = \sum_{\nu=0}^{n} \sum_{\rho<\nu} a_{\nu\rho} \left(x^\rho \mathbf{m}^{\mu\nu} \partial_\mu - x^\nu \mathbf{m}^{\mu\rho} \partial_\mu \right) + \mathbf{m}^{\mu\nu} b_\nu \partial_\mu$$

This shows that X is a linear combination of ∂_μ and $\Omega_{\mu\nu}$, where

$$\Omega_{\mu\nu} := \left(\mathbf{m}^{\rho\mu} x^\nu - \mathbf{m}^{\rho\nu} x^\mu \right) \partial_\rho.$$

Thus we obtain the following result on Killing vector fields.

Proposition 8

Any Killing vector field in $(\mathbb{R}^{1+n}, \mathbf{m})$ *can be written as a linear combination of the vector fields* ∂_μ, $0 \le \mu \le n$ *and*

$$\Omega_{\mu\nu} = \left(\mathbf{m}^{\rho\mu} x^\nu - \mathbf{m}^{\rho\nu} x^\mu \right) \partial_\rho, \quad 0 \le \mu < \nu \le n.$$

- Since $(\mathbf{m}^{\mu\nu}) = \text{diag}(-1, 1, \cdots, 1)$, the vector fields $\{\Omega_{\mu\nu}\}$ consist of the following elements

$$\Omega_{0i} = x^i \partial_t + t\partial_i, \quad 1 \le i \le n,$$
$$\Omega_{ij} = x^j \partial_i - x^i \partial_j, \quad 1 \le i < j \le n.$$

- When ${}^{(X)}\pi_{\mu\nu} = f\mathbf{m}_{\mu\nu}$ for some function f, the identity (26) can still be modified into a useful identity. To see this, we use (25) to obtain

$$\begin{aligned}\partial_\mu(\mathbf{m}^{\mu\nu} P_\nu) &= (\Box u) X u + \frac{1}{2} f \mathbf{m}^{\mu\nu} Q[u]_{\mu\nu} \\ &= (\Box u) X u + \frac{1-n}{4} f \mathbf{m}^{\mu\nu} \partial_\mu u \partial_\nu u.\end{aligned}$$

We can write

$$\begin{aligned} f\mathbf{m}^{\mu\nu}\partial_\mu u \partial_\nu u &= \mathbf{m}^{\mu\nu}\partial_\mu(fu\partial_\nu u) - \mathbf{m}^{\mu\nu} u \partial_\mu f \partial_\nu u - fu\Box u \\ &= \mathbf{m}^{\mu\nu}\partial_\mu(fu\partial_\nu u) - \mathbf{m}^{\mu\nu}\partial_\nu\left(\frac{1}{2}u^2\partial_\mu f\right) + \frac{1}{2}u^2\Box f - fu\Box u \\ &= \mathbf{m}^{\mu\nu}\partial_\mu\left(fu\partial_\nu u - \frac{1}{2}u^2\partial_\nu f\right) + \frac{1}{2}u^2\Box f - fu\Box u \end{aligned}$$

Therefore, by introducing

$$\widetilde{P}_\mu := P_\mu + \frac{n-1}{4} fu\partial_\mu u - \frac{n-1}{8}u^2\partial_\mu f$$

we obtain

$$\partial_\mu(\mathbf{m}^{\mu\nu}\widetilde{P}_\nu) = \Box u\left(Xu + \frac{n-1}{4}fu\right) - \frac{n-1}{8}u^2\Box f.$$

By integrating over $[t_0, t_1] \times \mathbb{R}^n$ as before, we obtain

Theorem 9

If X is a vector field in $(\mathbb{R}^{1+n}, \mathbf{m})$ with ${}^{(X)}\pi = f\mathbf{m}$, then for any smooth function u vanishing for large $|x|$ there holds

$$\int_{t=t_1} \widetilde{Q}(X, \partial_t)dx = \int_{t=t_0} \widetilde{Q}(X, \partial_t)dx - \frac{n-1}{8} \iint_{[t_0,t_1]\times\mathbb{R}^n} u^2 \Box f dxdt + \iint_{[t_0,t_1]\times\mathbb{R}^n} \left(Xu + \frac{n-1}{4} fu\right) \Box u dxdt,$$

where $t_0 \leq t_1$ and

$$\widetilde{Q}(X, \partial_t) := Q(X, \partial_t) + \frac{n-1}{4}\left(fu\partial_t u - \frac{1}{2}u^2\partial_t f\right).$$

- A vector field $X = X^\mu \partial_\mu$ in $(\mathbb{R}^{1+n}, \mathbf{m})$ is called conformal Killing if there is a function f such that ${}^{(X)}\pi = f\mathbf{m}$, i.e. $\partial_\mu X_\nu + \partial_\nu X_\mu = f\mathbf{m}_{\mu\nu}$.
- Any Killing vector field is conformal Killing. However, there are vector fields which are conformal Killing but not Killing.
  (i) Consider the vector field

$$L_0 = \sum_{\mu=0}^{n} x^\mu \partial_\mu = x^\mu \partial_\mu.$$

we have $(L_0)^\mu = x^\mu$ and so $(L_0)_\mu = \mathbf{m}_{\mu\nu} x^\nu$. Consequently

$$\begin{aligned} {}^{(L_0)}\pi_{\mu\nu} &= \partial_\mu (L_0)_\nu + \partial_\nu (L_0)_\mu = \partial_\mu(\mathbf{m}_{\nu\eta} x^\eta) + \partial_\nu(\mathbf{m}_{\mu\eta} x^\eta) \\ &= \mathbf{m}_{\nu\eta}\delta_\mu^\eta + \mathbf{m}_{\mu\eta}\delta_\nu^\eta = 2\mathbf{m}_{\mu\nu}. \end{aligned}$$

Therefore L_0 is conformal Killing and ${}^{(L_0)}\pi = 2\mathbf{m}$.

(ii) For each fixed $\mu = 0, 1, \cdots, n$ consider the vector field

$$K_\mu := 2\mathbf{m}_{\mu\nu} x^\nu x^\rho \partial_\rho - \mathbf{m}_{\eta\nu} x^\eta x^\nu \partial_\mu.$$

We have $(K_\mu)^\rho = 2\mathbf{m}_{\mu\nu} x^\nu x^\rho - \mathbf{m}_{\eta\nu} x^\eta x^\nu \delta_\mu^\rho$. Therefore

$$(K_\mu)_\rho = \mathbf{m}_{\rho\eta}(K_\mu)^\eta = 2\mathbf{m}_{\rho\eta}\mathbf{m}_{\mu\nu} x^\nu x^\eta - \mathbf{m}_{\rho\mu}\mathbf{m}_{\nu\eta} x^\nu x^\eta.$$

By direct calculation we obtain

$$^{(K_\mu)}\pi_{\rho\eta} = \partial_\rho (K_\mu)_\eta + \partial_\eta (K_\mu)_\rho = 4\mathbf{m}_{\mu\nu} x^\nu \mathbf{m}_{\rho\eta}.$$

Thus each K_μ is conformal Killing and $^{(K_\mu)}\pi = 4\mathbf{m}_{\mu\nu} x^\nu \mathbf{m}$. The vector field K_0 is due to Morawetz (1961).

All these conformal Killing vector fields can be found by looking at $X = X^\mu \partial_\mu$ with X^μ being quadratic.

- We can determine all conformal Killing vector fields in $(\mathbb{R}^{1+n}, \mathbf{m})$ when $n \geq 2$.

Proposition 10

Any conformal Killing vector field in $(\mathbb{R}^{1+n}, \mathbf{m})$ can be written as a linear combination of the vector fields

$$
\begin{aligned}
&\partial_\mu, \qquad\qquad 0 \leq \mu \leq n, \\
&\Omega_{\mu\nu} = (\mathbf{m}^{\rho\mu} x^\nu - \mathbf{m}^{\rho\nu} x^\mu)\partial_\rho, \quad 0 \leq \mu < \nu \leq n, \\
&L_0 = \sum_{\mu=0}^{n} x^\mu \partial_\mu, \\
&K_\mu = \mathbf{m}_{\mu\nu} x^\nu x^\rho \partial_\rho - \mathbf{m}_{\rho\nu} x^\rho x^\nu \partial_\mu, \quad \mu = 0, 1, \cdots, n.
\end{aligned}
$$

Proof. Let X be conformal Killing, i.e. there is a function f such that

$$ {}^{(X)}\pi_{\mu\nu} := \partial_\mu X_\nu + \partial_\nu X_\mu = f \mathbf{m}_{\mu\nu}. \tag{27} $$

We first show that f is an affine function. Recall that

$$ 2\partial_\mu \partial_\nu X_\rho = \partial_\mu \pi_{\nu\rho} + \partial_\nu \pi_{\rho\mu} - \partial_\rho \pi_{\mu\nu}. $$

Therefore

$$ 2\partial_\mu \partial_\nu X_\rho = \mathbf{m}_{\nu\rho} \partial_\mu f + \mathbf{m}_{\rho\mu} \partial_\nu f - \mathbf{m}_{\mu\nu} \partial_\rho f. $$

This gives

$$ 2\Box X_\rho = 2\mathbf{m}^{\mu\nu} \partial_\mu \partial_\nu X_\rho = (1-n)\partial_\rho f. \tag{28} $$

In view of (27), we have

$$(n+1)f = 2\mathbf{m}^{\mu\nu}\partial_\mu X_\nu$$

This together with (28) gives

$$(n+1)\Box f = 2\mathbf{m}^{\mu\nu}\partial_\mu \Box X_\nu = (1-n)\mathbf{m}^{\mu\nu}\partial_\mu\partial_\nu f = (1-n)\Box f.$$

So $\Box f = 0$. By using again (28) and (27) we have

$$\begin{aligned}(1-n)\partial_\mu\partial_\nu f &= \frac{1-n}{2}\left(\partial_\mu\partial_\nu f + \partial_\nu\partial_\mu f\right) = \partial_\mu \Box X_\nu + \partial_\nu \Box X_\mu \\ &= \Box\left(\partial_\mu X_\nu + \partial_\nu X_\mu\right) = \mathbf{m}_{\mu\nu}\Box f = 0.\end{aligned}$$

Since $n \geq 2$, we have $\partial_\mu\partial_\nu f = 0$. Thus f is an affine function, i.e. there are constants a_μ and b such that $f = a_\mu x^\mu + b$.

Consequently

$$^{(X)}\pi = (a_\mu x^\mu + b)\mathbf{m}.$$

Recall that $^{(L_0)}\pi = 2\mathbf{m}$ and $^{(K_\mu)}\pi = 4\mathbf{m}_{\mu\nu} x^\nu \mathbf{m}$. Therefore, by introducing the vector field

$$\widetilde{X} := X - \frac{1}{2} b L_0 - \frac{1}{4}\mathbf{m}^{\mu\nu} a_\nu K_\mu,$$

we obtain

$$^{(\widetilde{X})}\pi = {}^{(X)}\pi - \frac{1}{2} b\, {}^{(L_0)}\pi - \frac{1}{4}\mathbf{m}^{\mu\nu} a_\nu\, {}^{(K_\mu)}\pi = 0.$$

Thus $\widetilde{X}$ is Killing. We may apply Proposition 8 to conclude that $\widetilde{X}$ is a linear combination of ∂_μ and $\Omega_{\mu\nu}$. The proof is complete. $\square$

The formulation of Klainerman inequality involves only the constant vector fields

$$\partial_\mu, \quad 0 \le \mu \le n$$

and the homogeneous vector fields

$$\begin{aligned} L_0 &= x^\rho \partial_\rho, \\ \Omega_{\mu\nu} &= \left(\mathbf{m}^{\rho\mu} x^\nu - \mathbf{m}^{\rho\nu} x^\mu\right) \partial_\rho, \quad 0 \le \mu < \nu \le n. \end{aligned}$$

There are $m+1$ such vector fields, where $m = \frac{(n+1)(n+2)}{2}$. We will use Γ to denote any such vector field, i.e. $\Gamma = (\Gamma_0, \cdots, \Gamma_m)$ and for any multi-index $\alpha = (\alpha_0, \cdots, \alpha_m)$ we adopt the convention $\Gamma^\alpha = \Gamma_0^{\alpha_0} \cdots \Gamma_m^{\alpha_m}$.

Lemma 11 (Commutator relations)

Among the vector fields ∂_μ, $\Omega_{\mu\nu}$ and L_0 we have the commutator relations:

$$
\begin{aligned}
[\partial_\mu, \partial_\nu] &= 0, \\
[\partial_\mu, L_0] &= \partial_\mu, \\
[\partial_\rho, \Omega_{\mu\nu}] &= \left(\mathbf{m}^{\sigma\mu}\delta^\nu_\rho - \mathbf{m}^{\sigma\nu}\delta^\mu_\rho\right)\partial_\sigma, \\
[\Omega_{\mu\nu}, \Omega_{\rho\sigma}] &= \mathbf{m}^{\sigma\mu}\Omega_{\rho\nu} - \mathbf{m}^{\rho\mu}\Omega_{\sigma\nu} + \mathbf{m}^{\rho\nu}\Omega_{\sigma\mu} - \mathbf{m}^{\sigma\nu}\Omega_{\rho\mu}, \\
[\Omega_{\mu\nu}, L_0] &= 0.
\end{aligned}
$$

Therefore, the commutator between ∂_μ and any other vector field is a linear combination of ∂_ν, and the commutator of any two homogeneous vector fields is a linear combination of homogeneous vector fields.

Proof. These identity can be checked by direct calculation. As an example, we derive the formula for $[\Omega_{\mu\nu}, \Omega_{\rho\sigma}]$. Recall that

$$\Omega_{\mu\nu} = (\mathbf{m}^{\eta\mu} x^{\nu} - \mathbf{m}^{\eta\nu} x^{\mu})\, \partial_{\eta}.$$

Therefore

$$\begin{aligned}
[\Omega_{\mu\nu}, \Omega_{\rho\sigma}] &= \Omega_{\mu\nu} (\mathbf{m}^{\eta\rho} x^{\sigma} - \mathbf{m}^{\eta\sigma} x^{\rho})\, \partial_{\eta} - \Omega_{\rho\sigma} (\mathbf{m}^{\eta\mu} x^{\nu} - \mathbf{m}^{\eta\nu} x^{\mu})\, \partial_{\eta} \\
&= (\mathbf{m}^{\gamma\mu} x^{\nu} - \mathbf{m}^{\gamma\nu} x^{\mu}) \left(\mathbf{m}^{\eta\rho} \delta^{\sigma}_{\gamma} - \mathbf{m}^{\eta\sigma} \delta^{\rho}_{\gamma}\right) \partial_{\eta} \\
&\quad - (\mathbf{m}^{\gamma\rho} x^{\sigma} - \mathbf{m}^{\gamma\sigma} x^{\rho}) \left(\mathbf{m}^{\eta\mu} \delta^{\nu}_{\gamma} - \mathbf{m}^{\eta\nu} \delta^{\mu}_{\gamma}\right) \partial_{\eta} \\
&= \mathbf{m}^{\sigma\mu} (\mathbf{m}^{\eta\rho} x^{\nu} - \mathbf{m}^{\eta\nu} x^{\rho})\, \partial_{\eta} - \mathbf{m}^{\rho\mu} (\mathbf{m}^{\eta\sigma} x^{\nu} - \mathbf{m}^{\eta\nu} x^{\sigma})\, \partial_{\eta} \\
&\quad + \mathbf{m}^{\rho\nu} (\mathbf{m}^{\eta\sigma} x^{\mu} - \mathbf{m}^{\eta\mu} x^{\sigma})\, \partial_{\eta} - \mathbf{m}^{\sigma\nu} (\mathbf{m}^{\eta\rho} x^{\mu} - \mathbf{m}^{\eta\mu} x^{\rho})\, \partial_{\eta} \\
&= \mathbf{m}^{\sigma\mu} \Omega_{\rho\nu} - \mathbf{m}^{\rho\mu} \Omega_{\sigma\nu} + \mathbf{m}^{\rho\nu} \Omega_{\sigma\mu} - \mathbf{m}^{\sigma\nu} \Omega_{\rho\mu}.
\end{aligned}$$

This shows the result. □

Lemma 12

For any $0 \leq \mu, \nu \leq n$ *there hold*

$$[\Box, \partial_\mu] = 0, \quad [\Box, \Omega_{\mu\nu}] = 0, \quad [\Box, L_0] = 2\Box$$

Consequently, for any multiple-index α *there exist constants* $c_{\alpha\beta}$ *such that*

$$\Box\Gamma^\alpha = \sum_{|\beta|\leq|\alpha|} c_{\alpha\beta}\Gamma^\beta\Box. \tag{29}$$

Proof. Direct calculation. $\Box$

Let $\Lambda := \{(t, x) : t = |x|\}$ be the light cone. The following result says that the homogeneous vector fields span the tangent space of $\mathbb{R}^{1+n}_+$ at any point outside Λ.

Lemma 13

Let $r = |x|$. *In* $\mathbb{R}^{1+n}_+ \setminus \{0\}$ *there hold*

$$(t-r)\partial = \sum_{\Gamma} a_\Gamma(t,x)\Gamma,$$

where the sum involves only the homogeneous vector fields, the coefficients are smooth, homogeneous of degree zero, and satisfies, for any multi-index α, *the bounds*

$$|\partial^\alpha a_\Gamma(t,x)| \le C_\alpha (t+|x|)^{-|\alpha|}.$$

Proof. It suffices to show that

$$(t^2-r^2)\partial_j = t\Omega_{0j} + x^i\Omega_{ij} - x^j L_0, \quad j = 1,\cdots,n,$$
$$(t^2-r^2)\partial_t = tL_0 - x^i\Omega_{0i},$$

where we used Einstein summation convention, e.g. $x^i\Omega_{ij}$ means $\sum_{i=1}^n x^i\Omega_{ij}$. To see these identities, we use the definitions of L_0, Ω_{0i} and Ω_{ij} to obtain

$$\begin{aligned} x^i\Omega_{0i} &= r^2\partial_t + tx^i\partial_i = r^2\partial_t + t(L_0 - t\partial_t) = (r^2 - t^2)\partial_t + tL_0, \\ x^i\Omega_{ij} &= x^j x^i\partial_i - r^2\partial_j = x^j(L_0 - t\partial_t) - r^2\partial_j \\ &= x^j L_0 - t(\Omega_{0j} - t\partial_j) - t^2\partial_j = x^j L_0 - t\Omega_{0j} + (t^2 - r^2)\partial_j. \end{aligned}$$

The proof is thus complete. □

Let $\partial_r := r^{-1}\sum_{i=1}^n x^i\partial_i$. We have from the definition of L_0 and Ω_{0i} that

$$L_0 = t\partial_t + r\partial_r \quad \text{and} \quad x^i\Omega_{0i} = r^2\partial_t + rt\partial_r.$$

Therefore

$$rL_0 - \frac{t}{r}x^i\Omega_{0i} = (r^2 - t^2)\partial_r.$$

This gives the following result.

Lemma 14

Let $\partial_r := r^{-1}\sum_{i=1}^n x^i\partial_i$. Then in $\mathbb{R}^{1+n}_+ \setminus \{0\}$ there holds

$$(t - r)\partial_r = a_0(t,x)L_0 + \sum_{i=1}^n a_i(t,x)\Omega_{0i},$$

where a_i are smooth, homogenous of degree zero, and satisfies for any multi-index α the bounds of the form

$$|\partial^\alpha a_i(t,x)| \leq C_\alpha(t + |x|)^{-|\alpha|}$$

whenever $|x| > \delta t$ for some $\delta > 0$.

4.2. Klainerman-Sobolev inequality

It is now ready to state the Klainerman inequality of Sobolev type, which will be used in the proof of global existence.

Theorem 15 (Klainerman)

Let $u \in C^\infty([0,\infty) \times \mathbb{R}^n)$ vanish when $|x|$ is large. Then

$$(1+t+|x|)^{n-1}(1+|t-|x||)|u(t,x)|^2 \le C \sum_{|\alpha| \le \frac{n+2}{2}} \|\Gamma^\alpha u(t,\cdot)\|_{L^2}^2$$

for $t > 0$ and $x \in \mathbb{R}^n$, where C depends only on n.

In order to prove Theorem 15, we need some localized version of Sobolev inequality.

Lemma 16

Given $\delta > 0$, there is C_δ such that for all $f \in C^\infty(\mathbb{R}^n)$ there holds

$$|f(0)|^2 \leq C_\delta \sum_{|\alpha| \leq (n+2)/2} \int_{|y|<\delta} |\partial^\alpha f(y)|^2 dy.$$

We can take $C_\delta = C(1 + \delta^{-n-2})$ with C depending only on n.

Proof. Take $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\chi) \subset \{|y| \leq 1\}$ and $\chi(0) = 1$, and apply the Sobolev inequality to the function

$$\chi_\delta(y) f(y), \quad \text{where } \chi_\delta(y) := \chi(y/\delta),$$

to obtain

$$|f(0)|^2 \leq C \sum_{|\alpha| \leq (n+2)/2} \int_{\mathbb{R}^n} |\partial^\alpha (\chi_\delta(y) f(y))|^2 dy.$$

It is easy to see $|\partial^\alpha \chi_\delta(y)| \le C_\alpha \delta^{-|\alpha|}$ for any multi-index α. Since $\mathrm{supp}(\chi_\delta) \subset \{y : |y| \le \delta\}$, we have

$$|f(0)|^2 \le C(1+\delta^{-n-2}) \sum_{|\alpha| \le (n+2)/2} \int_{|y|\le\delta} |\partial^\alpha f(y)|^2 dy.$$

The proof is complete. □

Observe that, when restricted to $\mathbb{S}^{n-1}$, each Ω_{ij}, $1 \le i < j \le n$, is a tangent vector to $\mathbb{S}^{n-1}$ because it is orthogonal to the normal vector there. Moreover, one can show that $\{\Omega_{ij} : 1 \le i < j \le n\}$ spans the tangent space at any point of $\mathbb{S}^{n-1}$. Therefore, by using local coordinates on $\mathbb{S}^{n-1}$, we can obtain the following result.

Lemma 17

(a) *If $u \in C^\infty(\mathbb{S}^{n-1})$, then*

$$|u(\omega)|^2 \le C \sum_{|\alpha| \le \frac{n+1}{2}} \int_{\mathbb{S}^{n-1}} |(\partial_\eta^\alpha u)(\eta)|^2 d\sigma(\eta), \quad \forall \omega \in \mathbb{S}^{n-1},$$

where $\partial_\eta^\alpha = \Omega_{12}^{\alpha_1} \cdots \Omega_{n-1,n}^{\alpha_\mu}$ with $\mu = n(n-1)/2$.

(b) *Given $\delta > 0$, for all $v \in C^\infty(\mathbb{R} \times \mathbb{S}^{n-1})$*

$$|v(q,\omega)|^2 \le C_\delta \sum_{j+|\alpha| \le \frac{n+2}{2}} \int_{|p|<\delta} \int_{\eta \in \mathbb{S}^{n-1}} |\partial_q^j \partial_\eta^\alpha v(q+p,\eta)|^2 d\sigma(\eta) dp$$

where $\sup_{\delta \ge \delta_0} C_\delta < \infty$ for all $\delta_0 > 0$.

Proof of Theorem 15. If $t+|x|\leq 1$, the Sobolev inequality in Lemma 16 implies the inequality with Γ taking as ∂_μ, $0\leq\mu\leq n$. In what follows, we assume $t+|x|>1$.

Case 1. $|x|\leq\frac{t}{2}$ or $|x|\geq\frac{3t}{2}$. We first apply the Sobolev inequality in Lemma 16 to the function $y\to u(t,x+(t+|x|)y)$ to obtain

$$
\begin{aligned}
|u(t,x)|^2 &\leq C\sum_{|\alpha|\leq(n+2)/2}\int_{|y|<1/8}\left|\partial_y^\alpha\left(u(t,x+(t+|x|)y)\right)\right|^2dy\\
&= C\sum_{|\alpha|\leq(n+2)/2}(t+|x|)^{2|\alpha|-n}\int_{|y|<\frac{t+|x|}{8}}|(\partial_x^\alpha u)(t,x+y)|^2dy
\end{aligned}
$$

We will use Lemma 13 to control $(\partial_x^\alpha u)(t,x+y)$ in terms of $(\Gamma^\alpha u)(t,x+y)$ with Γ being homogeneous vector fields. This requires $(t,x+y)$ to be away from the light cone.

We claim that

$$|t - |x+y|| \geq \frac{3}{40}(t+|x|) \quad \text{if } |y| < \frac{1}{8}(t+|x|). \qquad (30)$$

Using this claim and Lemma 13 we have for $|y| < (t+|x|)/8$ that

$$|(\partial_x^\alpha u)(t, x+y)| \lesssim (t+|x|)^{-|\alpha|} \sum_{1 \leq |\beta| \leq |\alpha|} \left|(\Gamma^\beta u)(t, x+y)\right|.$$

Therefore

$$\begin{aligned}(t+|x|)^n |u(t,x)|^2 &\lesssim \sum_{|\alpha| \leq (n+2)/2} \int_{|y|<(t+|x|)/8} |(\Gamma^\alpha u)(t, x+y)|^2 \, dy \\ &\lesssim \sum_{|\alpha| \leq (n+2)/2} \|\Gamma^\alpha u(t,\cdot)\|_{L^2}^2.\end{aligned}$$

We show the claim (30). When $|x| \geq 3t/2$, we have

$$\frac{5}{2}t < t + |x| < \frac{5}{3}|x|.$$

So for $|y| < (t + |x|)/8$ there holds

$$|t - |x + y|| \geq |x| - |y| - t \geq \left(\frac{5}{5} - \frac{1}{8} - \frac{2}{5}\right)(t + |x|) = \frac{3}{40}(t + |x|).$$

On the other hand, when $|x| < t/2$ we have $3|x| < t + |x| < \frac{3}{2}t$.
So for $|y| < (t + |x|)/8$ there holds

$$|t - |x + y|| \geq t - |x| - |y| \geq \left(\frac{2}{3} - \frac{1}{3} - \frac{1}{8}\right)(t + |x|) = \frac{5}{24}(t + |x|).$$

Case 2. $t/2 \le |x| \le 3t/2$.

Since $t + |x| > 1$, we always have $t > 2/5$ and $|x| > 1/3$. We use polar coordinate $x = r\omega$ with $r > 0$ and $\omega \in \mathbb{S}^{n-1}$ and introduce

$$q = r - t$$

which is called the optical function. Then the light cone $\{t = |x|\}$ corresponds to $q = 0$. We define the function

$$v(t, q, \omega) := u(t, (t+q)\omega) \quad (= u(t, x))$$

It is easy to show that

$$\partial_q v = \partial_r u, \quad q\partial_q v = (r - t)\partial_r, \quad \partial_\omega^\alpha v = \partial_\omega^\alpha u. \qquad (31)$$

Since $t/2 \le |x| \le 3t/2 \Longleftrightarrow |q| < t/2$, it suffices to show that

$$t^{n-1}(1+|q_0|)|v(t,q_0,\omega)|^2 \lesssim \sum_{|\alpha|\le(n+2)/2} \|\Gamma^\alpha u(t,\cdot)\|_{L^2}^2 \qquad (32)$$

for all $|q_0| < t/2$ and $\omega \in \mathbb{S}^{n-1}$.

We first consider $|q_0| \le 1$. By the localized Sobolev inequality given in Lemma 17 on $\mathbb{R}\times\mathbb{S}^{n-1}$, we have

$$|v(t,q_0,\omega)|^2 \lesssim \int_{|q|<\frac{t}{4}} \int_{\mathbb{S}^{n-1}} \sum_{j+|\alpha|\le\frac{n+2}{2}} |\partial_q^j \partial_\eta^\alpha v(t,q_0+q,\eta)|^2 d\sigma(\eta)dq$$

$$\lesssim \int_{|q|<\frac{t}{4}} \int_{\mathbb{S}^{n-1}} \sum_{j+|\alpha|\le\frac{n+2}{2}} |(\partial_r^j \Gamma^\alpha u)(t,(t+q_0+q)\eta)|^2 d\sigma(\eta)dq,$$

where Γ denotes any vector fields Ω_{ij}, $1 \le i < j \le n$.

Let $r := t + q_0 + q$. Then $t/4 \leq r \leq 7t/4$. Thus

$$|v(t, q_0, \omega)|^2 \lesssim t^{1-n} \int_{\frac{t}{4}}^{\frac{7t}{4}} \int_{\mathbb{S}^{n-1}} \sum_{j+|\alpha| \leq \frac{n+2}{2}} |(\partial_r^j \Gamma^\alpha u)(t, r\eta)|^2 r^{n-1} d\sigma(\eta) dr$$

$$\lesssim t^{1-n} \int_{\frac{t}{4} \leq |y| \leq \frac{7t}{4}} \sum_{j+|\alpha| \leq \frac{n+2}{2}} |\partial_r^j \Gamma^\alpha u(t, y)|^2 dy.$$

Since $|y| > \frac{t}{4} \geq \frac{1}{10}$ and $\partial_r = \frac{y_k}{|y|}\partial_k$, we have $|\partial_r^j u| \lesssim \sum_{|\beta| \leq j} |\partial^\beta u|$. So

$$t^{n-1}|v(t, q_0, \omega)|^2 \lesssim \int_{\mathbb{R}^n} \sum_{|\alpha| \leq \frac{n+2}{2}} |\Gamma^\alpha u(t, y)|^2 dy.$$

We obtain (32) when $|q_0| \leq 1$.

Next consider the case $1 \leq |q_0| < t/2$. We choose $\chi \in C_0^\infty(-\frac{1}{2}, \frac{1}{2})$ with $\chi(0) = 1$, and define

$$V_{q_0}(t, q, \omega) := \chi((q - q_0)/q_0)v(t, q, \omega).$$

Then $V_{q_0}(t, q_0, \omega) = v(t, q_0, \omega)$ and

$$V_{q_0}(t, q, \omega) = 0 \quad \text{if } |q - q_0| > \frac{1}{2}|q_0|.$$

In order to get the factor $|q_0|$ in (32), we apply Sobolve inequality to the function $(q, \eta) \in \mathbb{R} \times \mathbb{S}^{n-1} \to V_{q_0}(t, q_0 + q_0 q, \eta)$ to obtain

$$\begin{aligned} &|v(t, q_0, \omega)|^2 = |V_{q_0}(t, q_0, \omega)|^2 \\ &\lesssim \int_{|q| \leq \frac{1}{2}} \int_{\mathbb{S}^{n-1}} \sum_{j + |\alpha| \leq \frac{n+2}{2}} \left| \partial_q^j \partial_\eta^\alpha \left(V_{q_0}(t, q_0 + q_0 q, \eta) \right) \right|^2 d\sigma(\eta) dq \end{aligned}$$

Consequently

$$
\begin{aligned}
&|v(t,q_0,\omega)|^2 \\
&\leq C \int_{|q|\leq \frac{1}{2}} \int_{\mathbb{S}^{n-1}} \sum_{j+|\alpha|\leq \frac{n+2}{2}} \left| \left((q_0\partial_q)^j \partial_\eta^\alpha V_{q_0} \right)(t, q_0 + q_0 q, \eta) \right|^2 d\sigma(\eta) dq \\
&= C|q_0|^{-1} \int_{|q-q_0|\leq \frac{|q_0|}{2}} \int_{\mathbb{S}^{n-1}} \sum_{j+|\alpha|\leq \frac{n+2}{2}} \left| (q_0\partial_q)^j \partial_\eta^\alpha V_{q_0}(t,q,\eta) \right|^2 d\sigma(\eta) dq.
\end{aligned}
$$

Since $\left|(q_0\partial_q)^j[\chi((q-q_0)/q_0)]\right| \lesssim 1$, we have for $|q| \sim |q_0|$ that

$$
\left| (q_0\partial_q)^j \partial_\eta^\alpha V_{q_0}(t,q,\eta) \right| \lesssim \sum_{k=1}^{j} \left| (q_0\partial_q)^k \partial_\eta^\alpha v(t,q,\eta) \right|
$$

Therefore

$$|q_0||v(t,q_0,\omega)|^2$$
$$\lesssim \int_{\frac{|q_0|}{2}\leq|q|\leq\frac{3|q_0|}{2}} \int_{\mathbb{S}^{n-1}} \sum_{j+|\alpha|\leq\frac{n+2}{2}} \left|(q_0\partial_q)^j\partial_\eta^\alpha v(t,q,\eta)\right|^2 d\sigma(\eta)dq.$$

For $|q|\sim|q_0|$, we have

$$\left|(q_0\partial_q)^j\partial_\eta^\alpha v\right| \lesssim \left|q^j\partial_q^j\partial_\eta^\alpha v\right| \lesssim \sum_{k=1}^{j}\left|(q\partial_q)^k\partial_\eta^\alpha v\right|.$$

Hence, by using $|q_0|<t/2$,

$$|q_0||v(t,q_0,\omega)|^2 \lesssim \int_{|q|\leq\frac{3t}{4}} \int_{\mathbb{S}^{n-1}} \sum_{j+|\alpha|\leq\frac{n+2}{2}} \left|(q\partial_q)^j\partial_\eta^\alpha v(t,q,\eta)\right|^2 d\sigma(\eta)dq.$$

Recall (31). We have with Γ denoting Ω_{ij}, $1 \le i < j \le n$, that

$$\begin{aligned}
&|q_0||v(t,q_0,\omega)|^2 \\
&\lesssim \int_{|q|\le \frac{3t}{4}} \int_{\mathbb{S}^{n-1}} \sum_{j+|\alpha|\le \frac{n+2}{2}} \left|(q\partial_r)^j \Gamma^\alpha u(t,(t+q)\eta)\right|^2 d\sigma(\eta)dq \\
&\lesssim \int_{r\ge \frac{t}{4}} \int_{\mathbb{S}^{n-1}} \sum_{j+|\alpha|\le \frac{n+2}{2}} \left|((r-t)\partial_r)^j \Gamma^\alpha u(t,r\eta)\right|^2 d\sigma(\eta)dr \\
&\lesssim t^{1-n} \int_{r\ge \frac{t}{4}} \int_{\mathbb{S}^{n-1}} \sum_{j+|\alpha|\le \frac{n+2}{2}} \left|((r-t)\partial_r)^j \Gamma^\alpha u(t,r\eta)\right|^2 r^{n-1} d\sigma(\eta)dr \\
&\lesssim t^{1-n} \int_{|y|\ge \frac{t}{4}} \sum_{j+|\alpha|\le \frac{n+2}{2}} \left|((r-t)\partial_r)^j \Gamma^\alpha u(t,y)\right|^2 dy. \qquad (33)
\end{aligned}$$

Since $|y| > t/4$ and $t > 2/5$, Lemma 14 gives

$$|((r - t)\partial_r)^j u(t, y)| \lesssim \sum_{|\alpha| \leq j} |\Gamma^\alpha u(t, y)|$$

where the sum only involves the homogeneous vector fields $\Gamma = L_0$ and $\Omega_{\mu\nu}$, $0 \leq \mu < \nu \leq n$. Combining this with (33) gives (32). □

5. Global Existence in higher dimensions

We consider in $\mathbb{R}^{1+n}$ the global existence of the Cauchy problem

$$\begin{aligned} &\Box u = F(\partial u) \\ &u|_{t=0} = \varepsilon f, \qquad \partial_t u|_{t=0} = \varepsilon g, \end{aligned} \tag{34}$$

where $n \geq 4$, $\varepsilon \geq 0$ is a number, and $F : \mathbb{R}^{1+n} \to \mathbb{R}$ is a given C^∞ function which vanishes to the second order at the origin:

$$F(0) = 0, \quad \mathbf{D}F(0) = 0. \tag{35}$$

The main result is as follows.

Theorem 18

Let $n \geq 4$ and let $f, g \in C_c^\infty(\mathbb{R}^n)$. If F is a C^∞ function satisfying (35), then there exists $\varepsilon_0 > 0$ such that (34) has a unique solution $u \in C^\infty([0,\infty) \times \mathbb{R}^n)$ for any $0 < \varepsilon \leq \varepsilon_0$.

Proof. Let

$$T_* := \{T > 0 : (34) \text{ has a solution } u \in C^\infty([0, T] \times \mathbb{R}^n)\}.$$

Then $T_* > 0$ by Theorem 7. We only need to show that $T_* = \infty$. Assume that $T_* < \infty$, then Theorem 7 implies

$$\sum_{|\alpha| \le (n+6)/2} |\partial^\alpha u(t, x)| \notin L^\infty([0, T_*) \times \mathbb{R}^n).$$

We will derive a contradiction by showing that there is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ there holds

$$\sup_{(t,x) \in [0, T_*) \times \mathbb{R}^n} \sum_{|\alpha| \le (n+6)/2} |\partial^\alpha u(t, x)| < \infty. \tag{36}$$

Step 1. We derive (36) by showing that there exist $A > 0$ and $\varepsilon_0 > 0$ such that

$$A(t) := \sum_{|\alpha| \leq n+4} \|\partial \Gamma^\alpha u(t,\cdot)\|_{L^2} \leq A\varepsilon, \quad 0 \leq t < T_* \qquad (37)$$

for $0 < \varepsilon \leq \varepsilon_0$, where the sum involves all invariant vector fields ∂_μ, L_0 and $\Omega_{\mu\nu}$.

In fact, by Klainerman inequality in Theorem 15 we have for any multi-index β that

$$|\partial \Gamma^\beta u(t,x)| \leq C(1+t)^{-\frac{n-1}{2}} \sum_{|\alpha| \leq (n+2)/2} \|\Gamma^\alpha \partial \Gamma^\beta u(t,\cdot)\|_{L^2}.$$

Since $[\Gamma, \partial]$ is either 0 or $\pm\partial$, see Lemma 11, using (37) we obtain for $|\beta| \leq (n+6)/2$ that

$$\begin{aligned} |\partial\Gamma^\beta u(t,x)| &\leq C(1+t)^{-\frac{n-1}{2}} \sum_{|\alpha|\leq n+4} \|\partial\Gamma^\alpha u(t,\cdot)\|_{L^2} \\ &= C(1+t)^{-\frac{n-1}{2}} A(t) \\ &\leq CA\varepsilon(1+t)^{-\frac{n-1}{2}}. \end{aligned} \tag{38}$$

To estimate $|\Gamma^\beta u(t,x)|$, we need further property of u. Since $f, g \in C_0^\infty(\mathbb{R}^n)$, we can choose $R > 0$ such that $f(x) = g(x) = 0$ for $|x| \geq R$. By the finite speed of propagation,

$$u(t,x) = 0, \qquad \text{if } 0 \leq t < T_* \text{ and } |x| \geq R + t.$$

To show (36), it suffices to show that

$$\sup_{0\le t<T_*,|x|\le R+t} |\Gamma^\alpha u(t,x)| < \infty, \quad \forall |\alpha| \le (n+6)/2.$$

For any (t,x) satisfying $0 \le t < T_*$ and $|x| < R+t$, write $x = |x|\omega$ with $|\omega| = 1$. Then

$$\Gamma^\alpha u(t,x) = \Gamma^\alpha u(t,|x|\omega) - \Gamma^\alpha u(t,(R+t)\omega)$$
$$= \int_0^1 \partial_j \Gamma^\alpha u(t,(s|x| + (1-s)(R+t))\omega) ds \; (|x| - R - t)\omega^j.$$

In view of (38), we obtain for all $|\alpha| \le (n+6)/2$ that

$$|\Gamma^\alpha u(t,x)| \le CA\varepsilon(1+t)^{-\frac{n-1}{2}}(R+t-|x|) \le CA\varepsilon(1+t)^{-\frac{n-3}{2}}.$$

Step 2. We prove (37).

- Since $u \in C^\infty([0, T_*) \times \mathbb{R}^n)$ and $u(t,x) = 0$ for $|x| \geq R + t$, we have $A(t) \in C([0, T_*))$.
- Using initial data we can find a large number A such that

$$A(0) \leq \frac{1}{4} A\varepsilon. \tag{39}$$

  By the continuity of $A(t)$, there is $0 < T < T_*$ such that $A(t) \leq A\varepsilon$ for $0 \leq t \leq T$.
- Let

$$T_0 = \sup\{T \in [0, T_*) : A(t) \leq A\varepsilon, \forall 0 \leq t \leq T\}.$$

  Then $T_0 > 0$. It suffices to show $T_0 = T_*$.

We show $T_0 = T_*$ be a contradiction argument. If $T_0 < T_*$, then $A(t) \leq A\varepsilon$ for $0 \leq t \leq T_0$. We will prove that for small $\varepsilon > 0$ there holds

$$A(t) \leq \frac{1}{2} A\varepsilon \quad \text{for } 0 \leq t \leq T_0.$$

By the continuity of $A(t)$, there is $\delta > 0$ such that

$$A(t) \leq A\varepsilon \quad \text{for } 0 \leq t \leq T_0 + \delta$$

which contradicts the definition of T_0.

Step 3. It remains only to prove that there is $\varepsilon_0 > 0$ such that

$$A(t) \leq A\varepsilon \text{ for } 0 \leq t \leq T_0 \Longrightarrow A(t) \leq \frac{1}{2} A\varepsilon \text{ for } 0 \leq t \leq T_0$$

for $0 < \varepsilon \leq \varepsilon_0$.

By Klainerman inequality and $A(t) \le A\varepsilon$ for $0 \le t \le T_0$, we have for $|\beta| \le (n+6)/2$ that

$$|\partial\Gamma^\beta u(t,x)| \le CA\varepsilon(1+t)^{-\frac{n-1}{2}}, \quad \forall (t,x) \in [0, T_0] \times \mathbb{R}^n. \quad (40)$$

To estimate $\|\partial\Gamma^\alpha u(t,\cdot)\|_{L^2}$ for $|\alpha| \le n+4$, we use the energy estimate to obtain

$$\|\partial\Gamma^\alpha u(t,\cdot)\|_{L^2} \le \|\partial\Gamma^\alpha u(0,\cdot)\|_{L^2} + C\int_0^t \|\Box\Gamma^\alpha u(\tau,\cdot)\|_{L^2} d\tau. \quad (41)$$

We write

$$\Box\Gamma^\alpha u = [\Box, \Gamma^\alpha]u + \Gamma^\alpha(F(\partial u))$$

and estimate $\|\Gamma^\alpha(F(\partial u))(\tau,\cdot)\|_{L^2}$ and $\|[\Box,\Gamma^\alpha]u(\tau,\cdot)\|_{L^2}$.

Since $F(0) = \mathbf{D}F(0) = 0$, we can write

$$F(\partial u) = \sum_{j,k=1}^{n} F_{jk}(\partial u)\partial_j u \partial_k u,$$

where F_{jk} are smooth functions. Using this it is easy to see that $\Gamma^\alpha(F(\partial u))$ is a linear combination of following terms

$$F_{\alpha_1 \cdots \alpha_m}(\partial u) \cdot \Gamma^{\alpha_1}\partial u \cdot \Gamma^{\alpha_2}\partial u \cdot \cdots \cdot \Gamma^{\alpha_m}\partial u$$

where $m \geq 2$, $F_{\alpha_1 \cdots \alpha_m}$ are smooth functions and $|\alpha_1| + \cdots + |\alpha_m|$ $= |\alpha|$ with at most one α_i satisfying $|\alpha_i| > |\alpha|/2$ and at least one α_i satisfying $|\alpha_i| \leq |\alpha|/2$.

- In view of (40), by taking ε_0 such that $A\varepsilon_0 \leq 1$, we obtain $\|F_{\alpha_1 \cdots \alpha_m}(\partial u)\|_{L^\infty} \leq C$ for $0 < \varepsilon \leq \varepsilon_0$ with a constant C independent of A and ε.

- Since $|\alpha|/2 \leq (n+4)/2$, using (40) all terms $\Gamma^{\alpha_j}\partial u$, except the one with largest $|\alpha_j|$, can be estimated as

$$\|\Gamma^{\alpha_j}\partial u(t,x)\|_{L^\infty([0,T_0]\times\mathbb{R}^n)} \leq CA\varepsilon(1+t)^{-\frac{n-1}{2}}$$

Therefore

$$\begin{aligned}\|\Gamma^{\alpha}(F(\partial u))(t,\cdot)\|_{L^2} &\leq CA\varepsilon(1+t)^{-\frac{n-1}{2}} \sum_{|\beta|\leq|\alpha|} \|\Gamma^{\beta}\partial u(t,\cdot)\|_{L^2} \\ &\leq CA\varepsilon(1+t)^{-\frac{n-1}{2}} A(t). \end{aligned} \tag{42}$$

Recall that $[\Box,\Gamma]$ is either 0 or $2\Box$. Thus

$$|[\Box,\Gamma^{\alpha}]u| \lesssim \sum_{|\beta|\leq|\alpha|} |\Gamma^{\beta}\Box u| \lesssim \sum_{|\beta|\leq|\alpha|} |\Gamma^{\beta}(F(\partial u))|.$$

Therefore

$$\|[\Box, \Gamma^\alpha]u(t,\cdot)\|_{L^2} \leq C \sum_{|\beta|\leq|\alpha|} \|\Gamma^\beta(F(\partial u))(t,\cdot)\|_{L^2} \leq CA\varepsilon(1+t)^{-\frac{n-1}{2}}A(t). \quad (43)$$

Consequently, it follows from (41), (42) and (43) that

$$\|\partial\Gamma^\alpha u(t,\cdot)\|_{L^2} \leq \|\partial\Gamma^\alpha u(0,\cdot)\|_{L^2} + CA\varepsilon \int_0^t \frac{A(\tau)}{(1+\tau)^{\frac{n-1}{2}}}d\tau$$

Summing over all α with $|\alpha| \leq n+4$ we obtain

$$A(t) \leq A(0) + CA\varepsilon \int_0^t \frac{A(\tau)}{(1+\tau)^{\frac{n-1}{2}}}d\tau \leq \frac{1}{4}A\varepsilon + CA\varepsilon \int_0^t \frac{A(\tau)}{(1+\tau)^{\frac{n-1}{2}}}d\tau.$$

By Gronwall inequality,

$$A(t) \le \frac{1}{4}A\varepsilon \exp\left(CA\varepsilon \int_0^t \frac{d\tau}{(1+\tau)^{(n-1)/2}} \right), \quad 0 \le t \le T_0.$$

For $n \ge 4$, $\int_0^\infty \frac{d\tau}{(1+\tau)^{(n-1)/2}} = \frac{2}{n+2} < \infty$. (This is the reason we need $n \ge 4$ for global existence). We now choose $\varepsilon_0 > 0$ so that

$$\exp\left(\frac{2}{n+2} CA\varepsilon_0 \right) \le 2.$$

Thus $A(t) \le A\varepsilon/2$ for $0 \le t \le T_0$ and $0 < \varepsilon \le \varepsilon_0$. The proof is complete. □

Remark. The proof does not provide global existence result when $n \leq 3$ in general. However, the argument can guarantee existence on some interval $[0, T_\varepsilon]$, where T_ε can be estimated as

$$T_\varepsilon \geq \begin{cases} e^{c/\varepsilon}, & n = 3, \\ c/\varepsilon^2, & n = 2, \\ c/\varepsilon, & n = 1. \end{cases} \tag{44}$$

In fact, let $A(t)$ be defined as before, the key point is to show that, for any $T < T_\varepsilon$,

$$A(t) \leq A\varepsilon \text{ for } 0 \leq t \leq T \Longrightarrow A(t) \leq \frac{1}{2}A\varepsilon \text{ for } 0 \leq t \leq T$$

The same argument as above gives

$$A(t) \leq \frac{1}{4} A\varepsilon \exp\left(CA\varepsilon \int_0^t \frac{d\tau}{(1+\tau)^{(n-1)/2}} \right), \quad 0 \leq t \leq T.$$

Thus we can improve the estimate to $A(t) \leq \frac{1}{2} A\varepsilon$ for $0 \leq t \leq T$ if T_ε satisfies

$$\exp\left(CA\varepsilon \int_0^{T_\varepsilon} \frac{d\tau}{(1+\tau)^{(n-1)/2}} \right) \leq 2$$

When $n \leq 3$, the maximal T_ε with this property satisfies (44).

Remark. For $n = 2$ or $n = 3$, the above argument can guarantee global existence when F satisfies stronger condition

$$F(0) = 0, \quad \mathbf{D}F(0) = 0, \quad \cdots, \quad \mathbf{D}^k F(0) = 0, \tag{45}$$

where $k = 5 - n$. Indeed, this condition guarantees that $F(\partial u)$ is a linear combination of the terms

$$F_{j_1 \cdots j_{k+1}}(\partial u)\partial_{j_1} u \cdots \partial_{j_{k+1}} u.$$

Thus $\Gamma^\alpha(F(\partial u))$ is a linear combination of the terms

$$f_{i_1 \cdots i_r}(\partial u)\Gamma^{\alpha_{i_1}}\partial u \cdot \ldots \cdot \Gamma^{\alpha_{i_r}}\partial u,$$

where $r \geq k + 1$, $|\alpha_1| + \cdots + |\alpha_r| = |\alpha|$ and $f_{i_1 \cdots i_r}$ are smooth functions; there are at most one α_i satisfying $\alpha_i > |\alpha|/2$ and at least k of α_i satisfying $|\alpha_i| \leq |\alpha|/2$.

We thus can obtain

$$\|\Gamma^\alpha(F(\partial u))(t,\cdot)\|_{L^2} \leq CA\varepsilon(1+t)^{-\frac{(n-1)k}{2}}A(t),$$
$$\|[\Box,\Gamma^\alpha]u(t,\cdot)\|_{L^2} \leq CA\varepsilon(1+t)^{-\frac{(n-1)k}{2}}A(t).$$

Therefore

$$A(t) \leq \frac{1}{4}A\varepsilon \exp\left(CA\varepsilon\int_0^t \frac{d\tau}{(1+\tau)^{((n-1)k)/2}}\right).$$

Since $k = 5-n$, $\int_0^\infty \frac{d\tau}{(1+\tau)^{((n-1)k)/2}}$ converges for $n=2$ or $n=3$.

The condition (45) is indeed too restrictive. In next lecture we relax it to include quadratic terms when $n=3$ using the so-call null condition introduced by Klainerman.

6. Null Conditions and Global Existence: $n = 3$

We have proved global existence of the nonlinear Cauchy problem

$$\Box u = F(\partial u)$$
$$u|_{t=0} = \varepsilon f, \quad \partial_t u|_{t=0} = \varepsilon g$$

in $\mathbb{R}^{1+n}$ with $n \geq 4$, for sufficiently small ε, where $F : \mathbb{R}^{1+n} \to \mathbb{R}$ is a given C^∞ function which vanishes to second order at origin, i.e.

$$F(0) = 0, \qquad \mathbf{D}F(0) = 0$$

This global existence result in general fails when $n \leq 3$ if there is no additional conditions on F.

Example. Fritz John (1981) proved that every smooth solution of

$$\Box u = (\partial_t u)^2$$

with nonzero initial data in $C_0^\infty(\mathbb{R}^3)$ must blow up in finite time.

For details please refer to

- F. John, *Blow-up for quasi-linear wave equations in three-space dimensions*, Comm. Pure Appl. Math., Vol. 34 (1981), 29–51.

Example. (Due to Klainerman and Nirenberg, 1980) On the other hand, for the equation

$$\Box u = (\partial_t u)^2 - \sum_{j=1}^{3} (\partial_j u)^2, \quad t \geq 0, x \in \mathbb{R}^3 \tag{46}$$

we have global smooth solutions for small data:

$$u|_{t=0} = \varepsilon f, \quad \partial_t u|_{t=0} = \varepsilon g, \tag{47}$$

where $f, g \in C_0^\infty(\mathbb{R}^3)$ and $\varepsilon > 0$ is sufficiently small.

To see this, let $v(t,x) = 1 - e^{-u(t,x)}$. Then v satisfies

$$\Box v = 0, \quad v|_{t=0} = 1 - e^{-\varepsilon f}, \quad \partial_t v|_{t=0} = \varepsilon g e^{-\varepsilon f} \tag{48}$$

which is a linear problem and thus has a global smooth solution. If $|v(t,x)| < 1$ for all (t,x), then

$$u(t,x) = -\log[1 - v(t,x)] \tag{49}$$

is a global solution of (46) and (47). To show $|v| < 1$, we can use the representation formula of solutions of $\Box v = 0$ to derive

$$\|v(t,\cdot)\|_{L^\infty} \le \frac{A}{1+t}, \qquad \forall t \ge 0,$$

where A is a constant depending only on L^∞ norm of $v|_{t=0}$ and $\partial v|_{t=0}$. In view of (48), it is easy to guarantee $A < 1$ if $\varepsilon > 0$ is sufficiently small. Hence $|v| < 1$.

6.1. Null forms in $\mathbb{R}^{1+n}$

- A covector $\xi = (\xi_\mu)$ in $(\mathbb{R}^{1+n}, \mathbf{m})$ is called null if

$$\mathbf{m}^{\mu\nu}\xi_\mu\xi_\nu = 0.$$

- A real bilinear form B in $(\mathbb{R}^{1+n}, \mathbf{m})$ is called a null form if

$$B(\xi, \xi) = 0 \quad \text{for all null covector } \xi.$$

Lemma 19

Any real null form in $(\mathbb{R}^{1+n}, \mathbf{m})$ is a linear combination of the following null forms

$$Q_0(\xi, \eta) = \mathbf{m}^{\mu\nu}\xi_\mu\eta_\nu, \tag{50}$$

$$Q_{\mu\nu}(\xi, \eta) = \xi_\mu\eta_\nu - \xi_\nu\eta_\mu, \quad 0 \le \mu < \nu \le n. \tag{51}$$

Proof. Let B be a null form. We can write $B(\xi,\eta) = B_s(\xi,\eta) + B_a(\xi,\eta)$, where

$$B_s(\xi,\eta) = \frac{1}{2}\left(B(\xi,\eta) + B(\eta,\xi)\right), \quad B_a(\xi,\eta) = \frac{1}{2}\left(B(\xi,\eta) - B(\eta,\xi)\right),$$

Then B_s is symmetric, B_a is skew-symmetric, and both are null forms. Therefore it suffices to show that

- If B symmetric, then it is a multiple of Q_0;
- If B skew-symmetric, then it is a linear combination of $Q_{\mu\nu}$.

When B is skew-symmetric, we can write $B(\xi,\eta) = b^{\mu\nu}\xi_\mu\eta_\nu$ with $b^{\mu\nu} = -b^{\nu\mu}$. Therefore

$$B(\xi,\eta) = \sum_{0\le\mu<\nu\le n} b^{\mu\nu}(\xi_\mu\eta_\nu - \xi_\nu\eta_\mu).$$

When B is a symmetric null-form, we can write $B(\xi, \eta) = b^{\mu\nu}\xi_\mu \eta_\nu$ with $b^{\mu\nu} = b^{\nu\mu}$. Then

$$b^{\mu\nu}\xi_\mu \xi_\nu = 0 \quad \text{for null covector } \xi = (\xi_\mu). \tag{52}$$

For any fixed $1 \le i \le n$, we take the null ξ with

$$\xi_0 = \pm 1, \quad \xi_i = 1 \quad \text{and} \quad \xi_j = 0 \text{ for } j \ne 0, i.$$

This gives $b^{00} \pm 2b^{0i} + b^{ii} = 0$. Consequently

$$b^{0i} = b^{i0} = 0 \quad \text{and} \quad b^{00} + b^{ii} = 0, \quad i = 1, \cdots, n. \tag{53}$$

Next for any fixed $1 \le i < j \le n$, we take null covector ξ with

$$\xi_0 = \sqrt{2}, \ \ \xi_i = \xi_j = 1 \quad \text{and} \quad \xi_k = 0 \text{ for } k \ne 0, i, j.$$

Using (52) and (53) we obtain $b^{ij} = 0$. Therefore

$$(b^{\mu\nu}) = b^{00}\text{diag}(1, -1, \cdots, -1).$$

Consequently $B(\xi, \eta) = -b^{00} Q_0(\xi, \eta)$ and the proof is complete. □

Recall that we have introduced in $(\mathbb{R}^{1+n}, \mathbf{m})$ the invariant vector fields ∂_μ, $\Omega_{\mu\nu}$ and L_0 which have been denoted as Γ. For each of them, we may replace ∂_μ by ξ_μ to obtain a function of (x, ξ), which is called the symbol of this vector field. Thus

- the symbol of ∂_μ is ξ_μ;
- the symbol of $\Omega_{\mu\nu}$ is $\Omega_{\mu\nu}(x, \xi) := (\mathbf{m}^{\rho\mu} x^\nu - \mathbf{m}^{\rho\nu} x^\mu)\xi_\rho$;
- the symbol of L_0 is $L_0(x, \xi) := x^\mu \xi_\mu$.

We then introduce the function

$$\Gamma(x,\xi) := \left(\sum_{0\le\mu<\nu\le n} \Omega_{\mu\nu}(x,\xi)^2 + L_0(x,\xi)^2 + \sum_{\mu=0}^{n} \xi_\mu^2 \right)^{1/2}$$

Let $|\xi|$ denote the Euclidean norm of ξ. Then we always have

$$|B(\xi,\eta)| \le C_0|\xi||\eta|, \quad \forall \xi,\eta \in \mathbb{R}^{1+n}, \tag{54}$$

where $C_0 := \max\{|B(\xi,\eta)| : |\xi| = |\eta| = 1\}$. The following result gives a decay estimate in $|x|$ when B is a null form.

Lemma 20

A bilinear form B in $(\mathbb{R}^{1+n}, \mathbf{m})$ is null if and only if

$$|B(\xi^1,\xi^2)| \le C(1+|x|)^{-1}|\Gamma(x,\xi^1)||\Gamma(x,\xi^2)|, \quad \forall x,\xi^i \in \mathbb{R}^{1+n}. \tag{55}$$

Proof. (55) $\Longrightarrow$ B is null. Let ξ be a nonzero null covector. We define $x = (x^\mu)$ by $x^\mu := \lambda \mathbf{m}^{\mu\nu}\xi_\nu$ with $\lambda > 0$. It is easy to see

$$L_0(x,\xi) = \lambda m^{\mu\nu}\xi_\mu\xi_\nu = 0 \quad \text{and} \quad \Omega_{\mu\nu}(x,\xi) = 0.$$

Thus $\Gamma(x,\xi) = |\xi|$. Consequently (55) gives

$$|B(\xi,\xi)| \le C(1+\lambda|\xi|)^{-1}|\xi|^2, \quad \forall \lambda > 0.$$

Taking $\lambda \to \infty$ gives $B(\xi,\xi) = 0$, i.e. B is null.

B is null $\Longrightarrow$ (55). It suffices to show that

$$\Gamma(x,\xi^1) = \Gamma(x,\xi^2) = 1 \Longrightarrow |B(\xi^1,\xi^2)| \le C(1+|x|)^{-1} \qquad (56)$$

Since $\Gamma(x,\xi^i) = 1$ implies $|\xi^i| \le 1$, we can obtain (56) from (54) if $|x| \le 1$. In what follows, we will assume $|x| > 1$.

Let $\xi^x := (\xi^x_\mu)$ with $\xi^x_\mu = \mathbf{m}_{\mu\nu} x^\nu$. We decompose

$$\xi^i = \eta^i + t_i \xi^x$$

with $\langle \eta^i, \xi^x \rangle = 0$ and $t_i \in \mathbb{R}$. Then

$$B(\xi^1, \xi^2) = B(\eta^1, \eta^2) + t_2 B(\eta^1, \xi^x) + t_1 B(\xi^x, \eta^2) + t_1 t_2 B(\xi^x, \xi^x).$$

In view of $|\xi^x| = |x|$, we have from (54) that

$$|B(\xi^1, \xi^2)| \le C_0 \left(|\eta^1||\eta^2| + |t_2||x||\eta^1| + |t_1||x||\eta^2| \right) + |t_1||t_2||B(\xi^x, \xi^x)|.$$

Since B is null, we have from Lemma 19 that

$$|B(\xi^x, \xi^x)| \le C_0 |Q_0(\xi^x, \xi^x)| = C_0 |\mathbf{m}^{\mu\nu} \xi^x_\mu \xi^x_\nu| = C_0 |\mathbf{m}_{\mu\nu} x^\mu x^\nu|.$$

Therefore

$$|B(\xi^1,\xi^2)| \le C_0\left(|\eta^1||\eta^2| + |t_2||x||\eta^1| + |t_1||x||\eta^2| + |t_1||t_2||\mathbf{m}_{\mu\nu}x^\mu x^\nu|\right).$$

We can complete the proof by showing that

$$|t_i| + |\eta^i| \lesssim |x|^{-1} \quad \text{and} \quad |t_i||\mathbf{m}_{\mu\nu}x^\mu x^\nu| \lesssim 1.$$

Observing that $\Gamma(x,\xi^i) = 1$ implies

$$|\xi^i| \le 1, \quad |L_0(x,\xi^i)| \le 1 \quad \text{and} \quad \sum_{0\le\mu<\nu\le n} \Omega_{\mu\nu}(x,\xi^i)^2 \le 1.$$

Using $\langle \eta^i, \xi^x\rangle = 0$ and $|\xi^i| \le 1$ we can derive that $t_i^2|\xi^x|^2 \le 1$. Thus $|t_i||x| = |t_i||\xi^x| \le 1$.

Since

$$L_0(x, \xi^i) = x^\mu \eta^i_\mu + t_i x^\mu \xi^x_\mu = x^\mu \eta^i_\mu + t_i \mathbf{m}_{\mu\nu} x^\mu x^\nu,$$

we have from $|L_0(x, \xi^i)| \le 1$ that

$$|t_i||\mathbf{m}_{\mu\nu} x^\mu x^\nu| \le 1 + |x||\eta^i|.$$

Thus $|t_i||\mathbf{m}_{\mu\nu} x^\mu x^\nu| \lesssim 1$ if we can show $|\eta^i| \lesssim |x|^{-1}$. It remains only to prove $|\eta^i| \lesssim |x|^{-1}$. Noticing that

$$\Omega_{\mu\nu}(x, \xi^x) = (\mathbf{m}^{\rho\mu} x^\nu - \mathbf{m}^{\rho\nu} x^\mu)\xi^x_\rho = (\mathbf{m}^{\rho\mu} x^\nu - \mathbf{m}^{\rho\nu} x^\mu)\mathbf{m}_{\rho\sigma} x^\sigma = 0.$$

This implies

$$\Omega_{\mu\nu}(x, \xi^i) = \Omega_{\mu\nu}(x, \eta^i) + t_i \Omega_{\mu\nu}(x, \xi^x) = \Omega_{\mu\nu}(x, \eta^i).$$

Therefore

$$\sum_{0\le\mu<\nu\le n} \Omega_{\mu\nu}(x,\eta^i)^2 = \sum_{0\le\mu<\nu\le n} \Omega_{\mu\nu}(x,\xi^i)^2 \le 1.$$

We will be able to obtain $|\eta^i| \le |x|^{-1}$ if we can show that

$$\sum_{0\le\mu<\nu\le n} \Omega_{\mu\nu}(x,\eta^i)^2 = |x|^2|\eta^i|^2. \tag{57}$$

To obtain (57), recall that $\xi_0^x = -x^0$ and $\xi_i^x = x^i$ for $1 \le i \le n$. Since $(\mathbf{m}^{\mu\nu}) = \mathrm{diag}(-1, 1, \cdots, 1)$, we obtain

$$\sum_{0\le\mu<\nu\le n} \Omega_{\mu\nu}(x,\eta^i)^2 = \sum_{0\le\mu<\nu\le n} (\xi_\mu^x \eta_\nu^i - \xi_\nu^x \eta_\mu^i)^2$$

By expanding the squares, we obtain

$$\begin{aligned}
&\sum_{0\le\mu<\nu\le n} \Omega_{\mu\nu}(x,\eta^i)^2 \\
&\qquad = \sum_{0\le\mu<\nu\le n} \left((\xi^x_\mu)^2(\eta^i_\nu)^2 + (\xi^x_\nu)^2(\eta^i_\mu)^2 - 2\xi^x_\mu\eta^i_\nu\xi^x_\nu\eta^i_\mu\right) \\
&\qquad = \sum_{0\le\mu\le n}\sum_{\nu\ne\mu}(\xi^x_\mu)^2(\eta^i_\nu)^2 - \sum_{0\le\mu\le n}\sum_{\nu\ne\mu}\xi^x_\mu\eta^i_\nu\xi^x_\nu\eta^i_\mu \\
&\qquad = |\xi^x|^2|\eta^i|^2 - \left(\sum_{\mu=0}^{n}\xi^x_\mu\eta^i_\mu\right)^2.
\end{aligned}$$

Since $\langle\xi^x,\eta^i\rangle = 0$, we obtain (57). □

6.2. Null condition and main result

We consider the Cauchy problem of a system of N equations

$$\begin{aligned} &\Box u^I = F^I(u, \partial u) \quad \text{in } \mathbb{R}^{1+3}_+, \quad I = 1, \cdots, N, \\ &u(0,\cdot) = \varepsilon f, \quad \partial_t u(0,\cdot) = \varepsilon g, \end{aligned} \tag{58}$$

where $\varepsilon > 0$, $f = (f^1, \cdots, f^N)$ and $g = (g^1, \cdots, g^N)$ are $C_0^\infty(\mathbb{R}^3)$, and $F = (F^1, \cdots, F^N)$ are C^∞. Of course, the unknown solution $u = (u^1, \cdots, u^N)$ is $\mathbb{R}^N$-valued. To obtain a global existence result, the so called null condition on the quadratic part of each F^I should be assumed.

- The quadratic part of a function F defined on $\mathbb{R}^M$ around $\mathbf{0}$ is

$$Q_F(z) := \sum_{|\alpha|=2} \frac{1}{\alpha!} \partial^\alpha F(0) z^\alpha, \quad \forall z \in \mathbb{R}^M.$$

Definition 21 (Klainerman, 1982)

$F := (F^1, \cdots, F^N)$ in (58) is said to satisfy the **null condition** if

(i) F vanishes to second order at the origin

$$F(0) = 0, \quad \mathbf{D}F(0) = 0.$$

(ii) The quadratic part of each F^I around $\mathbf{0}$ has the form

$$Q_{F^I}(\partial u) = \sum_{J,K=1}^{N} \sum_{\mu,\nu=0}^{3} a_{IJK}^{\mu\nu} \partial_\mu u^J \partial_\nu u^K,$$

where $a_{IJK}^{\mu\nu}$ are constants satisfying, for all $I, J, K = 1, \cdots, N$,

$$\sum_{\mu,\nu=0}^{3} a_{IJK}^{\mu\nu} \xi_\mu \xi_\nu = 0 \quad \text{for all null covector } \xi \in \mathbb{R}^{1+3}.$$

Klainerman (1986) and Christodoulou (1986) proved the following global existence result independently.

Theorem 22 (Klainerman, Christodoulou)

Assume that F in (58) satisfies the null condition. Then there exists $\varepsilon_0 = \varepsilon_0(f, g) > 0$ such that (58) has a global smooth solution provided $\varepsilon < \varepsilon_0$.

We first provide necessary ingredients toward proving Theorem 22.

The proof is carried out by the continuity method which is essentially based on suitable energy estimates and hence requires to handle $\Gamma^\alpha F(u, \partial u)$ for invariant vector fields ∂_μ, L_0 and $\Omega_{\mu\nu}$.

According to the null condition on F and Lemma 19, we have

Lemma 23

If F in (58) satisfies the null condition, then each component $F^I(u,\partial u)$ has the form

$$F^I(u,\partial u) = Q_{F^I}(\partial u) + R^I(u,\partial u),$$

where R^I is C^∞ and vanishes to third order at 0 and

$$Q_{F^I}(\partial u) = \sum_{J,K} a_{IJK} Q_0(\partial u^J, \partial u^K) + \sum_{J,K} \sum_{0\le\mu<\nu\le 3} b^{\mu\nu}_{IJK} Q_{\mu\nu}(\partial u^J, \partial u^K)$$

with constants a_{IJK} and $b^{\mu\nu}_{IJK}$.

The term $\Gamma^\alpha R^I$ is easy to handle. The term $\Gamma^\alpha Q_{F^I}(\partial u)$ needs some care; we need only consider $\Gamma^\alpha Q(\partial u^J, \partial u^K)$ for null forms Q.

Lemma 24

Let Q be one of the null forms in (19) and (23)

$$|Q(\partial v, \partial w)(t,x)| \le \frac{C}{1+t+|x|} \sum_{|\alpha|=1} |\Gamma^\alpha v(t,x)| \sum_{|\alpha|=1} |\Gamma^\alpha w(t,x)|$$

Proof. In view of Lemma 20, we have

$$|Q(\partial v, \partial w)| \le \frac{C}{1+t+|x|} |\Gamma(t,x,\partial v)\Gamma(t,x,\partial w)|.$$

Since $\Gamma(t,x,\partial v) = \sum_{|\alpha|=1} |\Gamma^\alpha v(t,x)|$, we obtain the result. □

Therefore, in order to estimate $\Gamma^\alpha Q(\partial v, \partial w)$ for a null form Q, it is useful to consider first the "commutator"

$$[\Gamma, Q](\partial v, \partial w) = \Gamma Q(\partial v, \partial w) - Q(\partial \Gamma v, \partial w) - Q(\partial v, \partial \Gamma w)$$

We have the following result.

Lemma 25

Let Q be any null form, let Q_0 and $Q_{\mu\nu}$ be the null forms given by (50) and (51). Then

$$[\partial_\mu, Q] = 0, \qquad [L_0, Q] = -2Q,$$
$$[\Omega_{\mu\nu}, Q_0] = 0,$$
$$[\Omega_{\mu\nu}, Q_{\rho\sigma}] = (\mathbf{m}^{\eta\mu}\delta_\sigma^\nu - \mathbf{m}^{\eta\nu}\delta_\sigma^\mu)Q_{\eta\rho} - (\mathbf{m}^{\eta\mu}\delta_\rho^\nu - \mathbf{m}^{\eta\nu}\delta_\rho^\mu)Q_{\eta\sigma}$$

Proof. All these identity can be derived by direct calculation. We derive $[\Omega_{\mu\nu}, Q_{\rho\sigma}]$ here. Let v and w be any two functions. Then

$$\begin{aligned}
[\Omega_{\mu\nu}, Q_{\rho\sigma}](\partial v, \partial w) &= \Omega_{\mu\nu}\left(\partial_\rho v \partial_\sigma w - \partial_\sigma w \partial_\rho v\right) \\
&\quad - \Big(\partial_\rho(\Omega_{\mu\nu} v)\partial_\sigma w - \partial_\sigma(\Omega_{\mu\nu} v)\partial_\rho w\Big) \\
&\quad - \Big(\partial_\rho v \partial_\sigma(\Omega_{\mu\nu} w) - \partial_\sigma v \partial_\rho(\Omega_{\mu\nu} w)\Big) \\
&= -[\partial_\rho, \Omega_{\mu\nu}]v \cdot \partial_\sigma w + [\partial_\sigma, \Omega_{\mu\nu}]v \cdot \partial_\rho w \\
&\quad - \partial_\rho v \cdot [\partial_\sigma, \Omega_{\mu\nu}]w + \partial_\sigma v \cdot [\partial_\rho, \Omega_{\mu\nu}]w.
\end{aligned}$$

Recall that

$$[\partial_\rho, \Omega_{\mu\nu}] = (\mathbf{m}^{\eta\mu}\delta_\rho^\nu - \mathbf{m}^{\eta\nu}\delta_\rho^\mu)\partial_\eta.$$

By substitution we obtain

$$[\Omega_{\mu\nu}, Q_{\rho\sigma}](\partial v, \partial w) = (\mathbf{m}^{\eta\mu}\delta_\sigma^\nu - \mathbf{m}^{\eta\nu}\delta_\sigma^\mu)Q_{\eta\rho}(\partial v, \partial w) - (\mathbf{m}^{\eta\mu}\delta_\rho^\nu - \mathbf{m}^{\eta\nu}\delta_\rho^\mu)Q_{\eta\sigma}(\partial v, \partial w).$$

The proof is complete. □

Proposition 26

For any null form Q, and any integer $M \geq 0$, we have

$$(1+|t|+|x|)\sum_{|\alpha|\leq M}|\Gamma^\alpha Q(\partial v, \partial w)| \leq C_M\Big(\sum_{1\leq|\alpha|\leq M+1}|\Gamma^\alpha v(t,x)|\Big)\Big(\sum_{1\leq|\alpha|\leq \frac{M}{2}+1}|\Gamma^\alpha w(t,x)|\Big) + C_M\Big(\sum_{1\leq|\alpha|\leq \frac{M}{2}+1}|\Gamma^\alpha v(t,x)|\Big)\Big(\sum_{1\leq|\alpha|\leq M+1}|\Gamma^\alpha w(t,x)|\Big).$$

Proof. By induction on M. For $M = 0$ it follows from Lemma 24. For a multi-index α with $|\alpha| = M \geq 1$, we can write $\Gamma^\alpha = \Gamma^\beta \Gamma$ with $|\beta| = M - 1$. In view of Lemma 25, we have

$$\Gamma^\alpha Q(\partial v, \partial w) = \Gamma^\beta \left([\Gamma, Q](\partial v, \partial w) + Q(\partial \Gamma v, \partial w) + Q(\partial v, \partial \Gamma w) \right).$$

Therefore

$$\begin{aligned} \sum_{|\alpha| \leq M} |\Gamma^\alpha Q(\partial v, \partial w)| \leq & \sum_{|\beta| \leq M-1} |\Gamma^\beta Q(\partial v, \partial w)| \\ & + \sum_{|\beta| \leq M-1} |\Gamma^\beta Q(\partial \Gamma v, \partial w)| \\ & + \sum_{|\beta| \leq M-1} |\Gamma^\beta Q(\partial v, \partial \Gamma w)|. \end{aligned}$$

By the induction hypothesis, we complete the proof. □

In order to apply Proposition 26, we need to know how to estimate

$$\sum_{|\alpha|\le M+1} \|\Gamma^\alpha u(t,\cdot)\|_{L^2}.$$

This will be achieved by considering a suitable conformal energy.

We have shown in Theorem 9 that if X is a conformal Killing vector field in $(\mathbb{R}^{1+n}, \mathbf{m})$ with ${}^{(X)}\pi = f\mathbf{m}$, then for any smooth function u vanishing for large $|x|$ there holds

$$\begin{aligned}\int_{t=t_1} \widetilde{Q}(X,\partial_t)dx = \int_{t=t_0} \widetilde{Q}(X,\partial_t)dx - \frac{n-1}{8} \iint_{[t_0,t_1]\times\mathbb{R}^n} u^2 \Box f dxdt \\ + \iint_{[t_0,t_1]\times\mathbb{R}^n} \left(Xu + \frac{n-1}{4} fu\right) \Box u dxdt, \qquad (59)\end{aligned}$$

where

$$\tilde{Q}(X, \partial_t) = Q(X, \partial_t) + \frac{n-1}{4}\left(fu\partial_t u - \frac{1}{2}u^2\partial_t f\right),$$
$$Q(X, \partial_t) = (Xu)\partial_t u - \frac{1}{2}\mathbf{m}(X, \partial_t)\mathbf{m}(\partial u, \partial u).$$

We have also determined all conformal Killing vector fields in $(\mathbb{R}^{1+n}, \mathbf{m})$. In particular, ∂_t is Killing and the Morawetz vector field

$$K_0 = (t^2 + |x|^2)\partial_t + 2tx^i\partial_i$$

is conformal Killing with ${}^{(K_0)}\pi = 4t\mathbf{m}$. Take $X = K_0 + \partial_t$. Then

$${}^{(X)}\pi = f\mathbf{m} \quad \text{with} \quad f = 4t.$$

Therefore

$$\begin{aligned} Q(X, \partial_t) &= \left[(1 + t^2 + |x|^2)\partial_t u + 2tx^i \partial_i u\right] \partial_t u \\ &\quad + \frac{1}{2}(1 + t^2 + |x|^2)\mathbf{m}(\partial u, \partial u) \\ &= \frac{1}{2}(1 + t^2 + |x|^2)|\partial u|^2 + 2tx^i \partial_i u \partial_t u. \end{aligned}$$

Consequently

$$\begin{aligned} \widetilde{Q}(X, \partial_t) &= \frac{1}{2}(1 + t^2 + |x|^2)|\partial u|^2 + 2tx^i \partial_i u \partial_t u + 2tu\partial_t u - u^2 \\ &= \frac{1}{2}\Big(|\partial u|^2 + |L_0 u|^2 + \sum_{0 \le \mu < \nu \le 3} |\Omega_{\mu\nu} u|^2\Big) + 2tu\partial_t u - u^2, \end{aligned}$$

where the second equality follows from some calculation.

We introduce the conformal energy

$$E_0(t) := \int_{\{t\}\times\mathbb{R}^3} \widetilde{Q}(X,\partial_t)dx,$$

According to the formula for $\widetilde{Q}(X,\partial_t)$ we have

$$E_0(t) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\partial u|^2 + |L_0 u|^2 + \sum_{0\le\mu<\nu\le 3}|\Omega_{\mu\nu}u|^2\right)dx \\ + \int_{\mathbb{R}^3}\left(2tu\partial_t u - u^2\right)dx. \tag{60}$$

We will show that $E(t)$ is nonnegative and is comparable with $\sum_{|\alpha|\le 1}\|\Gamma^\alpha u(t,\cdot)\|_{L^2}^2$, where the sum involves all vector fields ∂_μ, $\Omega_{\mu\nu}$ and L_0.

Lemma 27

$E(t) \geq 0$ and for $t \geq 0$ there holds

$$E(t)^{1/2} \leq E(0)^{1/2} + \int_0^t \|(1+\tau+|x|)\Box u(\tau,\cdot)\|_{L^2} d\tau$$

Proof. Observing that

$$\begin{aligned} 2tu\partial_t u &= 2u(L_0 u - x^i \partial_i u) = 2uL_0 u - x^i \partial_i (u^2) \\ &= 2uL_0 u + 3u^2 - \partial_i (x^i u^2). \end{aligned}$$

Therefore, by the divergence theorem, we have

$$\int_{\mathbb{R}^3} 2tu\partial_t u dx = \int_{\mathbb{R}^3} \left(2uL_0 u + 3u^2\right) dx. \tag{61}$$

Consequently

$$E_0(t) = \frac{1}{2}\int_{\mathbb{R}^3}\Big(|\partial u|^2 + |L_0 u|^2 + \sum_{0\le\mu<\nu\le 3}|\Omega_{\mu\nu}u|^2 + 4uL_0 u + 4u^2\Big)dx$$
$$= \frac{1}{2}\int_{\mathbb{R}^3}\Big(|\partial u|^2 + |L_0 u + 2u|^2 + \sum_{0\le\mu<\nu\le 3}|\Omega_{\mu\nu}u|^2\Big)dx, \quad (62)$$

which implies $E(t) \ge 0$.

To derive the estimate on $E(t)$, we use (59) to obtain

$$E_0(t) = E(0) + \int_0^t\int_{\mathbb{R}^3}(Xu + 2\tau u)\Box u dx d\tau,$$

Thus

$$\frac{d}{dt}E_0(t) = \int_{\mathbb{R}^3}(Xu + 2tu)\Box u dx.$$

Therefore

$$\frac{d}{dt}E_0(t) = \|(1+t+|x|)^{-1}(Xu+2tu)\|_{L^2}\|(1+t+|x|)\Box u(t,\cdot)\|_{L^2}.$$

In view of the definition of X, we have

$$\begin{aligned} Xu + 2tu &= (1+t^2+|x|^2)\partial_t u + 2tx^i\partial_i u + 2tu \\ &= \partial_t u + t(L_0 u + 2u) + x^i\Omega_{0i}. \end{aligned}$$

By Cauchy-Schwartz inequality it follows that

$$|Xu+2tu|^2 \le (1+t^2+|x|^2)\Big(|\partial_t u|^2 + |L_0 u + 2u|^2 + \sum_{i=1}^3 |\Omega_{0i}|^2\Big)$$

Hence

$$\|(1+t+|x|)^{-1}(Xu+2tu)\|_{L^2}^2 \le 2E_0(t).$$

Consequently

$$\frac{d}{dt}E_0(t) \leq \sqrt{2E_0(t)}\|(1+t+|x|)\Box u(t,\cdot)\|_{L^2}.$$

This implies that

$$\frac{d}{dt}E(t)^{1/2} \leq \|(1+t+|x|)\Box u(t,\cdot)\|_{L^2}$$

which gives the estimate by integration. □

Lemma 28

There is a constant $C \geq 1$ such that

$$C^{-1}\sum_{|\alpha|\leq 1}\|\Gamma^\alpha u(t,\cdot)\|_{L^2}^2 \leq E_0(t) \leq C\sum_{|\alpha|\leq 1}\|\Gamma^\alpha u(t,\cdot)\|_{L^2}^2,$$

where the sum involves all vector fields ∂_μ, L_0 and $\Omega_{\mu\nu}$.

Proof. In view of (62), the inequality on the right is obvious. Now we prove the inequality on left.

We will make use of (60) for $E(t)$. To deal with $\int 2tu\partial_t u dx$, we use Ω_{0i} to rewrite ∂_t. We have

$$x^i\Omega_{0i} = r^2\partial_t + tx^i\partial_i.$$

Thus, by introducing $\Omega_r := r^{-1}x^i\Omega_{0i}$, we have

$$\partial_t = r^{-1}\Omega_r - r^{-2}tx^i\partial_i.$$

Therefore

$$\int 2tu\partial_t u dx = \int 2r^{-1}tu\Omega_r u dx - t^2\int r^{-2}x^i\partial_i(u^2)dx.$$

Integration by parts gives

$$\int 2tu\partial_t u dx = \int \left(2r^{-1}tu\Omega_r u + r^{-2}t^2u^2\right) dx.$$

On the other hand, we obtained in (61) that

$$\int 2tu\partial_t u dx = \int \left(2uL_0 u + 3u^2\right) dx.$$

Therefore

$$\begin{aligned}
&\int (2tu\partial_t u - u^2) dx \\
&= \frac{3}{4}\int \left(2uL_0 u + 3u^2\right) dx + \frac{1}{4}\int \left(2r^{-1}tu\Omega_r u + r^{-2}t^2u^2\right) dx - \int u^2 dx \\
&= \int \left(\frac{3}{2}uL_0 u + \frac{5}{4}u^2 + \frac{1}{2}r^{-1}tu\Omega_r u + \frac{1}{4}r^{-2}t^2u^2\right) dx.
\end{aligned}$$

In view of (60) we obtain $E_0(t) = \frac{1}{2}\left(I_1 + I_2 + I_3\right)$, where

$$I_1 = \int \Big(|\partial u|^2 + \sum_{0\le\mu<\nu\le 3} |\Omega_{\mu\nu} u|^2 - |\Omega_r u|^2 \Big) dx,$$

$$I_2 = \int \left(|\Omega_r u|^2 + r^{-1} t u \Omega_r u + \frac{1}{2} r^{-2} t^2 u^2 \right) dx,$$

$$I_3 = \int \left(|L_0 u|^2 + 3 u L_0 u + \frac{5}{2} u^2 \right) dx.$$

By the definition of Ω_r and Cauchy-Schwartz inequality we have

$$|\Omega_r u|^2 = r^{-2} \left| \sum_{i=1}^{3} \Omega_{0i} u \right|^2 \le \sum_{i=1}^{3} |\Omega_{0i} u|^2$$

This implies $I_1 \ge 0$.

We also have $I_2 \geq 0$ because

$$|\Omega_r u|^2 + r^{-1} t u \Omega_r u + \frac{1}{2} r^{-2} t^2 u^2 = \frac{1}{2}\left(|\Omega_r u|^2 + \left|\Omega_r u + r^{-1} t u\right|^2\right) \geq 0.$$

Therefore $I_3 \leq 2E_0(t)$. It remains only to show that

$$\int \left(u^2 + |L_0 u|^2\right) dx \lesssim I_3.$$

To see this, we write

$$\begin{aligned}
&|L_0 u|^2 + 3uL_0 u + \frac{5}{2}u^2 \\
&= |aL_0 u + bu|^2 + (1-a^2)|L_0 u|^2 + \left(\frac{5}{2} - b^2\right) u^2 + (3 - 2ab) u L_0 u.
\end{aligned}$$

It is always possible to choose $a > 0$ and $b > 0$ such that

$$3 - 2ab = 0, \quad 1 - a^2 > 0, \quad \frac{5}{2} - b^2 > 0.$$

Thus

$$|L_0 u|^2 + 3uL_0 u + \frac{5}{2}u^2 \gtrsim |L_0 u|^2 + u^2.$$

This shows that $I_3 \gtrsim \int (u^2 + |L_0 u|^2) dx$. We therefore complete the proof. $\square$

We are now ready to derive, for any integer $M \geq 0$, the estimate on

$$\sum_{|\alpha| \leq M+1} \|\Gamma^\alpha u(t, \cdot)\|_{L^2}.$$

Proposition 29 (Energy estimates)

For any integer $M \geq 0$, there is a constant C such that

$$\sum_{|\alpha| \leq M+1} \|\Gamma^\alpha u(t, \cdot)\|_{L^2} \leq C \sum_{|\alpha| \leq M+1} \|\Gamma^\alpha u(0, \cdot)\|_{L^2}$$
$$+ C \sum_{|\alpha| \leq M} \int_0^t \|(1 + \tau + |\cdot|)\Gamma^\alpha \Box u(\tau, \cdot)\|_{L^2} d\tau$$

for all $t > 0$ and all $u \in C^\infty([0, \infty) \times \mathbb{R}^3)$ vanishing for large $|x|$.

Proof. The estimate for $M = 0$ follows from Lemma 27 and Lemma 28 immediately.

For the general case, let β be a multi-index and apply the estimate for $M = 0$ to $\Gamma^\beta u$ to obtain

$$\sum_{|\alpha|\le 1} \|\Gamma^\alpha \Gamma^\beta u(t,\cdot)\|_{L^2} \lesssim \sum_{|\alpha|\le 1} \|\Gamma^\alpha \Gamma^\beta u(0,\cdot)\|_{L^2} + \int_0^t \|(1+\tau+|\cdot|)\Box \Gamma^\beta u(\tau,\cdot)\|_{L^2} d\tau.$$

Since $[\Box, \Gamma]$ is either 0 or $2\Box$, we have

$$\|(1+\tau+|\cdot|)\Box \Gamma^\beta u(\tau,\cdot)\|_{L^2} \lesssim \sum_{|\gamma|\le|\beta|} \|(1+\tau+|\cdot|)\Gamma^\gamma \Box u(\tau,\cdot)\|_{L^2}$$

Therefore

$$\sum_{|\alpha|\le 1} \|\Gamma^\alpha \Gamma^\beta u(t,\cdot)\|_{L^2} \lesssim \sum_{|\alpha|\le 1} \|\Gamma^\alpha \Gamma^\beta u(0,\cdot)\|_{L^2} + \sum_{|\gamma|\le|\beta|} \int_0^t \|(1+\tau+|\cdot|)\Gamma^\gamma \Box u(\tau,\cdot)\|_{L^2} d\tau.$$

Summing over all β with $|\beta| \le M$ gives the desired estimate. $\Box$

6.3. Proof of Theorem 22: global existence

Let

$$T_* := \sup\{T > 0 : (58) \text{ has a solution } u \in C^\infty([0,T]\times\mathbb{R}^n\}.$$

By local existence theorem, $T_* > 0$, and, if $T_* < \infty$, then

$$\sum_{|\alpha|\le 4} |\partial^\alpha u| \notin L^\infty([0, T_*) \times \mathbb{R}^n).$$

On the other hand, we will show that there exist a large $A > 0$ and a small $\varepsilon_0 > 0$ so that

$$\sum_{|\alpha|\le 4} |\Gamma^\alpha u(t,x)| \le \frac{A\varepsilon}{1+t+|x|}, \quad \forall (t,x) \in [0, T_*) \times \mathbb{R}^n \qquad (63)$$

for $0 < \varepsilon \le \varepsilon_0$. This is a contradiction and hence $T_* = \infty$.

We will use the continuity method to obtain (63).

Since $f, g \in C_0^\infty(\mathbb{R}^n)$ and $F(0,0) = 0$, we can find a large $A > 0$ such that

$$\sum_{|\alpha|\leq 4} |\Gamma^\alpha u(0,x)| \leq \frac{1}{8} A\varepsilon, \quad \forall x \in \mathbb{R}^n.$$

We can find $R > 0$ such that $f(x) = g(x) = 0$ for $|x| \geq R$. By finite speed of propagation,

$$u(t,x) = 0 \quad \text{for } |x| \geq R + t.$$

Thus by continuity, there exists $T > 0$ such that

$$\sum_{|\alpha|\leq 4} |\Gamma^\alpha u(t,x)| \leq \frac{A\varepsilon}{1 + t + |x|}, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n. \qquad (64)$$

It remains only to show that there exists $\varepsilon_0 > 0$ such that if (64) holds for some $0 < T < T_*$ and $0 < \varepsilon \leq \varepsilon_0$, then there must hold

$$\sum_{|\alpha|\leq 4} |\Gamma^\alpha u(t,x)| \leq \frac{A\varepsilon}{2(1+t+|x|)}, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n. \quad (65)$$

We will show this by two steps.

Step 1. Show that there exists constants C_0 and C_1 such that

$$A(t) \leq C_0 (1+t)^{C_1 A\varepsilon} A(0), \quad 0 \leq t \leq T, \quad (66)$$

where

$$A(t) := \sum_{|\alpha|\leq 7} \|\Gamma^\alpha u(t,\cdot)\|_{L^2}.$$

To see this, we use Proposition 29 to obtain

$$A(t) \le CA(0) + C\int_0^t \sum_{|\alpha|\le 6} \|(1+\tau+|\cdot|)\Gamma^\alpha \Box u(\tau,\cdot)\|_{L^2} d\tau. \quad (67)$$

We need to estimate

$$\|(1+\tau+|\cdot|)\Gamma^\alpha \Box u(\tau,\cdot)\|_{L^2} = \|(1+\tau+|\cdot|)\Gamma^\alpha F(u,\partial u)(\tau,\cdot)\|_{L^2}.$$

Since F satisfies the null condition, we have

$$F(u,\partial u) = Q_F(\partial u) + R(u,\partial u), \quad (68)$$

where $Q_F(\partial u)$ is the quadratic part, and $R(u,\partial u)$ vanishes up to third order.

Therefore $R(u, \partial u)$ is a linear combination of the terms

$$R_{\beta_1\beta_2\beta_3}(u, \partial u)\partial^{\beta_1} u \partial^{\beta_2} u \partial^{\beta_3} u,$$

where each β_j is either 0 or 1. So $\Gamma^\alpha R(u, \partial u)$ is a linear combination of the terms

$$a(u, \partial u)\Gamma^{\alpha_1}\partial^{\beta_1} u \cdots \Gamma^{\alpha_m}\partial^{\beta_m} u, \tag{69}$$

where $a(\cdot, \cdot)$ are smooth functions, each β_j is either 0 or 1, $|\alpha_1| + \cdots + |\alpha_m| = |\alpha|$ with $m \geq 3$, and at most one α_j satisfies $|\alpha_j| > 3$. In view of (64),

$$|a(u, \partial u)(t, x)| \leq C, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n.$$

For all the terms $\Gamma^{\alpha_j}\partial^{\beta_j}u$ except the one with highest $|\alpha_j|$, we can use (64) to estimate them. We thus obtain

$$\sum_{|\alpha|\le 6}\|(1+\tau+|\cdot|)\Gamma^{\alpha_j}R(u,\partial u)(\tau,\cdot)\|_{L^\infty}$$
$$\le \frac{C(A\varepsilon)^2}{1+\tau}\sum_{|\alpha|\le 7}\|\Gamma^\alpha u(\tau,\cdot)\|_{L^2}=\frac{C(A\varepsilon)^2}{1+\tau}A(\tau),\quad 0\le\tau\le T.$$

For $\Gamma^\alpha Q_F(\partial u)$, we can use Proposition 26 and (64) to obtain

$$\sum_{|\alpha|\le 6}\|(1+\tau+|\cdot|)\Gamma^\alpha Q_F(\partial u)(\tau,\cdot)\|_{L^2}$$
$$\le C\sum_{|\alpha|\le 4}\|\Gamma^\alpha u(\tau,\cdot)\|_{L^\infty}\sum_{|\alpha|\le 7}\|\Gamma^\alpha u(\tau,\cdot)\|_{L^2}\le\frac{CA\varepsilon}{1+\tau}A(\tau).$$

Therefore

$$\sum_{|\alpha|\le 6} \|(1+\tau+|\cdot|)\Gamma^\alpha \Box u(\tau,\cdot)\|_{L^2} \le \frac{CA\varepsilon}{1+\tau} A(\tau), \quad 0 \le \tau \le T.$$

This together with (67) gives

$$A(t) \le CA(0) + CA\varepsilon \int_0^t \frac{A(\tau)}{1+\tau} d\tau, \quad 0 \le t \le T.$$

By Gronwall inequality,

$$A(t) \le CA(0) \exp\left(CA\varepsilon \int_0^t \frac{d\tau}{1+\tau} \right) = C(1+t)^{CA\varepsilon} A(0), \quad 0 \le t \le T.$$

This shows (66).

Step 2. We will show (65). We need the following estimate of Hórmander whose proof will be given later.

Theorem 30 (Hörmander)

There exists C such that if $F \in C^2([0,\infty)\times\mathbb{R}^3)$ and $\Box u = F$ with vanishing initial data at $t=0$, then

$$(1+t+|x|)|u(t,x)| \le C \sum_{|\alpha|\le 2} \int_0^t \int_{\mathbb{R}^3} |\Gamma^\alpha F(s,y)| \frac{dyds}{1+s+|y|}$$

In order to use Theorem 30 to estimate $|\Gamma^\alpha u(t,x)|$ with $|\alpha| \le 4$, we need $\Gamma^\alpha u(0,\cdot) = 0$ and $\partial_t \Gamma^\alpha u(0,\cdot) = 0$. So we define w_α by

$$\Box w_\alpha = 0, \quad w_\alpha|_{t=0} = (\Gamma^\alpha u)|_{t=0}, \quad \partial_t w_\alpha|_{t=0} = (\partial_t \Gamma^\alpha u)|_{t=0}.$$

We then apply Theorem 30 to $\Gamma^\alpha u - w_\alpha$ to obtain

$$(1+t+|x|) \sum_{|\alpha|\le 4} |\Gamma^\alpha u(t,x) - w_\alpha(t,x)| \le C \sum_{|\alpha|\le 4} \sum_{|\beta|\le 2} \int_0^t \int_{\mathbb{R}^3} |\Gamma^\beta \Box \Gamma^\alpha u(s,y)| \frac{dyds}{1+s}$$

Since $[\Box, \Gamma]$ is either 0 or $2\Box$, we have

$$(1+t+|x|) \sum_{|\alpha|\le 4} |\Gamma^\alpha u(t,x) - w_\alpha(t,x)|$$

$$\le C \sum_{|\alpha|\le 6} \int_0^t \int_{\mathbb{R}^3} |\Gamma^\alpha \Box u(s,y)| \frac{dyds}{1+s} = C \sum_{|\alpha|\le 6} \int_0^t \int_{\mathbb{R}^3} |\Gamma^\alpha F(u,\partial u)(s,y)| \frac{dyds}{1+s}.$$

We use again (68). For the quadratic term $Q_F(\partial u)$, we may use Proposition 26 to obtain

$$(1+s)\sum_{|\alpha|\le 6}|\Gamma^\alpha Q_F(\partial u)(s,y)| \le C\sum_{|\alpha|\le 7}|\Gamma^\alpha u(s,y)|^2.$$

This together with (66) gives

$$\begin{aligned}\int_{\mathbb{R}^n}\sum_{|\alpha|\le 6}|\Gamma^\alpha Q_F(\partial u)(s,y)|dy &\le \frac{C}{1+s}\sum_{|\alpha|\le 7}\|\Gamma^\alpha u(s,\cdot)\|_{L^2}^2\\ &\le CA(0)^2(1+s)^{-1+2C_1A\varepsilon}.\end{aligned}$$

For $\Gamma^\alpha R(u,\partial u)$, we use again (69). We use (64) to estimate all factors except the two factors with highest $|\alpha_j|$.

Then

$$\int_{\mathbb{R}^3} |\Gamma^\alpha R(u,\partial u)| dy \le \frac{CA\varepsilon}{1+s} \sum_{|\alpha|\le 7} \|\Gamma^\alpha u(s,\cdot)\|_{L^2}^2$$
$$\le CA\varepsilon A(0)^2 (1+s)^{-1+2C_1 A\varepsilon}.$$

Therefore

$$(1+t+|x|) \sum_{|\alpha|\le 4} |\Gamma^\alpha u - w_\alpha|(t,x) \le CA(0)^2 \int_0^t (1+s)^{-2+2C_1 A\varepsilon} ds.$$

It is easy to see that $A(0) = O(\varepsilon)$. We take $\varepsilon_0 > 0$ such that $4C_1 A\varepsilon < 1$. Then for $0 < \varepsilon \le \varepsilon_0$ there holds

$$(1+t+|x|) \sum_{|\alpha|\le 4} |\Gamma^\alpha u(t,x) - w_\alpha(t,x)| \le C\varepsilon^2$$

By shrinking $\varepsilon > 0$ if necessary, we can obtain

$$\sum_{|\alpha|\le 4} |\Gamma^\alpha u(t,x) - w_\alpha(t,x)| \le \frac{A\varepsilon}{4(1+t+|x|)}$$

This will complete the proof of (65) if we could show that

$$\sum_{|\alpha|\le 4} |w_\alpha(t,x)| \le \frac{A\varepsilon}{4(1+t+|x|)}. \tag{70}$$

To see (70), we observe that $|\Gamma^\alpha u(0,\cdot)| \le C_\alpha \varepsilon$ with C_α depending on α and f, g. Since w_α is the solution of a linear wave equation, by the representation formula, we can conclude

$$\sum_{|\alpha|\le 4} |w_\alpha(t,x)| \le \frac{C_\alpha \varepsilon}{1+t+|x|} \quad \forall (t,x) \in [0,\infty) \times \mathbb{R}^3.$$

By adjusting A to be a larger one, we obtain (70). □

6.4. Proof of Theorem 31: an estimate of Hörmander

Theorem 31 (Hörmander)

There exists C such that if $F \in C^2([0,\infty) \times \mathbb{R}^3)$ and $\Box u = F$ with vanishing initial data at $t = 0$, then

$$(1 + t + |x|)|u(t,x)| \leq C \sum_{|\alpha| \leq 2} \int_0^t \int_{\mathbb{R}^3} |\Gamma^\alpha F(s,y)| \frac{dyds}{1 + s + |y|} \quad (71)$$

We first indicate how to reduce the proof of Theorem 31 to some special cases. Take $\varphi \in C^\infty(\mathbb{R}^4)$ such that

$$\varphi(s,y) = \begin{cases} 0 & \text{when } s^2 + |y|^2 > 2/3 \\ 1 & \text{when } s^2 + |y|^2 < 1/3 \end{cases}$$

and write $F = F_1 + F_2$, where $F_1 = \varphi F$ and $F_2 = (1 - \varphi)F$.

Then

$$\mathrm{supp}(F_1) \subset B(0, 2/3) \quad \text{and} \quad \mathrm{supp}(F_2) \subset \mathbb{R}^4 \setminus B(0, 1/3).$$

Define u_1 and u_2 by $\Box u_j = F_j$ with vanishing Cauchy data, then $u = u_1 + u_2$. If the inequality in Theorem 31 holds true for u_1 and u_2, then it is also true for u, considering that $|\Gamma^\alpha \varphi| \leq C_\alpha$.

Therefore, we may assume either

- *F is zero in a neighborhood of the origin*, or
- *F is supported around the origin*.

We need the representation formula for u satisfying $\Box u = F$ with vanishing Cauchy data at $t = 0$.

Recall that the solution of the Cauchy problem $\Box u = 0$ with $u(0, \cdot) = 0$ and $\partial_t u(0, \cdot) = g$ is given by

$$u(t, x) = \frac{1}{4\pi t} \int_{|y-x|=t} g(y) d\sigma(y). \tag{72}$$

Lemma 32

The solution of $\Box u = F$ with vanishing Cauchy data at $t = 0$ is given by

$$u(t, x) = \frac{1}{4\pi} \int_{|y|<t} F(t - |y|, x - y) \frac{dy}{|y|}. \tag{73}$$

Proof. The Duhamel's principle says that $u(t, x) = \int_0^t v(t, x; s) ds$, where, for each fixed s, $v(t, x; s)$ satisfies

$$\partial_t^2 v - \Delta v = 0, \quad v(s, x; s) = 0, \quad \partial_t v(s, x; s) = F(s, x).$$

In view of the representation formula (72) we have

$$v(t, x; s) = \frac{1}{4\pi(t-s)} \int_{|y-x|=t-s} F(s, y) d\sigma(y).$$

Therefore

$$\begin{aligned} u(t, x) &= \frac{1}{4\pi} \int_0^t \int_{|y-x|=t-s} \frac{F(s, y)}{t-s} d\sigma(y) ds \\ &= \frac{1}{4\pi} \int_0^t \int_{|z|=\tau} \frac{F(t-\tau, x-z)}{\tau} d\sigma(z) d\tau \\ &= \frac{1}{4\pi} \int_{|z|<t} F(t-|z|, x-z) \frac{dz}{|z|}. \end{aligned}$$

This completes the proof. □

Corollary 33

(a) **Maximum Principle:** *Assume that u_1 and u_2 satisfy $\Box u_j = F_j$ with vanishing Cauchy data at $t = 0$. If $|F_1| \leq F_2$, then $|u_1| \leq u_2$.*

(b) *If F is spherically symmetric in the spatial variables, i.e $F(t,x) = \widetilde{F}(t,|x|)$, then the solution u of $\Box u = F$ with vanishing Cauchy data at $t = 0$ is also spherically symmetric, i.e. $u(t,x) = \widetilde{u}(t,|x|)$, where*

$$\widetilde{u}(t,r) = \frac{1}{2r} \int_0^t \int_{|r-(t-s)|}^{r+t-s} \widetilde{F}(s,\rho)\rho d\rho ds.$$

Proof. (a) follows immediately from (73) in Lemma 32.

(b) The spherical symmetry of u follows from the formula (73). Let $r = |x|$ and $e_3 = (0, 0, 1)$. Then

$$u(t, x) = u(t, re_3) = \frac{1}{4\pi} \int_{|y|<t} \widetilde{F}(t - |y|, |re_3 - y|) \frac{dy}{|y|}$$

Taking the polar coordinates $y = \tau(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and using $|re_3 - y| = \sqrt{r^2 - 2r\tau\cos\theta + \tau^2}$, we obtain

$$u(t, x) = \frac{1}{4\pi} \int_0^t \int_0^{2\pi} \int_0^{\pi} \widetilde{F}(t - \tau, \sqrt{r^2 - 2r\tau\cos\theta + \tau^2}) \tau \sin\theta d\theta d\phi d\tau.$$

Let $\rho = \sqrt{r^2 - 2r\tau\cos\theta + \tau^2}$. Since $\rho d\rho = r\tau\sin\theta d\theta$, we have

$$u(t, x) = \frac{1}{2r} \int_0^t \int_{|r-\tau|}^{r+\tau} \widetilde{F}(t - \tau, \rho) \rho d\rho d\tau.$$

This completes the proof by setting $s = t - \tau$. □

Lemma 34

There exists C such that if $\Box u = F$ with $F \in C^2([0,\infty) \times \mathbb{R}^3)$ and vanishing Cauchy data at $t = 0$ then

$$|x||u(t,x)| \leq C \int_0^t \int_{\mathbb{R}^3} \sum_{|\alpha| \leq 2} |\Gamma^\alpha F(s,y)| \frac{dyds}{|y|}$$

where the sum involves $\Gamma = \Omega_{ij}$, $1 \leq i < j \leq 3$ only.

Proof. Define the radial majorant of F by

$$F^*(t,r) := \sup_{\omega \in \mathbb{S}^2} |F(t, r\omega)|,$$

and let $u^*(t,x)$ solve $\Box u^*(t,x) = F^*(t,|x|)$ with vanishing Cauchy data at $t = 0$.

It follows from Corollary 33(a) that

$$|u(t,x)| \le u^*(t,x).$$

In view of Corollary 33(b) we then obtain with $r := |x|$ that

$$|x||u(t,x)| \le |x|u^*(t,x) = \frac{1}{2}\int_0^t \int_{|r-(t-s)|}^{r+(t-s)} F^*(s,\rho)\rho d\rho ds. \quad (74)$$

Using the Sobolev inequality on $\mathbb{S}^2$, see Lemma 17(a), we have

$$F^*(s,\rho) = \sup_{\omega\in\mathbb{S}^2} |F(s,\rho\omega)| \le C \sum_{|\alpha|\le 2} \int_{\mathbb{S}^2} |(\Gamma^\alpha F)(s,\rho\nu)| d\sigma(\nu),$$

where the sum involves only $\Gamma = \Omega_{ij}$ with $1 \le i < j \le 3$.

Combining this with (74) yields

$$
\begin{aligned}
|x||u(t,x)| &\leq C \sum_{|\alpha|\leq 2} \int_0^t \int_{|r-(t-s)|}^{r+(t-s)} \int_{\mathbb{S}^2} (\Gamma^\alpha F)(s,\rho\omega)|\rho d\sigma(\omega) d\rho ds \\
&\leq C \sum_{|\alpha|\leq 2} \int_0^t \int_0^\infty \int_{\mathbb{S}^2} (\Gamma^\alpha F)(s,\rho\omega)|\rho d\sigma(\omega) d\rho ds \\
&= C \sum_{|\alpha|\leq 2} \int_0^t \int_{\mathbb{R}^3} |(\Gamma^\alpha F)(s,y)| \frac{dyds}{|y|}.
\end{aligned}
$$

The proof is complete. □

Now we are ready to give the proof of Theorem 31. We first consider the case that F is supported around the origin.

Proposition 35

Let u satisfy $\Box u = F$ with $F \in C^2([0,\infty) \times \mathbb{R}^3)$ and vanishing Cauchy data at $t = 0$. If F is supported around the origin, say, $supp(F) \subset \{(s,y) : s + |y| < 1/3\}$, then

$$(1 + t + |x|)|u(t,x)| \leq C \int_0^t \int_{\mathbb{R}^3} \sum_{|\alpha| \leq 2} |\Gamma^\alpha F(s,y)| \frac{dyds}{1 + s + |y|}$$

where the sum only involves the vector fields $\Gamma = \partial_j$, $0 \leq j \leq 3$.

Proof. We claim that $u(t,x) = 0$ if $|t - |x|| > 1/3$. Indeed, recall that

$$u(t,x) = \frac{1}{4\pi} \int_{|y|<t} F(t - |y|, x - y) \frac{dy}{|y|}.$$

It is easy to see that for $|y| < t$ there hold

$$(t - |y|) + |x - y| \geq |t - |x||$$

Therefore when $|t - |x|| > 1/3$ we have

$$F(t - |y|, x - y) = 0 \quad \text{for all } |y| < t.$$

Consequently $u(t, x) = 0$ if $|t - |x|| > 1/3$.

Case 1. $|x| \leq t/2$. Since $t + |x| > 1$, we have $t > 2/3$. So

$$|t - |x|| = t - |x| > \frac{1}{2} t > \frac{1}{3}.$$

Consequently $u(t, x) = 0$ and the inequality holds trivially.

Case 2. $|x| > t/2$. We may use Lemma 34 to obtain

$$|x||u(t,x)| \le C \int_0^t \int_{\mathbb{R}^3} |F(s,y)| \frac{dyds}{|y|} + C \sum_{1 \le |\alpha| \le 2} \int_0^t \int_{\mathbb{R}^3} |(\Gamma^\alpha F)(s,y)| \frac{dyds}{|y|},$$

where the sum involves only $\Gamma = \Omega_{ij}$, $1 \le i < j \le 3$. Since

$$|\Omega_{ij} F(s,y)| \lesssim |y||\partial_y F(s,y)|$$

and $F(s,y) = 0$ for $s + |y| > 1/3$, we have

$$|\Gamma^\alpha F(s,y)| \le C|y| \sum_{1 \le |\beta| \le 2} |(\partial_y^\beta F)(s,y)|.$$

Therefore, using $|x| \geq (t+|x|)/3$, we have

$$(t+|x|)|u(t,x)| \leq C\int_0^t \int_{\mathbb{R}^3} |F(s,y)|\frac{dyds}{|y|} + C\sum_{1\leq|\alpha|\leq 2}\int_0^t\int_{\mathbb{R}^3}|(\partial_y^\alpha F)(s,y)|dyds. \quad (75)$$

In order to proceed further, we need

Lemma 36

If $\varphi(r)$ is C^1 and vanishes for large r, then

$$\int_0^\infty |\varphi(r)|rdr \leq \frac{1}{2}\int_0^\infty |\varphi'(r)|r^2 dr.$$

Using Lemma 36, we have

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{|F(s,y)|}{|y|} dy &= \int_{\mathbb{S}^2}\int_0^\infty |F(s,r\omega)| r dr d\sigma(\omega) \\ &\leq \frac{1}{2}\int_{\mathbb{S}^2}\int_0^\infty \left|\frac{\partial}{\partial r}(F(s,r\omega))\right| r^2 dr d\sigma(\omega) \\ &\leq \frac{1}{2}\int_{\mathbb{S}^2}\int_0^\infty |(\partial_y F)(s,r\omega)|\, r^2 dr d\sigma(\omega) \\ &= \frac{1}{2}\int_{\mathbb{R}^3} |(\partial_y F)(s,y)|\, dy.\end{aligned}$$

This, together with (75) and $F(s,y) = 0$ for $s + |y| > 1/10$, gives the desired inequality. □

Proof of Lemma 36. Since $|\varphi(\rho)|$ is Lipschitz, $\frac{d}{d\rho}|\varphi|$ exists a.e. and

$$\left|\frac{d}{d\rho}|\varphi(\rho)|\right| \leq |\varphi'(\rho)| \quad \text{a.e.}$$

Since $\varphi(\rho)$ vanishes for large ρ, we have

$$0 = \int_0^\infty \frac{d}{d\rho}\left(|\varphi(\rho)|\rho^2\right) d\rho = \int_0^\infty \left(2|\varphi(\rho)|\rho + \left(\frac{d}{d\rho}|\varphi(\rho)|\right)\rho^2\right) d\rho.$$

Therefore

$$2\int_0^\infty |\varphi(\rho)|\rho d\rho \leq \int_0^\infty \left|\frac{d}{d\rho}|\varphi(\rho)|\right| \rho^2 d\rho \leq \int_0^\infty |\varphi'(\rho)|\rho^2 d\rho.$$

The proof is complete. □

To complete the proof of Theorem 31, we remains only to consider the case that F vanishes in a neighborhood of the origin. We need a calculus lemma.

Lemma 37

For any $f \in C^1([a,b])$ there holds

$$|f(t)| \le \frac{1}{b-a}\int_a^b |f(s)|ds + \int_a^b |f'(s)|ds, \quad \forall t \in [a,b].$$

Proof. By the fundamental theorem of calculus we have

$$f(t) = f(s) + \int_s^t f'(\tau)d\tau, \quad \forall t, s \in [a,b]$$

which implies

$$|f(t)| \leq |f(s)| + \int_a^b |f'(\tau)| d\tau.$$

Integration over $[a, b]$ with respect to s yields the inequality. $\square$

Proposition 38

Let u satisfy $\Box u = F$ with $F \in C^2([0, \infty) \times \mathbb{R}^3)$ and vanishing Cauchy data at $t = 0$. If F vanishes in a neighborhood of the origin, say, $supp(F) \subset \{(s, y) : s + |y| > 1/6\}$, then

$$(1 + t + |x|)|u(t, x)| \leq C \int_0^t \int_{\mathbb{R}^3} \sum_{|\alpha| \leq 2} |\Gamma^\alpha F(s, y)| \frac{dyds}{1 + s + |y|}$$

where the sum only involves the homogeneous vector fields $\Gamma = L_0$ and Ω_{ij}, $0 \leq i < j \leq 3$.

Proof. Since $\mathrm{supp}(F) \subset \{(s, y) : s + |y| > 1/6\}$, it is equivalent to showing that

$$(t + |x|)|u(t, x)| \leq C \int_0^t \int_{\mathbb{R}^3} \sum_{|\alpha|\leq 2} |\Gamma^\alpha F(s, y)| \frac{dyds}{s + |y|}. \qquad (76)$$

We mention that it suffices to prove (76) for $t = 1$. In fact, if it is done for $t = 1$, we consider the function $u_\lambda(t, x) := u(\lambda t, \lambda x)$ for each $\lambda > 0$. Then

$$\Box u_\lambda = F_\lambda, \quad \text{with } F_\lambda(t, x) := \lambda^2 F(\lambda t, \lambda x).$$

We apply (76) to u_λ with $t = 1$ to obtain

$$(1 + |x|)|u_\lambda(1, x)| \leq C \int_0^1 \int_{\mathbb{R}^3} \sum_{|\alpha|\leq 2} |\Gamma^\alpha F_\lambda(s, y)| \frac{dyds}{s + |y|}.$$

Since Γ are homogeneous vector fields, we have

$$(\Gamma^\alpha F_\lambda)(s, y) = \lambda^2 (\Gamma^\alpha F)(\lambda s, \lambda y).$$

Since $u_\lambda(1, x) = u(\lambda, \lambda x)$, this and the above inequality imply

$$\begin{aligned}(1 + |x|)|u(\lambda, \lambda x)| &\le C \sum_{|\alpha| \le 2} \int_0^1 \int_{\mathbb{R}^3} \lambda^2 |(\Gamma^\alpha F)(\lambda s, \lambda y)| \frac{dyds}{s + |y|} \\ &= C\lambda^{-1} \sum_{|\alpha| \le 2} \int_0^\lambda \int_{\mathbb{R}^3} |(\Gamma^\alpha F)(\tau, z)| \frac{dzd\tau}{\tau + |z|}\end{aligned}$$

Therefore

$$(\lambda + |\lambda x|)|u(\lambda, \lambda x)| \le C \sum_{|\alpha| \le 2} \int_0^\lambda \int_{\mathbb{R}^3} |(\Gamma^\alpha F)(\tau, z)| \frac{dzd\tau}{\tau + |z|}.$$

Since $\lambda > 0$ is arbitrary and λx can be any point in $\mathbb{R}^3$, we obtain (76) for any $t > 0$.

In the following we will prove (76) for $t = 1$.

We need a reduction. By taking $\varphi \in C^\infty([0, \infty)$ with $\varphi(r) = 1$ for $0 \le r \le 1/3$ and $\varphi(r) = 0$ for $r \ge 1/2$, we can write $F = F_1 + F_2$, where

$$F_1(s, y) := \varphi(|y|/s)F(s, y), \quad F_2(s, y) := (1 - \varphi(|y|/s))F(s, y).$$

Since $\varphi(|y|/s)$ is homogeneous of degree 0, for any homogeneous vector field Γ we have $|\Gamma^\alpha \varphi| \lesssim 1$ for all $|\alpha| \le 2$. Consequently

$$\sum_{|\alpha|\le 2} (|\Gamma^\alpha F_1| + |\Gamma^\alpha F_2|) \lesssim \sum_{|\alpha|\le 2} |\Gamma^\alpha F|.$$

Thus, if (76) with $t = 1$ holds true for F_1 and F_2, it also holds true for F. Since

$$\text{supp}(F_1) \subset \{(s, y) : |y| \le s/2\}, \quad \text{supp}(F_2) \subset \{(s, y) : |y| \ge s/3\},$$

therefore, we need only consider two situations;

- $F(s, y) = 0$ when $|y| > s/2$; or
- $F(s, y) = 0$ when $|y| < s/3$.

(i) We first assume that $F(s, y) = 0$ when $|y| > s/2$. Using (73) it is easy to see that $u(1, x) = 0$ if $|x| > 1$. Thus, we may assume $|x| \le 1$. It then follows from (73) with $t = 1$ that

$$4\pi |u(1, x)| \le \int_{|y|<1} |F(1 - |y|, x - y)| \frac{dy}{|y|} = I_1 + I_2,$$

where

$$I_1 = \int_{\frac{1}{2}<|y|<1} |F(1-|y|, x-y)| \frac{dy}{|y|}, \quad I_2 = \int_{|y|\leq\frac{1}{2}} |F(1-|y|, x-y)| \frac{dy}{|y|}$$

To deal with I_1, By Lemma 37 we obtain

$$|F(1-|y|, x-y)| \lesssim \int_0^1 (|F(s, x-y)| + |\partial_s F(s, x-y)|)\, ds.$$

Therefore

$$\begin{aligned} I_1 &\lesssim \int_0^1 \int_{\frac{1}{2}<|y|<1} (|F(s, x-y)| + |\partial_s F(s, x-y)|)\, dyds \\ &\lesssim \int_0^1 \int_{\mathbb{R}^3} (|F(s, y)| + |\partial_s F(s, y)|)\, dyds \end{aligned}$$

Since $\text{supp}(F) \subset \{(s, y) : |y| < s/2\}$, from Lemma 13 it follows

$$|\partial_s F| \lesssim \frac{1}{s + |y|} \sum_{|\alpha|=1} |\Gamma^\alpha F|,$$

where the sum involves only the homogeneous vector fields. So

$$I_1 \lesssim \int_0^1 \int_{\mathbb{R}^3} \sum_{|\alpha| \le 1} |\Gamma^\alpha F(s, y)| \frac{dyds}{s + |y|}.$$

Next we consider I_2. We use Lemma 37 on $[1/2, 1]$ to derive that

$$|F(1 - |y|, x - y)| \lesssim \int_{\frac{1}{2}}^1 (|F(s, x - y)| + |\partial_s F(s, x - y)|)\, ds.$$

Thus

$$I_2 \lesssim \int_{\frac{1}{2}}^{1} \int_{|y| \leq \frac{1}{2}} (|F| + |\partial_s F|)(s, x - y) \frac{dyds}{|y|}$$

We may use Lemma 36 as before to derive that

$$I_2 \lesssim \int_{\frac{1}{2}}^{1} \int_{\mathbb{R}^3} (|\partial_y F| + |\partial_y \partial_s F|)(s, y) dyds.$$

Since $\text{supp}(F) \subset \{(s, y) : |y| < s/2\}$ and $1/2 < s < 1$, we have from Lemma 13 that

$$|\partial_s F| + |\partial_y \partial_s F| \lesssim \frac{1}{s + |y|} \sum_{1 \leq |\alpha| \leq 2} |\Gamma^\alpha F|.$$

Therefore

$$I_2 \lesssim \int_{\frac{1}{2}}^{1} \int_{\mathbb{R}^3} \sum_{1 \le |\alpha| \le 2} |\Gamma^\alpha F(s,y)| \frac{dyds}{s+|y|}$$

Combining the estimates on I_1 and I_2 we obtain the desired inequality.

(ii) Next we consider the case that $F(s,y) = 0$ when $|y| < s/3$.

If $|x| \ge 1/4$, then we have from Lemma 34 that

$$(1+|x|)|u(1,x)| \lesssim |x||u(1,x)| \lesssim \int_0^1 \int_{\mathbb{R}^3} |\Gamma^\alpha F(s,y)| \frac{dyds}{s+|y|}$$

as desired.

So we may assume $|x| < 1/4$. We will use (73). Observing that

$$(1-|y|, x-y) \in \mathrm{supp}(F) \Longrightarrow |x-y| > \frac{1}{3}(1-|y|)$$
$$\Longrightarrow \frac{4}{3}|y| > \frac{1}{3} - |x| > \frac{1}{12} \Longrightarrow |y| > \frac{1}{16}.$$

Therefore, it follows from (73) that

$$|u(1,x)| \lesssim \int_{\frac{1}{16}<|y|<1} |F(1-|y|, x-y)| dy.$$

Consider the transformation

$$\varphi(\tau, y) := \tau(1-|y|, x-y),$$

where $1/16 < |y| < 1$ and $1 < \tau < 16/15$.

By Lemma 37 we have

$$F(1-|y|,x-y) \le F(\varphi(\tau,y))|$$
$$\lesssim \int_1^{\frac{16}{15}} \left(|F(\varphi(\tau,y))| + \left| \frac{\partial}{\partial\tau}(F(\varphi(\tau,y))) \right| \right) d\tau$$

Observing that

$$\frac{\partial}{\partial\tau}(F(\varphi(\tau,y)) = \frac{1}{\tau}(L_0F)(\varphi(\tau,y)).$$

Therefore

$$|u(1,x)| \lesssim \int_0^{\frac{16}{15}} \int_{\frac{1}{16}<|y|<1} (|F| + |L_0F|)\,(\varphi(\tau,y))dyd\tau.$$

Under the transformation $(s, z) := \varphi(\tau, y)$, the domain

$$\{(\tau, y) : 1 < \tau < 16/15, 1/16 < |y| < 1\}$$

becomes a domain contained in

$$\{(s, z) : 0 < s < 1, |z| < 2\}.$$

The Jacobian of the transformation is $\tau^3(1 - x \cdot y/|y|)$ which is bounded below by $3/4$. Therefore

$$\begin{aligned}
|u(1, x)| &\lesssim \int_0^1 \int_{|z|\leq 2} (|F| + |L_0 F|)(s, z) dzds \\
&\lesssim \int_0^1 \int_{\mathbb{R}^3} (|F| + |L_0 F|)(s, z) \frac{dzds}{s + |y|}.
\end{aligned}$$

The proof is thus complete. □

7. Littlewood-Paley theory

Localization is a fundamental notion in analysis. Given a function, localization means restricting it to a small region in physical space, or frequency space.

- Physical space localization is the most familiar. To localize a function $f(x)$ on a open set, say, $B_r(x_0)$, in physical space, one can choose a C_0^∞ function χ supported on $B_r(x_0)$ which equals to 1 on $B_{r/2}(x_0)$. Then $\chi(x)f(x)$ gives the localization.
- Frequency space localization is an equally important notion. Let $\hat{f}(\xi)$ denote the Fourier transform of a function $f(x)$. Given a domain D in frequency space, one can choose a smooth function $\chi(\xi)$ supported on D and define a function $(\pi_D f)(x)$ with

$$\widehat{\pi_D f}(\xi) := \chi(\xi)\hat{f}(\xi).$$

  Then $\pi_D f$ is a frequency space localization of f over D.

Littlewood-paley decomposition of functions is based on frequency space localization.

7.1 Definition and basic properties

There is certain amount of flexibility in setting up the Littlewood -Paley decomposition on $\mathbb{R}^n$. One standard way is as follows:

- Let $\phi(\xi)$ be a real radial bump function with

$$\phi(\xi) = \begin{cases} 1, & |\xi| \le 1, \\ 0, & |\xi| \ge 2. \end{cases}$$

- Let $\psi(\xi)$ be the function

$$\psi(\xi) := \phi(\xi) - \phi(2\xi).$$

Then ψ is a bump function supported on $\{1/2 \leq |\xi| \leq 2\}$ and

$$\sum_{k \in \mathbb{Z}} \psi(\xi/2^k) = 1, \quad \forall \xi \neq 0. \tag{77}$$

- Define the *Littlewood-Paley (LP) projections* P_k and $P_{\leq k}$ by

$$\widehat{P_k f}(\xi) = \psi(\xi/2^k)\hat{f}(\xi), \quad \widehat{P_{\leq k} f}(\xi) = \phi(\xi/2^k)\hat{f}(\xi)$$

In physical space

$$P_k f = m_k * f, \tag{78}$$

where $m_k(x) := 2^{nk} m(2^k x)$ and $m(x)$ is the inverse Fourier transform of $\psi(\xi)$. Sometimes we write $f_k := P_k f$.

Using the Littlewood-Paley projections, we can decompose any L^2 function into the sum of frequency localized functions.

Lemma 39

For any $f \in L^2(\mathbb{R}^n)$ *there holds* $f = \sum_{k\in\mathbb{Z}} P_k f$.

Proof. By definition, we have for any $N, M > 0$ that

$$\sum_{-M\le k\le N} \widehat{P_k f}(\xi) = \sum_{-M\le k\le N} \left(\phi(\xi/2^k) - \phi(\xi/2^{k-1})\right)\hat{f}(\xi)$$
$$= \left(\phi(\xi/2^N) - \phi(\xi/2^{-M-1})\right)\hat{f}(\xi).$$

Therefore

$$\left\| f - \sum_{-M\le k\le N} P_k f \right\|_{L^2} = \left\| \hat{f} - \sum_{-M\le k\le N} \widehat{P_k f} \right\|_{L^2}$$
$$\le \|\phi(2^{M+1}\cdot)\hat{f}\|_{L^2} + \|(1-\phi(2^{-N}\cdot))\hat{f}\|_{L^2}.$$

Since $\phi(2^{M+1}\xi)$ is supported on $\{|\xi| \le 2^{-M}\}$ and $\phi(2^{-N}\xi) = 1$ on $\{|\xi| \le 2^N\}$. Therefore

$$\begin{aligned}\Big\| f - \sum_{-M\le k\le N} P_k f \Big\|_{L^2} &\lesssim \left(\int_{|\xi|\le 2^{-M}} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\ &\quad + \left(\int_{|\xi|\ge 2^{N}} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\ &\to 0 \quad \text{as } M, N \to \infty.\end{aligned}$$

This complete the proof. ◻

In the following we give some important properties of the LP projections. For any subset $J \subset \mathbb{Z}$, we define $P_J := \sum_{k\in J} P_k$.

Theorem 40

(i) *(Almost orthogonality) The operators P_k are selfadjoint and $P_{k_1}P_{k_2} = 0$ whenever $|k_1 - k_2| \geq 2$. In particular*

$$\|f\|_{L^2}^2 \approx \sum_k \|P_k f\|_{L^2}^2 \qquad \textbf{(LP1)}$$

(ii) *(L^p-boundedness) For any $1 \leq p \leq \infty$ and any interval $J \subset \mathbb{Z}$,*

$$\|P_J f\|_{L^p} \leq \|f\|_{L^p} \qquad \textbf{(LP2)}$$

(iii) *(Finite band property) There hold*

$$\|\partial P_k f\|_{L^p} \lesssim 2^k \|f\|_{L^p}, \quad 2^k \|P_k f\|_{L^p} \lesssim \|\partial f\|_{L^p}. \qquad \textbf{(LP3)}$$

*For any partial derivative $\partial P_k f$ there holds $\partial P_k f = 2^k \tilde{P}_k f$ where $\tilde{P}_k$ is a frequency cut-off operator associated to a different cut-off function $\tilde{\psi}$, which remains supported on $\{\frac{1}{2} \leq |\xi| \leq 2\}$ but may fail to satisfy (77). The operators $\tilde{P}_k$ satisfy (**LP2**).*

Theorem (Theorem 40 continued)

(iv) *(Bernstein inequality)* *For any* $1 \le p \le q \le \infty$ *there holds*

$$\|P_k f\|_{L^q} \lesssim 2^{kn(1/p-1/q)}\|f\|_{L^p}, \quad \|P_{\le 0} f\|_{L^q} \lesssim \|f\|_{L^p} \qquad \textbf{(LP4)}$$

(v) *(Commutator estimates)* *For* $f, g \in C_0^\infty(\mathbb{R}^n)$ *define the commutator* $[P_k, f]g = P_k(fg) - fP_k g$. *Then*

$$\|[P_k, f]g\|_{L^p} \lesssim 2^{-k}\|\nabla f\|_{L^\infty}\|g\|_{L^p}. \qquad \textbf{(LP5)}$$

(vi) *(Littlewood-Paley inequality).* *Let*

$$Sf(x) := \left(\sum_{k\in\mathbb{Z}} |P_k f(x)|^2\right)^{\frac{1}{2}}.$$

For every $1 < p < \infty$ *there holds*

$$\|f\|_{L^p} \lesssim \|Sf\|_{L^p} \lesssim \|f\|_{L^p}, \quad \forall f \in C_0^\infty(\mathbb{R}^n). \qquad \textbf{(LP6)}$$

Proof. (i) For any $f, g \in L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}\langle P_k f, g\rangle &= \langle \widehat{P_k f}, \hat{g}\rangle = \langle \psi(2^{-k}\cdot)\hat{f}, \hat{g}\rangle = \langle \hat{f}, \psi(2^{-k}\cdot)\hat{g}\rangle \\ &= \langle \hat{f}, \widehat{P_k g}\rangle = \langle f, P_k g\rangle.\end{aligned}$$

Therefore P_k is self-adjoint. Since $\psi(\xi/2^{k_1})\psi(\xi/2^{k_2}) = 0$ whenever $|k_1 - k_2| \geq 2$, we have

$$\widehat{P_{k_1} P_{k_2}} f(\xi) = \psi(\xi/2^{k_1})\psi(\xi/2^{k_2})\hat{f}(\xi) = 0.$$

So $P_{k_1} P_{k_2} f = 0$ whenever $|k_1 - k_2| \geq 2$. Next prove (**LP1**). We first have

$$\begin{aligned}\|f\|_{L^2}^2 &= \Big\| \sum_{k\in\mathbb{Z}} P_k f \Big\|_{L^2}^2 = \sum_{k,k'\in\mathbb{Z}} \langle P_k f, P_{k'} f\rangle = \sum_{|k-k'|\leq 1} \langle P_k f, P_{k'} f\rangle \\ &\leq \sum_{|k-k'|\leq 1} \|P_k f\|_{L^2} \|P_{k'} f\|_{L^2} \leq 3 \sum_k \|P_k f\|_{L^2}^2.\end{aligned}$$

On the other hand, since $\psi(\xi/2^k) = 0$ for $2^{k-1} \le |\xi| \le 2^{k+1}$, we have

$$\sum_{k\in\mathbb{Z}} \|P_k f\|_{L^2}^2 = \sum_{k\in\mathbb{Z}} \|\widehat{P_k f}\|_{L^2}^2 = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^n} |\psi(\xi/2^k)\hat{f}(\xi)|^2 d\xi$$
$$\lesssim \sum_{k\in\mathbb{Z}} \int_{2^{k-1}\le|\xi|\le 2^{k+1}} |\hat{f}(\xi)|^2 d\xi \lesssim \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi$$
$$= \|\hat{f}\|_{L^2}^2 = \|f\|_{L^2}^2.$$

(ii) It suffices to prove (**LP2**) for $J = (-\infty, k] \subset \mathbb{Z}$, i.e.

$$\|P_{\le k} f\|_{L^p} \lesssim \|f\|_{L^p}. \tag{79}$$

Let $\bar{m}(x)$ be the inverse Fourier transform of $\phi(\xi)$ and let $\bar{m}_k(x)$ $:= 2^{nk}\bar{m}(2^k x)$. Then

$$P_{\le k} f = \bar{m}_k * f.$$

Since $\|\bar{m}_k\|_{L^1} = \|\bar{m}\|_{L^1} \lesssim 1$, we have

$$\|P_{\leq k} f\|_{L^p} \lesssim \|\bar{m}_k\|_{L^1} \|f\|_{L^p} \lesssim \|f\|_{L^p}$$

where we used the Young's inequality: for $1 \leq p, q, r \leq \infty$ with $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$, there holds

$$\|k * f\|_{L^q} \leq \|k\|_{L^r} \|f\|_{L^p} \qquad \text{(Young)}$$

(iii) To prove (**LP3**), recall that $P_k f = m_k * f$, we have

$$\partial_j (P_k f) = 2^k (\partial_j m)_k * f,$$

where $(\partial_j m)_k(x) = 2^{nk} \partial_j m(2^k x)$. Since $\|(\partial_j m)_k\|_{L^1} = \|\partial_j m\|_{L^1} \lesssim 1$, by Young's inequality,

$$\|\partial_j (P_k f)\|_{L^p} \lesssim 2^k \|f\|_{L^p}.$$

Next we write

$$\hat{f}(\xi) = \sum_{j=1}^{n} \frac{\xi_j}{i|\xi|^2} \widehat{\partial_{x_j} f}(\xi), \quad \xi \neq 0.$$

Let $\chi_j(\xi) = \frac{\xi_j}{i|\xi|^2}\psi(\xi)$, we have

$$2^k \widehat{P_k f}(\xi) = \sum_{j=1}^{n} 2^k \frac{\xi_j}{i|\xi|^2} \psi(\xi/2^k) \widehat{\partial_{x_j} f}(\xi) = \sum_{j=1}^{n} \chi_j(\xi/2^k) \widehat{\partial_{x_j} f}(\xi).$$

Let h_j be inverse Fourier transform of χ_j and $(h_j)_k := 2^{nk} h_j(2^k x)$, then

$$2^k P_k f = \sum_{j=1}^{n} (h_j)_k * \partial_j f.$$

Therefore

$$2^k \|P_k f\|_{L^p} \le \sum_{j=1}^{n} \|h_j\|_{L^1} \|\partial_j f\|_{L^p} \lesssim \sum_{j=1}^{n} \|\partial_j f\|_{L^p} \lesssim \|\partial f\|_{L^p}.$$

(iv) To see (**LP4**), we use $P_k f = m_k * f$ and Young's inequality with $1 + q^{-1} = r^{-1} + p^{-1}$ to obtain

$$\|P_k f\|_{L^q} = \|m_k * f\|_{L^q} \lesssim \|m_k\|_{L^r} \|f\|_{L^p}.$$

The first inequality in (**LP4**) then follows, in view of

$$\|m_k\|_{L^r} = 2^{nk} \left(\int_{\mathbb{R}^n} |m(2^k x)|^r dx \right)^{\frac{1}{r}} = 2^{nk(1-\frac{1}{r})} \|m\|_{L^r} \lesssim 2^{nk(\frac{1}{p}-\frac{1}{q})}.$$

The second inequality in (**LP4**) follows directly from the first.

We remark that Bernstein inequality is a remedy for the failure of $W^{\frac{n}{p},p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$. It implies the Sobolev inequality for each LP component $P_k f$. The failure the Sobolev inequality for f is due to the divergence of the summation $f = \sum_k f_k$.

(v) We now prove (**LP5**). Since $P_k f = m_k * f$, we have

$$P_k(fg)(x) - f(x)P_k g(x) = \int_{\mathbb{R}^n} m_k(x-y)(f(y)-f(x))g(y)dy$$

Note that $|f(y)-f(x)| \le |x-y| \|\partial f\|_{L^\infty}$, we have

$$|P_k(fg)(x) - f(x)P_k g(x)| \lesssim 2^{-k} \|\partial f\|_{L^\infty} \int_{\mathbb{R}^n} |\bar{m}_k(x-y)g(y)|dy$$

where $\bar{m}(x) = |x|m(x)$ and $\bar{m}_k(x) = 2^{nk}\bar{m}(2^k x)$. (**LP5**) then follows by taking L^p-norm and using Young's inequality.

(vi) To prove (**LP6**), we need some Calderon-Zygmund theory.

Definition 41

A Calderon-Zygmund operator T is a linear operator on $\mathbb{R}^n$ of the form

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)dy$$

for some (possibly matrix valued) kernel K which obeys the bounds

$$|K(x,y)| \lesssim |x-y|^{-n}, \quad |\partial K(x,y)| \lesssim |x-y|^{-n-1}, \quad x \neq y \quad (80)$$

and $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is bounded.

Proposition 42

Calderon-Zygmund operators are bounded from L^p into L^p for any $1 < p < \infty$. They are not bounded, in general, for $p = 1$ and $p = \infty$.

We first prove $\|Sf\|_{L^p} \lesssim \|f\|_{L^p}$. To this end, we introduce the linear operator

$$\mathbf{S}f(x) = (P_k f(x))_{k\in\mathbb{Z}}.$$

It is easy to see that **S** has vector valued kernel

$$K(x,y) := \left(2^{nk} m(2^k(x-y))\right)_{k\in\mathbb{Z}},$$

where m is the inverse Fourier transform of ψ. Observing that m is a Schwartz function, (80) can be verified easily. Moreover, (**LP1**) implies that $\mathbf{S} : L^2 \to L^2$ is bounded. So **S** is a Calderon-Zygmund operator and Proposition 42 implies that

$$\|Sf\|_{L^p} = \||\mathbf{S}f|_{\ell^2}\|_{L^p} \lesssim \|f\|_{L^p}.$$

Next we prove $\|f\|_{L^p} \lesssim \|Sf\|_{L^p}$ by duality argument. For any Schwartz function g, by using $P_k P_{k'} = 0$ for $|k-k'| \geq 2$, the Cauchy-Schwartz inequality, and the Hölder inequality, we have

$$\begin{aligned}
\int f(x)g(x)dx &= \int \sum_{k,k' \in \mathbb{Z}} P_k f(x) P_{k'} g(x) dx \\
&= \int \sum_{|k-k'| \leq 1} P_k f(x) P_{k'} g(x) dx \\
&\lesssim \int \left(\sum_k |P_k f(x)|^2 \right)^{\frac{1}{2}} \left(\sum_{k'} P_{k'} g(x)|^2 \right)^{\frac{1}{2}} dx \\
&\lesssim \|Sf\|_{L^p} \|Sg\|_{L^{p'}} \lesssim \|Sf\|_{L^p} \|g\|_{L^{p'}},
\end{aligned}$$

where $1/p + 1/p' = 1$. This implies $\|f\|_{L^p} \lesssim \|Sf\|_{L^p}$. □

Spaces of functions

The Littlewood- Paley theory can be used to give alternative descriptions of Sobolev spaces and introduce new, more refined, spaces of functions. In view of **LP1**,

$$\|f\|_{L^2} \approx \sum_{k\in\mathbb{Z}} \|P_k f\|_{L^2}^2.$$

We can give a LP description of the homogeneous Sobolev norms $\|\cdot\|_{\dot{H}^s(\mathbb{R}^n)}$.

$$\|f\|_{\dot{H}^s}^2 \approx \sum_{k\in\mathbb{Z}} 2^{2ks}\|P_k f\|_{L^2}^2,$$

and for the H^s norms

$$\|f\|_{H^s}^2 \approx \sum_{k\in\mathbb{Z}} (1+2^k)^{2s}\|P_k f\|_{L^2}^2.$$

Definition 43

The Besov space $B^s_{2,1}$ is the closure of $C_0^\infty(\mathbb{R}^n)$ relative to the norm

$$\|f\|_{B^s_{2,1}} = \sum_{k\in\mathbb{Z}} (1+2^k)^s \|P_k f\|_{L^2}.$$

and the corresponding homogeneous Besov norm is defined by

$$\|f\|_{\dot{B}^s_{2,1}} = \sum_{k\in\mathbb{Z}} 2^{sk} \|P_k f\|_{L^2}.$$

Observe that $H^s \subset B^s_{2,1}$. We have the following embedding inequality by **LP4**

$$\|f\|_{L^\infty} \lesssim \|f\|_{\dot{B}^{\frac{n}{2}}_{2,1}}.$$

7.2 Product estimates

The LP calculus is particularly useful for nonlinear estimates. Let f, g be two functions on $\mathbb{R}^n$. Consider

$$P_k(fg) = P_k\left(\sum_{k',k''\in\mathbb{Z}} P_{k'}f \cdot P_{k''}g\right). \tag{81}$$

Now since $P_{k'}f$ has Fourier support $D' = \{2^{k'-1} \le |\xi| \le 2^{k'+1}\}$ and $P_{k''}f$ has Fourier support $D'' = \{2^{k''-1} \le |\xi| \le 2^{k''+1}\}$. It follows that $P_{k'}f \cdot P_{k''}g$ has Fourier support in $D' + D''$. We only get a nonzero contribution in the sum of (81) if $D' + D''$ intersects $\{2^{k-1} \le |\xi| \le 2^{k+1}\}$. Therefore, writing $f_k = P_k f$, $f_{<k} = P_{<k} f$, and $f_J := P_J f$ for any interval $J \subset \mathbb{Z}$, we can derive that

Proposition 44 (Trichotomy)

Given functions f, g we have the following decomposition

$$P_k(f \cdot g) = HH_k(f,g) + LL_k(f,g) + LH_k(f,g) + HL_k(f,g)$$

with

$$\begin{aligned} HH_k(f,g) &= \sum_{k',k''>k+5, |k'-k''|\leq 3} P_k(f_{k'} \cdot g_{k''}) \\ LL_k(f,g) &= P_k(f_{[k-5,k+5]} \cdot g_{[k-5,k+5]}) \\ LH_k(f,g) &= P_k(f_{\leq k-5} \cdot g_{[k-3,k+3]}) \\ HL_k(f,g) &= P_k(f_{[k-3,k+3]} \cdot g_{\leq k-5}), \end{aligned}$$

where LL_k consists of a finite number of terms, which can be typically ignored.

For applications, we can further simplify terms as follows,

$$HH_k(f,g) = P_k(\sum_{m>k} f_m \cdot g_m), \quad LH_k(f,g) = P_k(f_{<k} g_k),$$
$$HL_k(f,g) = P_k(f_k \cdot g_{<k}). \tag{82}$$

We now make use of Proposition 44 to prove a product estimate

Proposition 45

The following estimate holds true for all $s > 0$

$$\|fg\|_{H^s} \lesssim \|f\|_{L^\infty}\|g\|_{H^s} + \|g\|_{L^\infty}\|f\|_{H^s}. \tag{83}$$

Thus for all $s > n/2$,

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}. \tag{84}$$

Proof. Since $\|f\cdot g\|_{H^s}^2 \approx \sum_{k\in\mathbb{Z}}(1+2^k)^{2s}\|P_k(f\cdot g)\|_{L^2}^2$, it suffices to consider the higher frequency part

$$I = \sum_{k\geq 0} 2^{2ks}\|P_k(f\cdot g)\|_{L^2}^2$$

By using (82), we proceed by using **LP2** and Hölder's inequality

$$I_1 = \sum_{k\geq 0}\|2^{ks}HL_k(f,g)\|_{L^2}^2 \lesssim \|f\|_{H^s}^2\|g\|_{L^\infty}^2$$
$$I_2 = \sum_{k\geq 0}\|2^{ks}LH_k(f,g)\|_{L^2}^2 \lesssim \|f\|_{L^\infty}^2\|g\|_{H^s}^2$$

$$I_3 = \sum_{k\geq 0}\|2^{ks}HH_k(f,g)\|_{L^2}^2 \lesssim \|\sum_{m>k} 2^{(k-m)s}2^{ms}\|P_m f\|_{L^2}\|_{l_k^2}^2\|g\|_{L^\infty}^2$$
$$\lesssim \|f\|_{H^s}^2\|g\|_{L^\infty}^2$$

where we employed Young's inequality to derive the last inequality. By combining I_1, I_2 and I_3, we complete the proof.

8 Strichartz estimates

We will prove some Strichartz estimates for linear wave equation and derive a global existence result for a semilinear wave equation. Given a function $u(t,x)$ defined on $\mathbb{R} \times \mathbb{R}^n$, for any $q, r \geq 1$ we use the notation

$$\|u\|_{L_t^q L_x^r} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |u(t,x)|^r dx \right)^{\frac{q}{r}} dt \right)^{\frac{1}{q}}.$$

8.1 Homogeneous Strichartz estimates

We start with the homogeneous linear wave equation

$$\begin{aligned} &\Box u = 0 \quad \text{ on } \mathbb{R}^{1+n} \text{ with } n \geq 2, \\ &u(0,\cdot) = f, \quad \partial_t u(0,\cdot) = g. \end{aligned} \tag{85}$$

Theorem 46

Let u be the solution of (85). There holds

$$\|u\|_{L_t^q L_x^r} \leq C(\|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}}) \tag{86}$$

where $s = \frac{n}{2} - \frac{1}{q} - \frac{n}{r}$ *for any pair* (q, r) *that is wave admissible , i.e.*

$$2 \leq q \leq \infty, \quad 2 \leq r < \infty, \quad \text{and } \frac{2}{q} \leq \frac{n-1}{2}\left(1 - \frac{2}{r}\right).$$

We will prove Theorem 46 except the so-called endpoint cases

$$1 = \frac{2}{q} = \frac{n-1}{2}\left(1 - \frac{2}{r}\right).$$

One may refer to (Keel-Tao, Amer J. Math., 1998) for a proof.

The proof of Theorem 46 is based the Littlewood-Paley theory and consists of several steps.

Step 1 Applying the Littlewood Paley projection P_k to (85), and using the commutativity between P_k and $\Box$, we obtain

$$\begin{aligned} &\Box P_k u = 0 \qquad \text{on } \mathbb{R}\times\mathbb{R}^n \\ &P_k u|_{t=0} = P_k f, \quad \partial_t P_k u|_{t=0} = P_k g. \end{aligned} \tag{87}$$

We claim that it suffices to show

$$\|P_k u\|_{L_t^q L_x^r} \lesssim 2^{sk}\|P_k f\|_{L_x^2} + 2^{(s-1)k}\|P_k g\|_{L_x^2}, \quad \forall k\in\mathbb{Z}, \tag{88}$$

where $s = \frac{n}{2} - \frac{n}{r} - \frac{1}{q}$.

In fact, since $r \geq 2$, $q \geq 2$, and $u = \sum_{k\in\mathbb{Z}} P_k u$, by using Theorem 40 (vi) and the Minkowski inequality we have

$$\|u\|_{L_t^q L_x^r} \lesssim \left\| \left(\sum_{k\in\mathbb{Z}} |P_k u|^2 \right)^{1/2} \right\|_{L_t^q L_x^r} \lesssim \left(\sum_{k\in\mathbb{Z}} \|P_k u\|_{L_t^q L_x^r}^2 \right)^{1/2}$$
$$\lesssim \left(\sum_{k\in\mathbb{Z}} \left(2^{2sk} \|P_k f\|_{L_x^2}^2 + 2^{2(s-1)k} \|P_k g\|_{L_x^2}^2 \right) \right)^{1/2}$$
$$\lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}}.$$

Step 2. We next show that (88) can be derive from the estimate

$$\|P_0 u\|_{L_t^q L_x^r} \lesssim \|P_0 f\|_{L_x^2} + \|P_0 g\|_{L_x^2} \tag{89}$$

for any solution u of (85).

In fact, by letting

$$\begin{aligned} u_k(t,x) &:= u(2^{-k}t, 2^{-k}x), \\ f_k(x) &:= f(2^{-k}x), \\ g_k(x) &:= 2^{-k}g(2^{-k}x). \end{aligned}$$

Then there holds

$$\begin{aligned} &\Box u_k = 0 \qquad \text{on } \mathbb{R}\times\mathbb{R}^n, \\ &u_k(0,\cdot) = f_k, \quad \partial_t u_k(0,\cdot) = g_k. \end{aligned}$$

Therefore (89) can be applied for u_k to obtain

$$\|P_0 u_k\|_{L_t^q L_x^r} \lesssim \|P_0 f_k\|_{L_x^2} + \|P_0 g_k\|_{L_x^2}. \tag{90}$$

By straightforward calculation we have

$$\|P_0 u_k\|_{L_t^q L_x^r} = 2^{(\frac{n}{r}+\frac{1}{q})k} \|P_k u\|_{L_t^q L_x^r},$$
$$\|P_0 f_k\|_{L_x^2} = 2^{\frac{nk}{2}} \|P_k f\|_{L_x^2},$$
$$\|P_0 g_k\|_{L_x^2} = 2^{(\frac{n}{2}-1)k} \|P_k g\|_{L_x^2}.$$

These identities together with (90) give (88).

Step 3. It remains only to prove (89) for any solution u of (85). Let $\hat{u}(t,\xi)$ be the Fourier transform of $x \to u(t,x)$. Then

$$\partial_t^2 \hat{u} + |\xi|^2 \hat{u} = 0, \quad \hat{u}(0,\cdot) = \hat{f}, \quad \partial_t \hat{u}(0,\cdot) = \hat{g}.$$

This show that

$$\hat{u}(t,\xi) = \frac{1}{2}\left(\hat{f}(\xi) + \frac{\hat{g}(\xi)}{i|\xi|}\right) e^{it|\xi|} + \frac{1}{2}\left(\hat{f}(\xi) - \frac{\hat{g}(\xi)}{i|\xi|}\right) e^{-it|\xi|},$$

i.e. $\hat{u}(t,\xi)$ is a linear combination of $e^{\pm it|\xi|}\hat{f}(\xi)$ and $e^{\pm it|\xi|}\frac{\hat{g}(\xi)}{|\xi|}$.
Define $e^{it\sqrt{-\Delta}}$ by

$$\widehat{e^{it\sqrt{-\Delta}}f}(\xi) = e^{it|\xi|}\hat{f}(\xi).$$

Then, it suffices to show

$$\|P_0 e^{it\sqrt{-\Delta}} f\|_{L_t^q L_x^r} \lesssim \|f\|_{L^2(\mathbb{R}^n)} \tag{91}$$

To derive (91) we need to employ a $\mathcal{T}\mathcal{T}^*$ argument. Recall that, for $1 \leq p < \infty$,

$$\|f\|_{L^p} = \sup\{|\langle f, \varphi\rangle| : \varphi \in \mathcal{S}, \|\varphi\|_{L^{p'}} \leq 1\},$$

where p' denotes the conjugate exponent of p, i.e. $1/p + 1/p' = 1$.

Similarly, for $1 \leq q, r < \infty$, one has for the mixed norms,

$$\|F\|_{L_t^q L_x^r} = \sup\{|\langle F, \Phi\rangle| : \Phi \in \mathcal{S}, \|\Phi\|_{L_t^{q'} L_x^{r'}} \leq 1\}. \qquad (92)$$

Lemma 47 (TT^* argument)

The following statements are equivalent:

(i) $\mathcal{T} : L_x^2 \to L_t^q L_x^r$ *is bounded,*

(ii) $\mathcal{T}^* : L_t^{q'} L_x^{r'} \to L_x^2$ *is bounded,*

(iii) $\mathcal{T}\mathcal{T}^* : L_t^{q'} L_x^{r'} \to L_t^q L_x^r$ *is bounded.*

Proof. For any $f \in L_x^2$ and $F \in L_t^q L_x^r$ we have

$$|\langle \mathcal{T} f, F\rangle| = |\langle f, \mathcal{T}^* F\rangle| \leq \|f\|_{L_x^2} \|\mathcal{T}^* F\|_{L_x^2},$$

It follows from (92) that (ii) implies (i), and the converse follows from

$$|\langle f, \mathcal{T}^* F\rangle| = |\langle \mathcal{T} f, F\rangle| \leq |\mathcal{T} f\|_{L_t^q L_x^r} \|F\|_{L_t^{q'} L_x^{r'}}.$$

Obviously (i) and (ii) together imply (iii). Since

$$\|\mathcal{T}^* F\|_{L^2}^2 = \langle \mathcal{T}^* F, \mathcal{T}^* F\rangle = \langle F, \mathcal{T}\mathcal{T}^* F\rangle \leq \|F\|_{L_t^{q'} L_x^{r'}} \|\mathcal{T}\mathcal{T}^* F\|_{L_t^q L_x^r},$$

we conclude (iii) implies (ii). □

Return to the proof of (91). We define $\mathcal{T} : L^2 \to L_t^q L_x^r$ by

$$\mathcal{T} f := P_0 e^{it\sqrt{-\Delta}} f = \int_{\mathbb{R}^n} e^{i(t|\xi| + x\cdot\xi)} \psi(\xi) \hat{f}(\xi) d\xi, \qquad (93)$$

where $\psi(\xi)$ is the symbol of the Littlewood Paley projections.

Let $\mathcal{T}^*: L_t^{q'} L_x^{r'} \to L_x^2$ be the formal adjoint of $\mathcal{T}$. By Lemma 47, to show $\|\mathcal{T} f\|_{L_t^q L_x^r} \lesssim \|f\|_{L^2}$, it suffices to show

$$\|\mathcal{T}\mathcal{T}^*\|_{L_t^{q'} L_x^{r'} \to L_t^q L_x^r} \lesssim 1.$$

We need to calculate $\mathcal{T}^* F$. By definition,

$$\begin{aligned}\langle f, \mathcal{T}^* F\rangle_{L_x^2} &= \int_{\mathbb{R}}\int_{\mathbb{R}^n} \mathcal{T} f \cdot \bar{F} dx dt = \int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{it|\xi|} \psi(\xi) \hat{f}(\xi) \overline{\hat{F}(t,\xi)} d\xi dt \\ &= \int_{\mathbb{R}^n} f(x) \left(\int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi} e^{it|\xi|} \psi(\xi) \overline{\hat{F}(t,\xi)} d\xi dt \right) dx.\end{aligned}$$

This shows that

$$\mathcal{T}^* F(x) = \int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|)} \bar{\psi}(\xi) \hat{F}(t,\xi) d\xi dt.$$

Therefore

$$\widehat{\mathcal{T}\mathcal{T}^*F}(t,\xi) = e^{it|\xi|}\psi(\xi)\widehat{\mathcal{T}^*F}(\xi) = \int_{\mathbb{R}} e^{i(t-s)}|\psi(\xi)|^2\hat{F}(s,\xi)ds$$

Let

$$K_t(x) = K(t,x) := \int_{\mathbb{R}^n} e^{i(x\cdot\xi+t|\xi|)}|\psi(\xi)|^2 d\xi.$$

Then

$$\mathcal{T}\mathcal{T}^*F(t,x) = \int_{\mathbb{R}} K(t-s,\cdot) * F(s,\cdot)(x)ds.$$

where $K(t-s,\cdot) * F(s,\cdot)(x) := \int_{\mathbb{R}^n} K(t-s,y)F(s,x-y)dy$. We claim

$$\|K(t-s,\cdot) * F(s,\cdot)\|_{L^2_x} \leq C\|F(s,\cdot)\|_{L^2_x} \tag{94}$$

$$\|K(t-s,\cdot) * F(s,\cdot)\|_{L^\infty_x} \leq \frac{C\|F(s,\cdot)\|_{L^1_x}}{(1+|t-s|)^{\frac{n-1}{2}}} \tag{\textbf{Disp}}$$

Assuming the claim, by interpolation we have for $r \geq 2$ that

$$\|K(t-s,\cdot) * F(s,\cdot)\|_{L^r_x} \lesssim \frac{\|F(s,\cdot)\|_{L^{r'}_x}}{(1+|t-s|)^{\gamma(r)}} \tag{95}$$

with $\gamma(r) = \frac{n-1}{2}(1-\frac{2}{r})$. Thus we have

$$\begin{aligned}\|\mathcal{T}\mathcal{T}^* F(t,\cdot)\|_{L^r_x} &= \int \|K(t-s,\cdot) * F(s,\cdot)\|_{L^r_x} ds \\ &\lesssim \int \frac{\|F(s,\cdot)\|_{L^{r'}_x}}{(1+|t-s|)^{\gamma(r)}} ds. \end{aligned} \tag{96}$$

It remains to take L^q_t, for which we consider two cases $2/q < \gamma(r)$ and $2/q = \gamma(r)$.

Case 1. $2/q < \gamma(r)$. Note that $(1+|t|)^{-\gamma(r)}$ is $L^{\frac{q}{2}}(\mathbb{R})$. We need to use the Young's inequality

$$\|f * g\|_{L^q} \leq \|f\|_{L^a}\|g\|_{L^b} \tag{97}$$

where $1 \leq a, b, q \leq \infty$ satisfy $1 + \frac{1}{q} = \frac{1}{a} + \frac{1}{b}$.

We apply (97) with $f = (1+|t|)^{-\gamma(r)}$, $g = \|F(s)\|_{L_x^{r'}}$, $a = q/2$ and $b = q'$. It then follows that

$$\|\mathcal{T}\mathcal{T}^* F\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{q'} L_x^{r'}}.$$

Case 2. $2/q = \gamma(r)$. We need the Hardy-Littlewood inequality.

Theorem 48 (Hardy-Littlewood inequality)

Let $0 \le \lambda < 1$. Assume that $\frac{1}{a} + \frac{1}{b} + \frac{\lambda}{n} = 2$, there holds

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)|x-y|^{-\lambda} g(y)dxdy \le \|f\|_{L^a}\|g\|_{L^b}. \qquad (98)$$

We now take any $\varphi(t) \in L^q(\mathbb{R})$. It then follows from (96) and (98) with $f = \|F(s,\cdot)\|_{L_x^{r'}}$, $g = |\varphi|$, $a = b = q'$, $\lambda = \gamma(r)$ and $n = 1$ that

$$\int_{\mathbb{R}} \|\mathcal{T}\mathcal{T}^* F(t,\cdot)\|_{L_x^r}\varphi(t)dt \lesssim \int_{\mathbb{R}}\int_{\mathbb{R}} \|F(s,\cdot)\|_{L_x^{r'}}|t-s|^{-\gamma(r)}|\varphi(t)|dsdt$$
$$\lesssim \|\|F(s,\cdot)\|_{L_x^{r'}}\|_{L_t^{q'}}\|\varphi\|_{L_t^{q'}}.$$

Therefore

$$\|\mathcal{T}\mathcal{T}^* F\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{q'} L_x^{r'}}.$$

Remark. *(98) does not work for the end-point case that* $\frac{2}{q} = \gamma(r) = 1$*, which is settled by using atomic decomposition See Keel-Tao (1998).*

Step 4. Now we prove (94) and (**Disp**). Recall that

$$K_t(x) = K(t,x) = \int_{\mathbb{R}^n} e^{it|\xi|} e^{ix\cdot\xi} |\psi(\xi)|^2 d\xi.$$

We have

$$\|\hat{K}_t \cdot \hat{f}\|_{L^2} \lesssim \|e^{it|\xi|} |\psi(\xi)|^2 \hat{f}(\xi)\|_{L^2} \lesssim \|\hat{f}\|_{L^2}.$$

By Planchrel, we can obtain

$$\|K(t,\cdot)*f(\cdot)\|_{L^2_x} \leq C\|f\|_{L^2_x},$$

which gives (94).

Next we prove (**Disp**). It suffices to show that

$$|K(t,x)| \lesssim (1+|t|+|x|)^{-\frac{n-1}{2}}, \quad \forall (t,x). \tag{99}$$

It is easy to see that $|K(t,x)| \lesssim 1$ for any (t,x). Therefore it remains to consider $|t|+|x| \geq 1$. By using polar coordinates $\xi = \rho\omega$ and $\omega \in \mathbb{S}^{n-1}$, we have with $a(\rho) := \rho^{n-1}\psi(\rho)^2$ that

$$\begin{aligned} K(t,x) &= \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{i\rho(t+x\cdot\omega)} a(\rho)\,d\rho\, d\sigma(\omega) \\ &= \int_0^\infty e^{it\rho}\hat{\sigma}(\rho x) a(\rho)\,d\rho \end{aligned} \tag{100}$$

where $\hat{\sigma}(\xi) = \int_{\mathbb{S}^{n-1}} e^{i\xi\cdot\omega} d\sigma(\omega)$. We claim

$$|\hat{\sigma}(\xi)| \leq C(1+|\xi|)^{-\frac{n-1}{2}}, \quad \xi \in \mathbb{R}^n \tag{101}$$

Assume (101), we proceed to complete the proof of (99).

Case 1. $|t| < 2|x|$**.** We have

$$\begin{aligned} K(t,x) &= \int_0^\infty |\hat{\sigma}(\rho x)| a(\rho)^2 d\rho \lesssim \int_0^\infty |\rho x|^{-\frac{n-1}{2}} a(\rho) d\rho \\ &\lesssim |x|^{-\frac{n-1}{2}} \int_0^\infty \rho^{-\frac{n-1}{2}} a(\rho) d\rho. \end{aligned}$$

Note that $a(\rho)$ is supported within $\{\frac{1}{2} < \rho < 2\}$, thus we obtain

$$|K(t,x)| \lesssim |x|^{-\frac{n-1}{2}} \lesssim (|x|+t+1)^{-\frac{n-1}{2}}.$$

Case 2. $|t| \geq 2|x|$. Since $a(\rho)$ is supported within $\{\frac{1}{2} < \rho < 2\}$, by integration by parts we have

$$\begin{aligned} K(t,x) &= \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{i\rho(t+x\cdot\omega)} a(\rho) d\rho d\sigma(\omega) \\ &= \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{a(\rho)}{i(t+x\cdot\omega)} \frac{d}{d\rho} \left(e^{i\rho(t+x\cdot\omega)} \right) d\rho d\sigma(\omega) \\ &= -\int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{1}{i(t+x\cdot\omega)} e^{i\rho(t+x\cdot\omega)} a'(\rho) d\sigma(\omega) d\rho \end{aligned}$$

Repeating the procedure, we have

$$|K(t,x)| \lesssim |t|^{-N}$$

for any $N \in \mathbb{N}$, which shows it decays faster than $(|x| + t + 1)^{-\frac{n-1}{2}}$.

To complete the proof of (**Disp**), it remains to check (101). For simplicity, we only consider $n = 3$.
By rotational symmetry it suffices to take $\xi = (0, 0, \rho)$, $\rho = |\xi|$. Then using spherical coordinates on $\mathbb{S}^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$

$$\omega = \begin{cases} x = \sin\phi\cos\theta \\ y = \sin\phi\sin\theta \\ z = \cos\phi \end{cases}$$

where $0 < \phi < \pi, 0 < \theta < 2\pi$, we have

$$\begin{aligned} \hat{\sigma}(0, 0, \rho) &= \int_0^{\pi}\int_0^{2\pi} e^{-i\rho\cos\phi}\sin\phi d\theta d\phi \\ &= 2\pi\int_{-1}^{1} e^{i\rho r}dr = 4\pi\frac{\sin\rho}{\rho} \end{aligned}$$

Strichartz estimates for inhomogeneous wave equations

Consider the solution of inhomogeneous wave equation

$$\begin{aligned} \Box u = F \qquad & \text{on } \mathbb{R}^{1+n}, \ n \geq 2, \\ u|_{t=0} = f, \quad & \partial_t u|_{t=0} = g. \end{aligned} \tag{102}$$

By using Duhamel's principle and Theorem 46 we can obtain the Strichartz estimate for the solution of (102).

Theorem 49

Let (q, r) be wave admissible as defined in Theorem 46 and $s = \frac{n}{2} - \frac{1}{q} - \frac{r}{n}$. Then for any solution of (102) there holds

$$\|u\|_{L_t^q L_x^r} \leq C(\|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}} + \|F\|_{L_t^{q'} L_x^{r'}}) \tag{103}$$

An example

Now we consider the semi-linear wave equation

$$\begin{aligned} \Box u = u^3 \qquad & \text{on } \mathbb{R}^{1+3}, \\ (u, \partial_t u)|_{t=0} = (f, g) \in \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}} \end{aligned} \tag{104}$$

A function $u \in L_t^q L_x^r(\mathbb{R}^{1+n})$ with $3 \le q, r < \infty$ is called a weak solution of (104) if for any $\varphi \in C_0^(\mathbb{R}^{1+n})$ there holds

$$\int_0^\infty \int_{\mathbb{R}^n} u \Box \varphi dxdt + \int_{\mathbb{R}^n} \left[f \partial_t \varphi(0, \cdot) - g \varphi(0, \cdot) \right] dx = \int_0^\infty \int_{\mathbb{R}^n} u^3 \varphi dxdt.$$

In the following we will show that if

$$E_0 := \|f\|_{\dot{H}^{\frac{1}{2}}} + \|g\|_{\dot{H}^{-\frac{1}{2}}}$$

is sufficiently small, (104) has a global solution in $u \in L_t^4 L_x^4(\mathbb{R}^{1+n})$.

To see this, we define $u_{-1} \equiv 0$ and

$$\begin{aligned} \Box u_j &= u_{j-1}^3 \qquad \text{on } \mathbb{R}^{1+3}, \\ u_j(0,\cdot) &= f, \quad \partial_t u_j(0,\cdot) = g. \end{aligned} \tag{105}$$

Let

$$X(u_j) := \|u_j\|_{L_t^4 L_x^4} + \|u_j(t,\cdot)\|_{\dot{H}^{\frac{1}{2}}} + \|\partial_t u_j(t,\cdot)\|_{\dot{H}^{-\frac{1}{2}}}$$

Then it follows from (103) that

$$\begin{aligned} X(u_j) &\leq C\left(\|f\|_{\dot{H}^{\frac{1}{2}}} + \|g\|_{\dot{H}^{-\frac{1}{2}}} + \|u_{j-1}^3\|_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}}\right) \\ &\leq C\left(\|f\|_{\dot{H}^{\frac{1}{2}}} + \|g\|_{\dot{H}^{-\frac{1}{2}}} + \|u_{j-1}\|^3_{L_t^4 L_x^4}\right) \\ &\leq C\left(E_0 + X(u_{j-1})^3\right) \end{aligned} \tag{106}$$

By using $u_{-1} = 0$ and an induction argument, it is straightforward to show that

$$X(u_j) \le 2CE_0, \qquad j = 0, 1, \cdots \tag{107}$$

provided that $8C^3E_0^2 \le 1$.

Next we apply (103) to

$$\Box(u_{j+1} - u_j) = u_j^3 - u_{j-1}^3 = (u_j - u_{j-1})(u_j^2 + u_j u_{j-1} + u_{j-1}^2)$$

with vanishing initial data, and use (103) to obtain

$$\begin{aligned} X(u_{j+1} - u_j) &\le C_1 \|(u_j - u_{j-1})(u_j^2 + u_j u_{j-1} + u_{j-1}^2)\|_{L_t^{4/3} L_x^{4/3}} \\ &\le C_1 \|u_j - u_{j-1}\|_{L_t^4 L_x^4} \|u_j^2 + u_j u_{j-1} + u_{j-1}^2\|_{L_t^2 L_x^2} \\ &\le C_1 (X(u_j)^2 + X(u_{j-1}^2)) X(u_j - u_{j-1}). \end{aligned}$$

In view of (107), we obtain

$$X(u_{j+1}-u_j)\le C_2E_0^2X(u_j-u_{j-1})\le \frac{1}{2}X(u_j-u_{j-1})$$

provided E_0 is sufficiently small. So $\{u_j\}$ is a Cauchy sequence according to the norm $X(\cdot)$ with limit u. Since each u_j satisfies

$$\int_0^\infty\int_{\mathbb{R}^n} u_j\Box\varphi dxdt+\int_{\mathbb{R}^n}\left[f\partial_t\varphi(0,\cdot)-g\varphi(0,\cdot)\right]dx=\int_0^\infty\int_{\mathbb{R}^n}u_j^3\varphi dxdt$$

for all $\varphi\in C_0^\infty(\mathbb{R}^{1+n})$. By taking $j\to\infty$ we obtain

$$\int_0^\infty\int_{\mathbb{R}^n} u\Box\varphi dxdt+\int_{\mathbb{R}^n}\left[f\partial_t\varphi(0,\cdot)-g\varphi(0,\cdot)\right]dx=\int_0^\infty\int_{\mathbb{R}^n}u^3\varphi dxdt,$$

i.e. u is a globally defined weak solution of (104).